Basis tools to study Ordinary Differential Equation models

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What is mathematical modeling?

The translation of our beliefs about how a system functions into the language of mathematics.

This has many advantages:

- Mathematics is a very precise language.
- Mathematics is a concise language, with well rules for manipulations.
- Many results and theorems are available.
- Computers can be used to perform numerical calculations.

Some elements of compromising in mathematical modeling:

- The majority of interacting systems in the real world are too complicated to model in their entirety; Only most important factors are usually identified and considered.
- Restrictions (or assumptions) are often applied in the mathematical analysis.

Objectives of mathematical modeling

- Developing scientific understanding (through quantitative expression of current knowledge of a system).
- Testing the effect of changes in a system.
- Aiding decisions making (tactical decisions for managers and strategic decisions for planners).

Stages in Modelling

- Building
- Analysis
- Testing, interpreting, recommendations

Outline

Ordinary differential equations ODE

Notations for differentiation ($y = f(t)$):

- Leibniz's notation: $\frac{dy}{dt}$, $\frac{d^2y}{dt^2}$, ..., $\frac{d^{(n)}y}{dt^{(n)}}$
- Lagrange's notation: $y' = y^{(1)}$, $y'' = y^{(2)}$, ..., $y^{(n)}$
- Newton's notation: $\dot{y}$, $\ddot{y}$, $\dddot{y}$, ...

General form of ODEs of order n

Explicit form

$$y^{(n)} = F(t, y, y', y'', ..., y^{(n-1)}),$$

implicit form

$$F(t, y, y', y'', ..., y^{(n-1)}, y^{(n)}) = 0,$$

where F is a function of t, $y = y(t)$, and the derivatives of y.

Definition

Let $D \subset \mathbb{R}^{n+1}$, $J = (a, b) \subset \mathbb{R}$ and $F \in \mathcal{C}(D, \mathbb{R})$ (set of continuous functions $f : D \to \mathbb{R}$). **A solution of the ODE**

$$y^{(n)} = F(t, y, y', y'', ..., y^{(n-1)})$$

on J is a function $\phi \in \mathcal{C}^n(J, \mathbb{R})$ such that

$$(t, \phi(t), \phi'(t), \ldots, \phi^{(n-1)}(t)) \in D$$

and

$$\phi^{(n)}(t) = F(t, \phi(t), \phi'(t), \ldots, \phi^{(n-1)}(t)),$$

for all $t \in J$.

$$F(t, y, y', y'', ..., y^{(n-1)}, y^{(n)}) = 0$$

Classification of ODEs

- **Linear**:

  $$F = a_0 y + a y' + a_2 y'' + \dots a_{n-1} y^{(n-1)} + a_n y^{(n)} + r(t)$$

  coefficients $a_i = a(t)$.
  - **Homogeneous**: $r(t) = 0$ for all t.
  - **Nonhomogeneous**: $r(t) \neq 0$ for some t.
- **Nonlinear:** Not linear.
- **Autonomous:** F does not explicitly depend on t.

The order of an ODE is the highest order derivative.

Applications

Physical interpretation of the derivative

- $Q(t)$: density/size/concentration of a quantity Q at time t
- $\Delta Q = Q(t + \Delta t) - Q(t)$: change in Q over the time Δt
- $\frac{\Delta Q}{\Delta t}$: rate of change of Q in the time Δt
- $Q'(t) = \frac{dQ}{dt}$: instantaneous rate of change of Q w.r.t t

Example (Applications)

A drug decays in the body at a rate proportional to its present mass concentration. If the initial mass concentration is 4 g/ml and the half-life span is 80 years:

a. How much will be left after 50 years?

b. How long will it take for the mass concentration to be 0.75 g/ml?

- Drug decays at a rate proportional to its present mass

$$\underbrace{Q'(t)}_{\text{rate of change}} = -\underbrace{k}_{\text{cons of proportion}} \times \underbrace{Q(t)}_{\text{present mass con}}$$

- Initial mass is 4 g/ml and half-life span is 80 days

$$Q(0) = 4 \quad \text{and} \quad Q(80) = \frac{1}{2}Q(0) = 2$$

- How much will be left after 50 days?

$$Q(t) = Q(0)e^{-kt} = 4e^{-kt}$$

$$Q(80) = 4e^{-80k} = 2 \implies k = \frac{1}{80}\ln 2 = 0.00866$$

$$\mathrm{Q(50) = 4e^{-0.00866\times 50} = 2.594\ g/ml}$$

- Drug decays at a rate proportional to its present mass

$$\underbrace{Q'(t)}_{\text{rate of change}} = -\underbrace{k}_{\text{cons of proportion}} \times \underbrace{Q(t)}_{\text{present mass con}}$$

- Initial mass is 4 g/ml and half-life span is 80 years

$$Q(0) = 4 \quad \text{and} \quad Q(80) = \frac{1}{2}Q(0) = 2$$

- How long will it take for the mass to be 0.75 g/ml?

$$\mathrm{Q(t) = 4e^{-0.00866t} = 0.75 \implies t = 20days}$$

Applications: SIR model

- $S(t)$: number of susceptible individuals at time t
- $I(t)$: number of infectious individuals at time t
- $R(t)$: number of removed individuals at time t
- β: rate of infection of susceptible

$$\underbrace{S(t+\Delta)}_{\text{at time } t+\Delta t} = S(t) - \overbrace{\underbrace{\beta S(t)I(t)\Delta t}}^{\text{newly infected during } \Delta t}$$

$$I(t+\Delta) = I(t) + \beta S(t)I(t)\Delta t - \overbrace{\underbrace{\gamma I(t)\Delta t}}^{\text{newly removed during } \Delta t}$$

$$R(t+\Delta t) = R(t) + \gamma I(t)\Delta t$$

$$
\begin{aligned}
\frac{S(t+\Delta t)-S(t)}{\Delta t} &= -\underbrace{\beta S(t)I(t)}_{incidence}, \\
\frac{I(t+\Delta t)-I(t)}{\Delta t} &= \beta S(t)I(t)-\gamma I(t) \\
\frac{R(t+\Delta t)-R(t)}{\Delta t} &= \gamma R(t)
\end{aligned}
$$

SIR model (small time interval $\Delta t \to 0$)

$$
\begin{aligned}
s'(t) &= -\beta s(t)i(t) \\
i'(t) &= \beta s(t)i(t)-\gamma i(t) \\
r'(t) &= \gamma i(t)
\end{aligned}
$$

Single species model

- $N(t)$ be a population size at time t
- $b(N)$ and $d(N)$ the birth and death rates respectively
- $f(N)$ the variation within the population (resulting from immigration, emigration, natural disaster, etc.)

After a time Δt,

$$N(t+\Delta t) = N(t) + \underbrace{(b(N)N(t))\Delta t}_{newbirths} - \underbrace{(d(N)N(t))\Delta}_{newdeaths} t + f(N(t))\Delta t.$$

Then

$$\frac{N(t+\Delta t) - N(t)}{\Delta t} = (b(N) - d(N))N(t) + f(N(t)).$$

Single species model

For small time interval Δt (i.e., when $\Delta t \to 0$), we obtain the general single specie model

$$\frac{dN}{dt} = r(N)N + f(N), \tag{1}$$

which describes the rate of change in the population, with $r(N)$ being the growth rate.

- Constant growth (Malthus model): $r(N) = r$, r growth rate per capita
- Linear growth (logistic equation): $r(N) = r\left(1 - \frac{N}{K}\right)$
- cubic growth (Allee effect): $r(N) = r\left(1 - \frac{N}{K}\right)\left(\frac{N}{K_1} - 1\right)$

Analytic solutions

Methods to solve 1^{st} order (scalar case) ODE

- Linear equations: integrating factor
- Nonlinear equations:
  - Separable variables
  - Substitution methods (change of variables)

Method to solve a system of 1^{st} order linear ODE: Fundamental matrix

Linear 1^{st} order ODE

How to solve the linear ODE $y' + p(x)y = f(x)$?

1. Consider the complementary equation (CE):

$$y_c' + p(x)y_c = 0$$

2. Find one (nontrivial) solution y_1 of the CE
3. The general solution is $y = uy_1$, where u is solution of

$$u(x) = \int \frac{f(x)}{y_1(x)} dx.$$

Nonlinear - separable ODE

A 1^{st} ODE is said to be **separable** if it can be written in the form

$$\frac{dy}{dx} = f(x)g(y).$$

How do we solve the separable ODE?

1. Divide both sides by $g(y)$

$$\frac{1}{g(y)}\frac{dy}{dx} = f(x)$$

2. Integrate both sides w.r.t x

$$\int \frac{1}{g(y)}\frac{dy}{dx}dx = \int f(x)dx \implies \int \frac{1}{g(u)}du = \int f(x)dx$$

$u = y(x)$

3. Solve the integral equation to get an implicit solution.

Nonlinear - Bernoulli

The **Bernoulli's equation** is a nonlinear ODE of the form

$$y' + p(x)y = f(x)y^r,$$

where $r \in \mathbb{R}$ is a parameter, with $r \neq 0$ and $r \neq 1$.

How do we solve Bernoulli's equations?

- Find a non-trivial solution y_1 of the CE

$$y' + p(x)y = 0.$$

- Let $y = uy_1$ be a solution of the Bernoulli's equation.
- Substitute $y = uy_1$ in the Bernoulli's equation and generate a separable DE in u.
- Solve the separable DE to find u.
- The general solution of the Bernoulli equation is $y = uy_1$.

System of 1^{st} order linear ODE

Consider the n^{th}-dimensional IVP

$$\frac{d}{dt}\boldsymbol{X}(t) = \boldsymbol{A}(t)\boldsymbol{X}(t) + \boldsymbol{B}(t), \quad \boldsymbol{X}(t_0) = \boldsymbol{X}_0.$$

Its solution can be expressed as

$$\boldsymbol{X}(t) = \Phi(t)\Phi^{-1}(t_0)\boldsymbol{X}_0 + \Phi(t)\int_{t_0}^{t} \Phi^{-1}(s)\boldsymbol{B}(s)ds$$

where $\Phi(t)$ is a fundamental matrix of the corresponding homogeneous system.

Definition

A **fundamental matrix** of solutions of the homogeneous system $\boldsymbol{X}'(t) = \boldsymbol{A}(t)\boldsymbol{X}(t)$ is $\Phi(t) = (\boldsymbol{X}_1(t), \dots, \boldsymbol{X}_n(t))$, where the columns of $\Phi(t)$ are the n linearly independent solution vectors $\boldsymbol{X}_i(t)$.

Homogeneous linear system with constant coefficients

$$\frac{d}{dt}\boldsymbol{X}(t) = \boldsymbol{A}\boldsymbol{X}(t), \quad \boldsymbol{X}(t_0) = \boldsymbol{X}_0$$

where $\boldsymbol{A} = (a_{i,j})$ is a $n \times n$ constant matrix with real elements.

- If $\det(\boldsymbol{A}) \neq 0$, the unique equilibrium solution is $\boldsymbol{X}(t) = 0$, $\forall t \in \mathbb{R}$.
- The general solution is $\boldsymbol{X}(t) = e^{\boldsymbol{A}t}\boldsymbol{C}$. $\forall t \in \mathbb{R}$, where $e^{\boldsymbol{A}t}$ (matrix exponent) is an $n \times n$ matrix, and $\boldsymbol{C}$ an arbitrary constant vector.
- $\Phi(t) = e^{\boldsymbol{A}t}$ is the fundamental matrix and $\Phi(0) = \boldsymbol{I}_n$.
- $e^{\boldsymbol{A}t} = \boldsymbol{I}_n + \boldsymbol{A}t + \frac{t^2}{2!}\boldsymbol{A}^2 + \frac{t^3}{3!}\boldsymbol{A}^3 + \cdots = \sum_{i=0}^{\infty} \frac{t^i}{i!}\boldsymbol{A}^i$, $\forall t \in \mathbb{R}$.

Homogeneous linear system with constant coefficients

$$\frac{d}{dt}\boldsymbol{X}(t) = \boldsymbol{A}\boldsymbol{X}(t)$$

where $\boldsymbol{A} = (a_{ij})$ is a $n \times n$ constant matrix with real elements.

Instead of computing $e^{\boldsymbol{A}t}$, we can find n linearly independent solutions $\boldsymbol{X}_i(t)$ (to form a fundamental matrix)

- Let $\boldsymbol{X}_i(t) = e^{\lambda_i t}\boldsymbol{u_i}$ (λ_i= unknown scalar, $\boldsymbol{u_i}$ = unknown $n \times 1-$vector).
- So $\boldsymbol{A}\boldsymbol{u_i} = \lambda_i \boldsymbol{u_i}$ where λ_i is an eigenvalue of $\boldsymbol{A}$ and $\boldsymbol{u_i}$ is an eigenvector associated to λ_i.
- To find λ_i / u_i ($i \in 1, \dots, n$), solve

$$\det(\boldsymbol{A} - \lambda \boldsymbol{I}_n) = 0, \ / \ (\boldsymbol{A} - \lambda_i \boldsymbol{I_n})\,\boldsymbol{u_i} = 0.$$

Homogeneous linear system with constant coefficients : n distinct eigenvalues

Theorem

Let $\lambda_1, \ldots, \lambda_n$ be n distinct eigenvalues of the coefficient matrix $\boldsymbol{A}$ of the homogeneous system

$$\frac{d}{dt}\boldsymbol{X} = \boldsymbol{A}\boldsymbol{X},$$

and let $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_n$ be the corresponding eigenvectors. Then the general solution of the homogeneous system on the interval $(-\infty, \infty)$ is given by

$$\boldsymbol{X}(t) = c_1\boldsymbol{u}_1 e^{\lambda_1 t} + \cdots + c_n\boldsymbol{u}_n e^{\lambda_n t}$$

with $c_1, \ldots, c_n$ arbitrary constants.

Complex conjugate eigenvalues

Theorem

Let $\boldsymbol{A}$ the matrix with real entries of the homogeneous system $\frac{d}{dt}\boldsymbol{X} = \boldsymbol{A}\boldsymbol{X}$, and let $\boldsymbol{u}_1$ be an eigenvector corresponding to the complex eigenvalue $\lambda_1 = \alpha + i\beta$, with α and β real. Then,

$$\boldsymbol{X}_1(t) = \boldsymbol{u}_1 e^{\lambda_1 t}, \qquad \boldsymbol{X}_2(t) = \bar{\boldsymbol{u}}_1 e^{\bar{\lambda}_1 t}$$

are solutions of the homogeneous system.
Moreover, if $\boldsymbol{u}_1 = \boldsymbol{a} + i\boldsymbol{b}$, then

$$\begin{aligned}
\boldsymbol{X}_1(t) &= (\boldsymbol{a}\cos(\beta t) - \boldsymbol{b}\sin(\beta t))\, e^{\alpha t}, \\
\boldsymbol{X}_2(t) &= (\boldsymbol{b}\cos(\beta t) + \boldsymbol{a}\sin(\beta t))\, e^{\alpha t}
\end{aligned}$$

are linearly independent solutions of the homogeneous system on $\mathbb{R}$.

Example

Consider the ODE

$$x'(t) = \sin x.$$

Solution

$$\int \frac{dx}{\sin x} = \int dt \implies -\ln|\csc x + \cot x| + c = t.$$

For a given initial condition $x(0) = x_0$, we obtain the implicit solution

$$\ln\left|\frac{\csc x_0 + \cot x_0}{\csc x + \cot x}\right| = t.$$

Does a solution exist? Is it unique? Can we describe the feature of the solution for a given x_0 (say, $x_0 = \pi/4$)?

Well-posedness

In mathematics, a problem is said to be **well-posed** (in the sense of Hadamard [2]) if the following properties hold:

- The problem has a solution
- The solution is unique
- The solution's behavior changes continuously with the initial conditions

Existence

Consider the the initial value problem

$$x'(t) = f(t, x(t)), \quad (2)$$
$$x(t_0) = x_0, \quad (3)$$

where $I \subset \mathbb{R}$ is an interval, $\omega \subset \mathbb{R}^n$ is open, $f : I \times \Omega \longrightarrow \mathbb{R}^n$ a map, and $(t_0, x_0) \in I \times \Omega$.

Cauchy-Peanot's theorem(Simpson, 1984)

Assume that for every $x \in \omega$, there exist $\delta > 0$, $c \in L^1(I, [0, \infty))$ and a non-decreasing function $\omega : [0, \infty) \longrightarrow [0, \infty)$ with $\lim_{h \to 0} w(h) = 0$ such that

$$\|f(t, y) - f(t, z)\| \leq c(t) w\left(\|y - z\|\right), \quad \|f(t, y)\| \leq c(t)$$

for a.e $t \in I$ and $y, z \in B(x, \delta)$. Then the solution of (2)-(3) exists locally in the interval $(t_0 - \epsilon, t_0 + \epsilon)$, with $\epsilon > 0$.

Sketch of the proof

- Subdivide $[0, \epsilon]$ into N sub-intervals $[t_i, t_{i+1}]$, $i = 0 \ldots N-1$ (N fixed)
- Construct a equicontinuous sequence $\{x^N\}$ solution of the problem

$$\begin{aligned} x'(t) =& f(t, x_{k-1}(t)), \quad \text{on } [t_0, t_k] \\ x(t_{k-1}) =& x_{k-1}, \quad k = 1, \ldots N \end{aligned}$$

- Apply Artela-Ascoli theorem to show the existence of a convergent subsequence $\{x^{N_k}\}$ of $\{x^N\}$
- Show that $\{x^{N_k}\}$ is the solution of the IVP

Uniqueness

Consider the the initial value problem

$$x'(t) = f(t, x(t)), \quad (4)$$
$$x(t_0) = x_0, \quad (5)$$

where $I \subset \mathbb{R}$ is an interval, $\omega \subset \mathbb{R}^n$ is open, $f : I \times \Omega \longrightarrow \mathbb{R}^n$ a map, and $(t_0, x_0) \in I \times \Omega$.

Cauchy-Caratheodory's theorem [Caratheodory, 1963]

Assume that for all $x \in \omega$, there exist $\delta > 0$, $c \in L^1(I, [0, \infty))$ such that

$$\|f(t, y) - f(t, z)\| \leq c(t)\|y - z\|, \quad \|f(t, y)\| \leq c(t) \quad (6)$$

for almost every $t \in I$ and $y, z \in B(x, \delta)$. Then there is a unique solution of the Cauchy problem (4)-(5).

Corrolary

- Picard–Lindelöf's theorem: f is continuous and Lipschitz in x (the second variable).
- Picard's theorem: f and $\frac{\partial f}{\partial x}$ are continuous on some open rectangle.

Example

The IVP

$$x'(t) = \sin x, \quad x(0) = x_0$$

has a unique solution in the interval $[0, \epsilon]$, with $\epsilon > 0$, since the function $f(x) = \sin x$ is of class $\mathcal{C}^1$. Implicit solution

$$\ln \left| \frac{\csc x_0 + \cot x_0}{\csc x + \cot x} \right| = t.$$

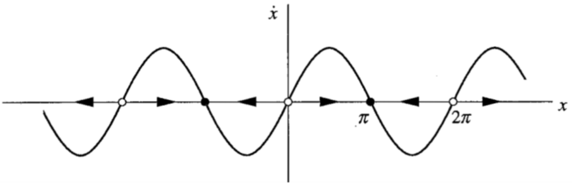
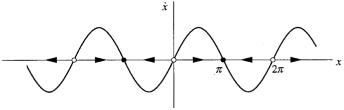
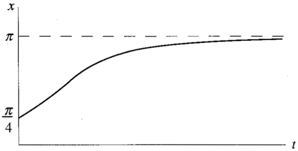
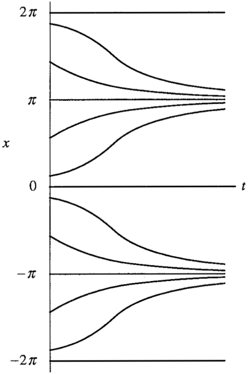


Figure: Phase portrait


$\dot{x}$
x
π
2π


x
π
$\frac{\pi}{4}$
t


2π
π
x
0
t
$-\pi$
-2π

Consider the autonomous system

$$x'(t) = f(x(t)), \tag{7}$$

where $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}^n$. Assume that $f \in \mathcal{C}^1$.

Definition

An equilibrium point x^* of the system (7) is a real solution of the equation $f(x) = 0$.

Example (SI model with demography)

$$s'(t) = \Pi - \beta s(t)i(t) + \gamma i(t) - \mu s(t)$$
$$i'(t) = \beta s(t)i(t) - \gamma i(t) - \mu i(t)$$

Equilibrium points: $(\frac{\Pi}{\mu}, 0)$ or (s^*, i^*)

Stability: An equilibrium point x^* is said to be

- Lyapunov stable if for all $\epsilon > 0$, there exists $\delta > 0$ such that if $\|x(0) - x^*\| < \delta$, then $\|x(t) - x\| < \epsilon$.
- Asymptotically stable if it is Lyapunov stable and for all $\epsilon > 0$, there exists $\delta > 0$ such that if $\|x(0) - x^*\| < \delta$, then $\lim_{x\to\infty} \|x(t) - x^*\| = 0$.
- Globally stable if $\lim_{x\to\infty} x(t) = x^*$ for all $t \geq 0$.
- Unstable if it is not stable.

Example (SI model with demography)

$$s'(t) = \Pi - \beta s(t)i(t) + \gamma i(t) - \mu s(t)$$
$$i'(t) = \beta s(t)i(t) - \gamma i(t) - \mu i(t)$$

Stability

Disease free equilibrium $(\frac{\Pi}{\mu}, 0)$
Endemic equilibrium (s^*, i^*)

Linear systems

Asymptotic behavior of solutions of

$$\frac{dx}{dt} = Ax, \tag{8}$$

where $x = (x_1, \ldots, x_n)^T$ and $A = (a_{i,j}) \in \mathcal{M}(n \times n)$.

Theorem

If all the roots of the eigenvalues of A have negative real part, then given any solution x(t) of (8), there exist positive constants M and b such that

$$||x(t)|| \leq Me^{-bt}, \quad \forall t > 0$$

and

$$\lim_{t \to \infty} ||x(t)|| = 0.$$

Linear systems

Consider the 2-dimensional autonomous linear system

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix}, \quad A = (a_{i,j}) \in \mathcal{M}(2 \times 2). \tag{9}$$

Let λ_1 and λ_2 be the eigenvalues of the matrix A.
Case 1: Assume $\lambda_1 \neq \lambda_2$. Then A is diagonalisable and can be written in the form $A = PDP^{-1}$. Then (9) can be reduced to

$$\begin{pmatrix} u' \\ v' \end{pmatrix} = D \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}.$$

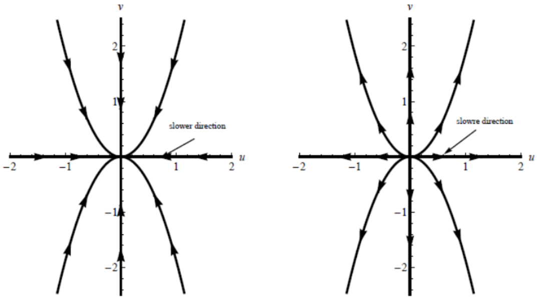
$$\frac{v'}{v} = \frac{\lambda_2}{\lambda_1} \frac{u'}{u} = \lambda \frac{u'}{u} \implies v = ku^{\lambda}.$$

$$u(t) = u(0)e^{\lambda_1 t}, v(t) = v(0)e^{\lambda_2 t}.$$

Linear systems

$$v = ku^{\lambda} \quad u(t) = u(0)e^{\lambda_1 t}, v(t) = v(0)e^{\lambda_2 t}.$$

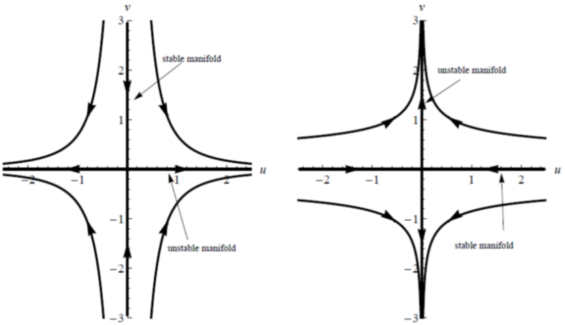
Case 1a $\lambda_1, \lambda_2 \in \mathbb{R}^*$ and have same signs: (0,0) is a node. It is stable when $\lambda_1 < 0$ and unstable when $\lambda_1 > 0$.



Linear systems

$$v = ku^{\lambda} \quad u(t) = u(0)e^{\lambda_1 t}, v(t) = v(0)e^{\lambda_2 t}.$$

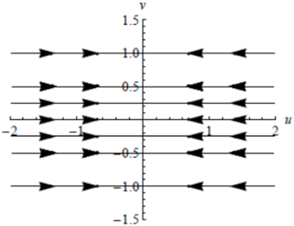
Case 1b $\lambda_1, \lambda_2 \in \mathbb{R}$ and opposite signs: (0,0) is a saddle point.



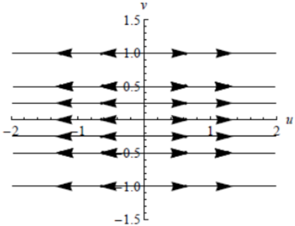
Linear systems

$$v = ku^{\lambda} \quad u(t) = u(0)e^{\lambda_1 t}, v(t) = v(0)e^{\lambda_2 t}.$$

Case 1c If $\lambda_1 = 0$ or $\lambda_2 = 0$, then the origin is a non isolated equilibrium point; we have a whole line of equilibrium points.



$\lambda_1 < 0, \lambda_2 = 0$



$\lambda_1 > 0, \lambda_2 = 0$

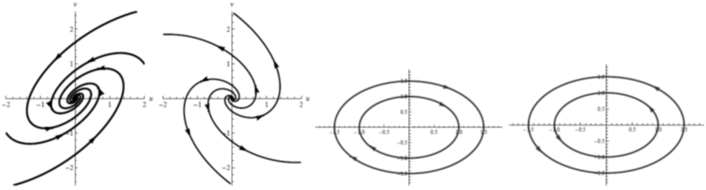
Linear systems

$$u' = \lambda_1 u \qquad (10)$$

Case 1d $\lambda_1, \lambda_2 \in \mathbb{C}$, with $\lambda_1 = \alpha + i\beta$.
Assume u of the form $u = re^{i\theta}$ and substitute into (10)

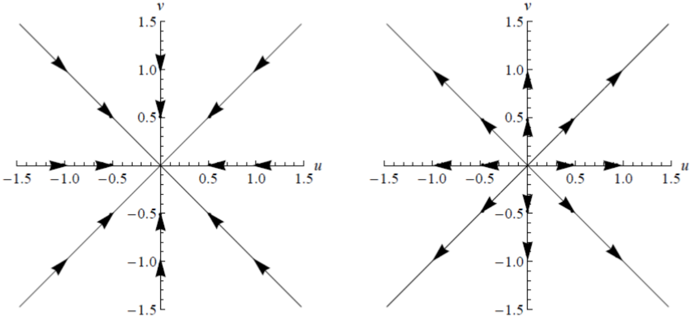
$$r' = \alpha r, \quad \theta' = \beta.$$

$$r(t) = r_0 e^{\alpha t} = r_0 e^{\alpha\left(\frac{\theta - \theta_0}{\beta}\right)} = r_0 e^{b\theta}, \theta = \beta t + \theta_0.$$



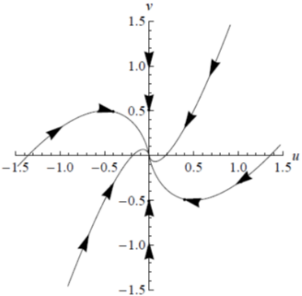
(a) $\alpha < 0$ (b) $\alpha > 0$ (c) $\alpha = 0$ and $\beta < 0$ (d) $\alpha = 0$ and $\beta > 0$

Case 2a: Assume $\lambda_1 = \lambda_2 = \lambda$ and there are two independent eigenvectors associated with λ: (0,0) is a star.
It is stable when $\lambda < 0$ and unstable when $\lambda > 0$.



(a) Case $\lambda < 0$ (b) Case $\lambda > 0$

Case 2b: Assume $\lambda_1 = \lambda_2 = \lambda$ and there is only one eigenvector associated with λ: (0,0) is a degenerate node.
It is stable when $\lambda < 0$ and unstable when $\lambda > 0$.



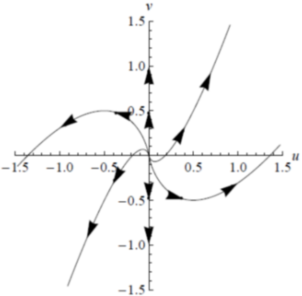
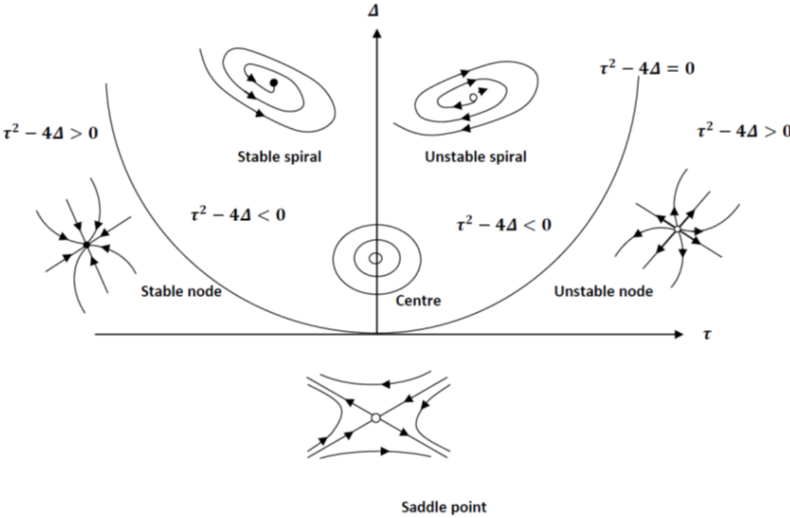


Diagram of the classification of equilibrium points



Nonlinear systems

Consider the nonlinear autonomous system

$$x'(t) = f(x(t)), \tag{11}$$

where $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ and $f = (f_1, \dots, f_n) : \mathbb{R}^n \to \mathbb{R}^n$. Assume that $f \in \mathcal{C}^1$. Let $x^* = (x_1^*, \dots, x_n^*)$ be an equilibrium point of the system (11).
Consider the small perturbation $u = x - x^*$ near x^*. By the Taylor's expansion we get:

$$\begin{aligned} u_i'(t) &= x_i'(t) \\ &= f_i(u + x^*) = f_i(u_1 + x_1^*, \dots, u_n + x_n^*) \\ &= u_1 \frac{\partial f_i}{\partial x_1}(x^*) + \dots u_n \frac{\partial f_i}{\partial x_n}(x^*) + \mathcal{O}(|u|^2). \end{aligned}$$

In the matrix form

$$u'(t) = Au + \mathcal{O}(|u|^2)$$

where A is the Jacobian matrix of the system (11) evaluated at x^*, given by

$$A = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(x^*) & \dots & \frac{\partial f_1}{\partial x_2}(x^*) \\ \vdots & & \vdots \\ \frac{\partial f_n}{\partial x_1}(x^*) & \dots & \frac{\partial f_n}{\partial x_1}(x^*) \end{pmatrix}$$

The system

$$u'(t) = Au$$

is called the linearized system associated to (11) near x^*.

Consider the nonlinear autonomous system

$$x'(t) = f(x), \tag{12}$$

where $x \in \mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}^n$ of class $\mathcal{C}^1$.

Hartman–Grobman theorem (linearisation theorem) [4, 3]

If the system (12) has a **hyperbolic equilibrium point** x^*, then in a neighborhood of x^*, the system (12) and its associated linearized system $u' = Au$ are **topological conjugate** (there exists a homeomorphism that will conjugate the one into the other).
$\exists N = \mathcal{V}(x^*)$, $\exists h : N \to \mathbb{R}^n$ homeomorphism with $h(u^*) = 0$, such that in N the flow $x' = f(x)$ is topological conjugate by the continuous map $v = h(u)$ to the flow $v' = Av$.

Planar systems (with constant coefficients)

$$\frac{dx}{dt} = f(x, y)$$
$$\frac{dy}{dt} = g(x, y)$$

Theorem

Assume that f and g are of class C^1 in some open set containing the equilibrium $(\bar{x}, \bar{y})$ of the system. Then the equilibrium is locally asymptotically stable if

$$\mathrm{tr}(J_{(\bar{x},\bar{y})}) < 0 \quad \text{and} \quad \det(J_{(\bar{x},\bar{y})}) > 0,$$

where $J_{(\bar{x},\bar{y})}$ is the Jacobian matrix evaluated at the equilibrium. In addition, the equilibrium is unstable if either $\mathrm{tr}(J_{(\bar{x},\bar{y})}) > 0$ or $\det(J_{(\bar{x},\bar{y})}) > 0$.

Tools to determine the sign of eigenvalues

Consider polynomial

$$p(\lambda) = a_0\lambda^n + a_1\lambda^{n-1} + \ldots + a_{n-1}\lambda + a_n,$$

where the $a_i's$, $i = 1, \ldots, n$, are real constants coefficients and $a_0 > 0$. Define the n Hurwitz matrices:

$$H_1 = (a_1), H_2 = \begin{pmatrix} a_1 & a_0 \\ a_3 & a_2 \end{pmatrix}, H_3 = \begin{pmatrix} a_1 & a_0 & 0 \\ a_3 & a_2 & a_1 \\ a_5 & a_4 & a_3 \end{pmatrix},$$

$$H_n = \begin{pmatrix} a_1 & a_0 & 0 & \vdots & 0 \\ a_3 & a_2 & a_1 & \vdots & 0 \\ a_5 & a_4 & a_3 & \vdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \vdots & a_n \end{pmatrix}$$

All the roots of $p(\lambda)$ have negative real parts if and only if $a_0 > 0$ and the determinants of the n Hurwitz matrices are positive, i.e.

$$\Delta_1 = a_1 > 0, \Delta_2 = \det(H_2) > 0, \cdots, \Delta_n = \det(H_n) > 0.$$

In particular,

i. When $n = 1$: $a_1 > 0$.
ii. When $n = 2$: $a_1 > 0$ and $a_2 > 0$.
iii. When $n = 3$: $a_1 > 0$, $a_2 > 0$ and $a_1 a_2 - a_0 a_3 > 0$.
iv. When $n = 4$: $a_1 > 0$, $a_2 > 0$, $a_1 a_2 - a_0 a_3 > 0$ and $a_1 a_2 a_3 - a_1^2 a_4 - a_0 a_3^2 > 0$.

Global stability: Planar systems

$$\frac{dx}{dt} = f(x, y)$$
$$\frac{dy}{dt} = g(x, y)$$

with the initial conditions $X_0 = (x(t_0), y(t_0)) = (x_0, y_0)$.

$\Rightarrow$ Poincaré-Bendixson Theorem (for global stability analysis)

Planar systems

$$\frac{dx}{dt} = f(x, y)$$
$$\frac{dy}{dt} = g(x, y)$$

with initial conditions $X_0 = (x(t_0), y(t_0))^T = (x_0, y_0)^T$.

- $\Gamma(X_0, t)$: solution trajectory (as a function of time) starting at X_0
- $\Gamma^+(X_0, t)$: part of solution trajectory where $t \geq t_0$ (positive orbit)
- $\Gamma^-(X_0, t)$: part of solution trajectory where $t \leq t_0$ (negative orbit)
- $\alpha-$limit set, $\alpha(X_0)$: set of points in the plane that are approached by the negative orbit $\Gamma^-(X_0, t)$, as $t \to -\infty$
- $\omega-$limit set, $\omega(X_0)$: set of points in the plane that are approached by the positive orbit $\Gamma^+(X_0, t)$, as $t \to +\infty$

Definition

A periodic solution $X(t)$ of $\frac{dX}{dt} = f(X)$ is a non-constant solution satisfying $X(t+T) = X(t)$ for all t on the interval of existence ($T > 0$ is called the period).

(No periodic solutions in autonomous scalar differential equations)

Definition

A limit cycle is the orbit of an isolated periodic solution.

Existence of periodic solutions

Poincaré-Bendixson theorem

Let $\Gamma^+(X_0, t)$ be a positive orbit of

$$\frac{dx}{dt} = f(x, y)$$
$$\frac{dy}{dt} = g(x, y)$$

that remains in a closed and bounded region of the plane. Suppose that the $\omega-$limit set does not contain any equilibria. Then either

- $\Gamma^+(X_0, t)$ is a periodic orbit ($\Gamma^+(X_0, t) = \omega(X_0)$),
- or $\omega-$limit set, $\omega(X_0)$, is a periodic orbit.

Theorem

Every periodic orbit (closed orbit) must enclose an equilibrium

Poincaré-Bendixson trichotomy

Let $\Gamma^+(X_0, t)$ be a positive orbit of

$$\begin{aligned}\frac{dx}{dt} &= f(x,y)\\ \frac{dy}{dt} &= g(x,y)\end{aligned}$$

that remains in a closed and bounded region B of the plane. Suppose B contains only a finite number of equilibria. Then the $\omega-$limit set takes ones of the following 3 forms:

- $\omega(X_0)$ is an equilibrium,
- $\omega(X_0)$ is a periodic orbit,
- $\omega(X_0)$ (cycle graph) contains a finite number of equilibria and a set of trajectories Γ_i whose $\alpha-$ and $\omega-$limit sets consist of one of these equilibria for each trajectory Γ_i.

Dulac's criterion

Suppose D is a simply connected open subset of the plane and $\beta(x,y)$ is a real-valued continuously differentiable function in D. If

$$\frac{\partial(\beta f)}{\partial x} + \frac{\partial(\beta g)}{\partial y}$$

is not identically zero and does not change sign in D, then there is no periodic solutions in D of the autonomous system

$$\frac{dx}{dt} = f(x,y), \quad \frac{dy}{dt} = g(x,y). \tag{13}$$

Definition

A region D of the plane is said to be simply connected if every closed loop within D can be shrunk to a point without leaving D.

Bendixson's criterion

Suppose D is a simply connected open subset of the plane. If

$$\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}$$

is not identically zero and does not change sign in D, then there is no periodic solutions of the autonomous system

$$\begin{aligned} \frac{dx}{dt} &= f(x, y) \\ \frac{dy}{dt} &= g(x, y) \end{aligned}$$

in D.

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