A short introduction to partial differential equations with applications in life sciences

J. Banasiak

Department of Mathematics and Applied Mathematics

University of Pretoria, Pretoria, South Africa

Basic facts from Calculus. In one-dimensional space $\mathbb{R}$, typical sets with which we will be concerned are open intervals $]a, b[$, where $-\infty \leq a < b \leq +\infty$. For $-\infty < a < b < +\infty$, by $[a, b]$ we will denote the closed interval with endpoints a, b. In this case, we say that (a, b) is the interior of the interval, $[a, b]$ is its closure and the two-point set consisting of $\{a\}$ and $\{b\}$ constitutes the boundary.

For a general set $\Omega \in \mathbb{R}^n$, by $\mathring{\Omega}$ we denote its interior, by $\overline{\Omega}$ its closure, and by $\partial\Omega$ its boundary.

In typical cases considered here, the boundary $\partial\Omega$ of a two-dimensional region Ω is a closed curve. The two most used analytic descriptions of curves in $\mathbb{R}^2$ are:

(i) as a level curve of a function of two variables

$$F(x_1, x_2) = c,$$

(ii) by using two functions of a single variable

$$x_1(t) = f(t), \quad x_2(t) = g(t),$$

where $t \in [t_0, t_1]$ (parametric description).

Note that if the curve is to be closed, we must have $f(t_0) = f(t_1)$ and $g(t_0) = g(t_1)$.

When $\Omega \in \mathbb{R}^3$, typically $\partial\Omega$ is a two-dimensional surface. This surface can be analytically described as a level surface of a function of three variables

$$F(x_1, x_2, x_3) = c,$$

or parametrically by, this time, three functions of two variables each:

$$x_1(t,s) = f(t,s), \quad x_2(t,s) = g(t,s), \quad x_3(t,s) = h(t,s),$$

$t \in [t_0, t_1],\ s \in [s_0, s_1]$.

Examples. Let us consider the elliptical disk $\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} \leq 1$. The boundary is then the ellipse, given either as the level curve

$$F(x_1, x_2) = \frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} = 1.$$

or

$$x_1(t) = a\cos t, \qquad x_2(t) = b\sin t,$$

with $t \in [0, 2\pi]$.

In $\mathbb{R}^3$, the boundary of $\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} + \frac{x_3^2}{c^2} \leq 1$ is given as the level curve

$$F(x_1, x_2, x_3) = \frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} + \frac{x_3^2}{c^2} = 1.$$

or parametrically as

$$f(t, s) = a\cos t \sin s, \quad g(t, s) = b\sin t \sin s, \quad h(t, s) = c\cos s,$$

where $t \in [0, 2\pi], s \in [0, \pi]$.

Vanishing theorems

Theorem 1

Let f be a continuous function in $\bar{\Omega}$, where $\Omega \subset \mathbb{R}^d$ is a bounded domain. Assume that $\forall_{\boldsymbol{x}\in\bar{\Omega}}\, f(\boldsymbol{x}) \geq 0$ and $\int\limits_{\Omega} f(\boldsymbol{x})d\boldsymbol{x} = 0$. Then $f \equiv 0$ in $\bar{\Omega}$.

Theorem 2

Let f be a continuous function in a domain Ω such that $\int\limits_{\Omega_0} f(\boldsymbol{x})d\boldsymbol{x} = 0$ for any $\Omega_0 \subset \Omega$. Then $f \equiv 0$ in Ω.

In the proofs of both theorems the crucial role is played by the theorem on local sign preservation by continuous functions.

Normal vectors. For a given point $\boldsymbol{p} \in \partial\Omega$ the outward normal vector is the vector $\boldsymbol{n}$, normal (perpendicular) to the boundary at p, pointing outside Ω, and having unit length.

If the boundary of set $\Omega \in \mathbb{R}^n, n \geq 2$, is given as a level set of a function F, then the vector given by

$$\boldsymbol{N}(\boldsymbol{p}) = \nabla F(\boldsymbol{p}),$$

is perpendicular to the boundary at $\boldsymbol{p}$. However, it is not necessarily unit, nor outward. To make it a unit vector, we divide $\boldsymbol{N}$ by its length; then the unit outward normal is either $\boldsymbol{n} = \boldsymbol{N}/\|\boldsymbol{N}\|$, or $\boldsymbol{n} = -\boldsymbol{N}/\|\boldsymbol{N}\|$ and the proper sign must be selected by inspection.

Example. To find the unit outward normal to the ellipsoid

$$F(x_1, x_2, x_3) = x_1^2 + \frac{x_2^2}{4} + \frac{x_3^2}{9} = 1,$$

at the point $\boldsymbol{p} = (1/\sqrt{2}, 0, 3/\sqrt{2})$, we have

$\nabla F(x_1, x_2, x_3) = (2x_1, x_2/2, 2x_3/9)$, hence

$$\boldsymbol{N}(\boldsymbol{p}) = \nabla F(\boldsymbol{p}) = (2/\sqrt{2}, 0, 2/3\sqrt{2})$$

with

$$\|\boldsymbol{N}(\boldsymbol{p})\| = \sqrt{2 + 2/9} = \sqrt{20/9} = 2\sqrt{5}/3.$$

Since the vector pointing outside the ellipsoid must necessarily

point away from the origin, we obtain

$$\boldsymbol{n}(\boldsymbol{p}) = (3/\sqrt{10}, 0, 1/\sqrt{10}).$$

Normal derivative of a function. Let now f be defined in a neighbourhood of a point $\boldsymbol{p} \in \partial\Omega$. The *normal derivative* of f at $\boldsymbol{p}$ is defined as the derivative of f in the direction of $\boldsymbol{n}(\boldsymbol{p})$:

$$\frac{\partial f}{\partial \boldsymbol{n}}(\boldsymbol{p}) = f_{\boldsymbol{n}}(\boldsymbol{p}) = \nabla f|_{\boldsymbol{p}} \cdot \boldsymbol{n}(\boldsymbol{p}).$$

Example. Let us consider the spherical coordinates

$$x_1 = r\cos\theta\sin\phi, \quad x_2 = r\sin\theta\sin\phi, \quad x_3 = r\cos\phi.$$

Any function $f(x_1, x_2, x_3)$ of three variables can be expressed in the spherical coordinates as

$$F(r,\theta,\phi) = f(r\cos\theta\sin\phi, r\sin\theta\sin\phi, r\cos\phi) = f(x_1, x_2, x_3).$$

Using the Chain Rule we have

$$\frac{\partial F}{\partial r} = \frac{\partial f}{\partial x_1}\frac{\partial x_1}{\partial r} + \frac{\partial f}{\partial x_2}\frac{\partial x_2}{\partial r} + \frac{\partial f}{\partial x_3}\frac{\partial x_3}{\partial r}.$$

Since, for $i = 1, 2, 3$, $\partial x_i/\partial r = x_i/r$, we can write

$$\frac{\partial F}{\partial r} = \frac{1}{r}\nabla f \cdot (x_1, x_2, x_3) = \frac{1}{r}\nabla f \cdot \boldsymbol{r}. \tag{1}$$

Assume now that f (and thus F) be given in some neighbourhood of the sphere

$$F(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2 = R^2.$$

To find the outward unit normal to this sphere we note that $\nabla F = (2x_1, 2x_2, 2x_3)$ and $\|\nabla F\| = 2\sqrt{x_1^2 + x_2^2 + x_3^2} = 2R$. Thus,

$$\boldsymbol{n} = \frac{1}{R}(x_1, x_2, x_3) = \frac{1}{R}\boldsymbol{r}. \tag{2}$$

Geometrically, $\boldsymbol{n}$ is parallel to the radius but of unit length.

Combining (1) with (2), we see that the normal derivative of f at any point of the sphere is given by

$$\frac{\partial f}{\partial \boldsymbol{n}} = \nabla f \cdot \boldsymbol{n} = \frac{\partial F}{\partial r}.$$

Flux of a vector field. Vector field is a function $\phi : \Omega \to \mathbb{R}^d$, where $\Omega \subset \mathbb{R}^d$, where $d = 1, 2, 3...$, that is, a vector field assigns a vector to each point of a subset of the space.

Definition 3

The flux of a vector field ϕ across the boundary $\partial\Omega$ of a domain $\Omega \subset \mathbb{R}^d$, $d \geq 2$ is

$$\int_{\partial\Omega} \phi \cdot \boldsymbol{n} dS.$$

Here, if $d = 2$, then $\partial\Omega$ is a closed curve and the integral above is the line integral (of the second kind). The arc length element dS is to be calculated according to the description of $\partial\Omega$.

If $\partial\Omega$ is described parametrically by $\boldsymbol{r}(s) = (f(s), g(s))$, $s \in [s_0, s_1]$, running in the clockwise direction, then $dS = \sqrt{(f')^2 + (g')^2} ds$ and

$$\int_{\partial\Omega} \boldsymbol{\phi} \cdot \boldsymbol{n} dS = \int_{s_0}^{s_1} \boldsymbol{\phi}(s) \cdot (-g'(s), f'(s)) \sqrt{(f')^2(s) + (g')^2(s)} ds.$$

When $d = 3$, then $\partial\Omega$ is a surface and the integral above is the surface integral. If $\partial\Omega$ is given in a parametric form $\boldsymbol{r}(t) = (f(u,s), g(u,s), h(u,s))$, $u \in [u_0, u_1], s \in [s_0, s_1]$. Then $dS = |\boldsymbol{r}_u \times \boldsymbol{r}_s| du ds$ and, using the formula for the normal vector,

$$\int_{\partial\Omega} \boldsymbol{\phi} \cdot \boldsymbol{n} dS = \pm \int_{u_0}^{u_1} \int_{s_0}^{s_1} \boldsymbol{\phi}(u,s) \cdot \boldsymbol{r}_u(u,s) \times \boldsymbol{r}_s(u,s) du ds.$$

where the sign is determined by the orientation of the boundary.

Remark. In 1D, let $\Omega = [a, b]$ with $\partial\Omega = \{a\} \cup \{b\}$. A vector field in one-dimension is just a scalar function. The unit outward normal at $\{a\}$ is -1, and at $\{b\}$ is 1. Thus $\boldsymbol{f}\boldsymbol{n}(a) = f(a)(-1)$ and $\boldsymbol{f}\boldsymbol{n}(b) = f(b)(1)$ and the flux across the boundary of Ω is

$$\boldsymbol{f} \cdot \boldsymbol{n}(a) + \boldsymbol{f} \cdot \boldsymbol{n}(b) = f(b) - f(a). \tag{3}$$

Example. Consider a fluid moving in a certain domain of space with velocity $\boldsymbol{v}(\boldsymbol{p})$ at point $\boldsymbol{p}$. Thus, we have the *velocity field* of the fluid.

In 1D, if $v(x) > 0$, then the fluid flows to the right at x, and if $v(x) < 0$, then it flows to the left. Let the points $x = a$ and $x = b$ be the end-points of a section of the pipe and consider the new field $f(x) = \rho(x)v(x)$, where ρ is the (linear) density of the fluid at point x. The flux of f, as defined by (3), is

$$f(b) - f(a) = \rho(b)v(b) - \rho(a)v(a).$$

In 3D, if fluid of density $\rho(\boldsymbol{p})$ moves in Ω with velocity $\boldsymbol{v}(\boldsymbol{p})$ at point $\boldsymbol{p}$, we define the mass-velocity field $\boldsymbol{f}(\boldsymbol{p}) = \rho(\boldsymbol{p})\boldsymbol{v}(\boldsymbol{p})$.

Imagine a small portion $\Delta\sigma$ of $\partial\Omega$, which could be considered flat with normal $\boldsymbol{n}$ and consider the rate at which the fluid crosses $\Delta\sigma$. The mass of fluid crossing $\Delta\sigma$ in time Δt is given by

$$\Delta m = \rho(\boldsymbol{v} \cdot \boldsymbol{n})\Delta t \Delta\sigma.$$

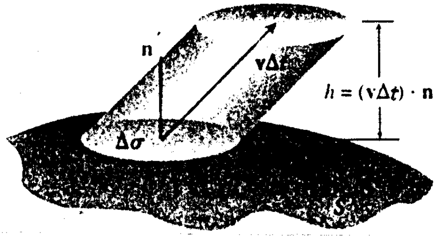


Fig. 3.2 The fluid that flows through the patch $\Delta\sigma$ in a short time Δt fills a slanted cylinder whose volume is approximately equal to the base times height, $\mathbf{v} \cdot \mathbf{n}\Delta\sigma\Delta t$. The mass of the fluid in the cylinder is then $\rho\mathbf{v} \cdot \mathbf{n}\Delta\sigma\Delta t$.

Thus, the rate at which the fluid is crossing the whole boundary $\partial\Omega$ is obtained by summing up all the contributions Δm over all patches $\Delta\sigma$, that is, the flux of $\rho\boldsymbol{v}$ is

$$\int_{\partial\Omega}(\rho\boldsymbol{v}\cdot\boldsymbol{n})d\sigma.$$

Gauss theorem

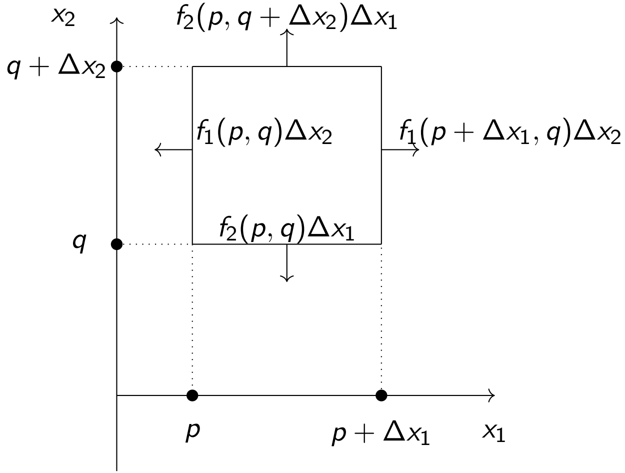


Figure: Flux across the boundary of a box.

Outflow across the boundary

$$
\begin{aligned}
&= (f_1(p+\Delta x_1, q) - f_1(p,q))\Delta x_2 + (f_2(p, q+\Delta x_2) - f_2(p,q))\Delta x_1 \\
&\approx \left(\frac{\partial f_1}{\partial x_1}(p,q) + \frac{\partial f_2}{\partial x_2}(p,q)\right)\Delta x_1 \Delta x_2.
\end{aligned}
$$

Theorem 4

Let Ω be a bounded domain in $\mathbb{R}^d$, $d \geq 1$, with a piecewise C^1 boundary $\partial\Omega$. Let $\boldsymbol{n}$ be the unit outward normal vector on $\partial\Omega$. Let $\boldsymbol{f}(\boldsymbol{x}) = (f_1(\boldsymbol{x}), \cdots, f_n(\boldsymbol{x}))$ be any C^1 vector field on $\bar{\Omega} = \Omega \cup \partial\Omega$. Then

$$
\int_{\Omega} \operatorname{div} \boldsymbol{f}\, d\boldsymbol{x} = \int_{\partial\Omega} \boldsymbol{f} \cdot \boldsymbol{n} d\sigma. \tag{4}
$$

Conservation laws. Consider $u = u(x,t)$, $x \in \Omega \subset \mathbb{R}$, $t > 0$. We assume that u is a density of certain quantity Q such as population, mass, energy.

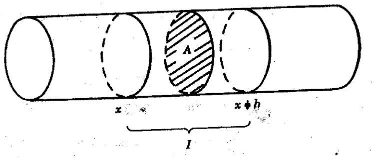


Figure: Tube $\mathcal{I}$.

If $\mathcal{I}$ is the section of the tube between x and $x+h$, then

$$\text{Total amount of quantity in } \mathcal{I} = A\int\limits_{x}^{x+h} u(s,t)ds,$$

where A is the area of the cross section of $\mathcal{I}$.

We define the *flux density* of u at t, x to be the scalar function $\phi(x, t, u)$ equal to the amount of the quantity u passing through the cross section at x at time t, per unit area, per unit time. By convention, the flux density at x is positive if the flow at x is in the positive x direction. Then

$$\text{Net rate that } Q \text{ flows into } \mathcal{I} = A(\phi(x, t, u(x, t)) - \phi(x+h, t, u(x+h, t))).$$

Earlier, we considered $\phi = vu$, where v is the velocity of the flow. Here we the flux density of an arbitrary quantity (arbitrary one-dimensional vector field).

The considered quantity may be destroyed or created inside $\mathscr{I}$, e.g., by a chemical reaction, or by birth/death processes. We introduce the *source function* $f(x, t, u)$ giving the rate at which u is created or destroyed at x at time t, per unit volume. Note that f may depend on u itself (e.g., the rate of chemical reactions is determined by concentration of the chemicals). Then

$$\text{Rate that } Q \text{ is produced in } \mathscr{I} \text{ by sources} = A\int\limits_{x}^{x+h} f(s, t, u(s, t))ds.$$

The fundamental conservation law can be formulated as follows: for any section $\mathscr{I}$

$$\begin{aligned}\text{The rate of change of the total amount of } Q \text{ in } \mathscr{I} \\ = \text{net rate that } Q \text{ flows into } \mathscr{I} \\ + \text{rate that } Q \text{ is produced in } \mathscr{I}\end{aligned}$$

Thus, mathematically,

$$\frac{d}{dt}\int\limits_{x}^{x+h} u(s,t)ds = \phi(x,t,u(x,t)) - \phi(x+h,t,u(x+h,t)) + \int\limits_{x}^{x+h} f(s,t,u)ds. \tag{5}$$

The equation above is called a *conservation law in integral form* and holds even if u, f, ϕ are not smooth functions.

This form is useful in many cases but rather difficult to handle, therefore it is convenient to reduce it to a differential equation. If all involved functions are continuously differentiable, then we can rewrite (5) in the form

$$\int_{x}^{x+h} (u_t(s,t) + \phi_s(s,t,u(s,t)) - f(s,t,u))ds = 0. \qquad (6)$$

Since this equation is valid for any interval $I = [x, x+h]$, we can use Theorem 2 to infer that the integral must vanish identically; that is, changing the independent variable back into x we must have

$$u_t(x,t) + \phi_x(x,t,u(x,t)) = f(x,t,u) \qquad (7)$$

for any $x \in \Omega$ and $t > 0$.

Note that in (7) we have two unknown functions: u and ϕ; function f is assumed to be given. Function ϕ is usually to be determined from empirical considerations. Equations resulting from such considerations, which specify ϕ, are often called *constitutive relations* or *equations of state*.

Conservation laws in higher dimensions. Let $u = u(\mathbf{x}, t)$, $\mathbf{x} \in \mathbb{R}^3$ be a scalar density function of some quantity Q distributed in $\mathbb{R}^3$. Let $\Omega \subset \mathbb{R}^3$ be an arbitrary region with a smooth boundary $\partial\Omega$. The total amount of Q in Ω at time t is given by

$$\int_{\Omega} u(\boldsymbol{x}, t) d\boldsymbol{x},$$

and the rate that the quantity is produced in Ω is given by

$$\int_{\Omega} f(\boldsymbol{x}, t, u) d\boldsymbol{x},$$

where f is the rate at which the quantity is being produced in Ω.

Here, the flow can occur in any direction, the flux density is given by a vector $\mathbf{\Phi}$ (earlier, it was given by $\rho(\boldsymbol{x})\boldsymbol{v}(\boldsymbol{x})$), and the net outward flux of Q through the boundary $\partial\Omega$ is given by the surface integral

$$\int\limits_{\partial\Omega} \mathbf{\Phi}(\boldsymbol{x}, t, \boldsymbol{u}(\boldsymbol{x}, t)) \cdot \boldsymbol{n}(\boldsymbol{x}) d\sigma.$$

Finally, the conservation law for u is given by

$$\frac{d}{dt} \int\limits_{\Omega} u \, d\boldsymbol{x} = -\int\limits_{\partial\Omega} \mathbf{\Phi} \cdot \boldsymbol{n} d\sigma + \int\limits_{\Omega} f d\boldsymbol{x}. \tag{8}$$

The minus sign at the flux term occurs because the outward flux decreases the amount of u in Ω.

If all functions are sufficiently smooth, the Gauss theorem yields

$$\int_{\Omega} (u_t + \operatorname{div} \mathbf{\Phi} - f)\, d\mathbf{x} = 0$$

for any subregion Ω. Using the vanishing theorem (Theorem 2) we finally obtain the differential form of the general conservation law in higher-dimensional spaces,

$$u_t(\mathbf{x}, t) + \operatorname{div} \mathbf{\Phi}(\mathbf{x}, t, u(\mathbf{x}, t)) = f(\mathbf{x}, t, u(\mathbf{x}, t)). \qquad (9)$$

Constitutive relations and examples.

- Conservation law is a mathematical expression of fundamental laws,
- Constitutive relation – originates in empirics.

Transport equation. A substance of density u moving through a surrounding medium with a velocity which may depend on $\boldsymbol{x}$ and u. Then

$$\boldsymbol{\Phi}(\boldsymbol{x}, u) = \boldsymbol{v}(\boldsymbol{x}, u)u,$$

and, if there are no sources or sinks, the transport equation is given by

$$u_t + \operatorname{div}(\boldsymbol{v}u) = \boldsymbol{v} \cdot \nabla u + u\operatorname{div} \boldsymbol{v} = 0. \tag{10}$$

McKendrick partial differential equation. Transport in the derivation of conservation law can occur in other state space. Consider an age-structured population, described by the density of the population $n(a,t)$ with respect to age a and look at the population as if it was 'transported' through stages of life. The number of individuals in the age group $[a, a+\Delta a)$ at time t is $n(a,t)\Delta a$ and the rate of change is

$$\text{change in } [a, a+\Delta a)$$
$$= \text{amount entering at } a - \text{amount exiting at } a+\Delta a - \text{deaths}.$$

Denoting per capita mortality rate for individuals by $\mu(a,t)$, the last term is simply $-\mu(a,t)n(a,t)\Delta t$.

Thus, over Δt,

$$(n(a, t + \Delta t) - n(a, t))\Delta a$$
$$= n(a - \Delta t, t)\Delta t - n(a + \Delta a - \Delta t)\Delta t - \mu(a, t)n(a, t)\Delta a\Delta t$$

and, dividing by $\Delta a\Delta t$ and passing to the limit with $\Delta t, \Delta a \to 0$, we obtain

$$\frac{\partial n(a, t)}{\partial t} + \frac{\partial n(a, t)}{\partial a} = -\mu(a, t)n(a, t). \qquad (11)$$

This equation is defined for $a > 0$ and the flow is to the right hence we need a boundary condition. In this model the birth rate enters here: the number of neonates ($a = 0$) is the number of births across the whole age range:

$$n(0,t) = \int_0^\omega n(a,t)\beta(a,n(a,t))da,$$

where β is the maternity function. Eq. (11) also must be supplemented by the initial condition

$$n(a,0) = n_0(a)$$

describing the initial age distribution.

Diffusion/heat equation. Assume at first that there are no sources, and write the basic conservation law in one-dimension as

$$u_t + \phi_x = 0 \tag{12}$$

In many problems the substance moves from the regions of higher concentration to regions of lower concentration and the larger difference, the more rapid flow is observed. Large differences can be expressed as a large gradient of the concentration u, so it is reasonable to assume that

$$\phi(x,t) = -F(u_x(x,t)),$$

where F is an increasing function passing through $(0,0)$.

The minus sign is due to the fact that the flow occurs in the opposite direction of of the gradient. The simplest increasing function passing through $(0,0)$ is a linear function with positive leading coefficient, and this assumption gives *Fick's law*:

$$\phi(x,t) = -Du_x(x,t) \tag{13}$$

where D is a positive diffusion constant. Thus, we get the one dimensional *diffusion equation*

$$u_t - Du_{xx} = 0, \tag{14}$$

which governs conservative processes, when the flux is specified by Fick's law.

Multidimensional case. If the medium is isotropic, that is, the process of diffusion is independent of the orientation in space, then Fick's law states that the flux density is proportional to the gradient of u, that is,

$$\mathbf{\Phi} = -D\nabla u$$

and the diffusion equation, obtained from the conservation law (9) in the absence of sources, reads

$$u_t = \mathrm{div}(D\nabla u) = D\Delta u, \tag{15}$$

where the second equality is valid if D is a constant.

Diffusion through correlated random walk. Consider particle starting at the origin of the x-axis which jumps the length δ to the right with probability p or to the left with probability q. Let x_i be a random variable that assumes the value δ if the particle moves to the right at the ith step and $-\delta$ if it moves to the left. Assume that each step is independent, so that the x_is are identically distributed independent random variables. Then

$$P(x_i = \delta) = p, \quad P(x_i = -\delta) = q$$

for each i. If the particle cannot rest, then $p + q = 1$. The position of the particle after n jumps is given by the random variable

$$X_n = x_1 + x_2 + \ldots + x_n.$$

The expected value of x_i is

$$E(x_i) = \langle x_i \rangle = (p - q)\delta$$

and, since the expectation is linear

$$E(X_n) = \langle X_n \rangle = (p - q)n\delta.$$

For the variance we have

$$V(X_n) = E(X_n - E(X_n))^2 = E(X_n^2) - (E(X_n))^2.$$

Since $E(x_i^2) = \delta^2(p+q) = \delta^2$, again by linearity of E, we see that

$$\begin{aligned} V(X_n) &= \sum_{i=1}^{n}(\langle x_i^2\rangle - \langle x_i\rangle^2) = \sum_{i=1}^{n}(\delta^2 - (p-q)^2\delta^2) \\ &= n\delta^2(1-(p-q)^2) = 4pqn\delta^2, \end{aligned}$$

upon using $p + q = 1$.

Random walk is a model for a Brownian motion, where the movement is caused by collisions of the particle with the particles of the fluid and, in one dimension, each collision results in a small jump by δ of the particle to the right or to the left. Many such collisions occur in a unit time.

Experimentally, we can observe the average displacement per unit time, denoted by c and the variance of the observed displacement around the average, which we denote by $D > 0$. Thus, after n collisions in unit time, should have

$$c \approx (p - q)\delta n, \tag{16}$$

and

$$D \approx 4pq\delta^2 n. \tag{17}$$

Since the motion appears to be continuous, we have to consider the limit of the above equations as $\delta \to 0$ while $n \to \infty$ in such a way that D and c given above remain constant.

• If $p \neq q$ and $p-q$ does not tend to zero as $\delta \to 0, n \to \infty$ we have

$$\delta n \to \frac{c}{p-q}$$

but then $4pq\delta^2 n \to 0$ yielding $D = 0$ in which case the motion would be deterministic.

• If we want $D \neq 0$, then $p - q \to 0$ yielding $p, q \to 1/2$. If $p = q = 1/2$ in the discrete case, then $c = 0$. However, if $p - q \neq 0$, then $c \neq 0$ and we have a drift. This conditions can be realized if

$$p = \frac{1 + b\delta}{2}, q = \frac{1 - b\delta}{2}$$

for some yet unspecified b chosen so that $0 \leq p, q \leq 1$.

These leads to $p \to 1/2, q \to 1/2$ and

$$(p-q)\delta n = b\delta^2 n$$

so that, to be consistent with (17), we must have

$$\delta^2 n \to D. \tag{18}$$

and thus $b = c/D$, yielding

$$(p-q)\delta n \to c. \tag{19}$$

Let us derive the equation governing the random walk in the continuous limit as $\delta \to 0, n \to \infty$ in such a way that (18) holds. For n steps to occur in a unit time, one step must occur in $\tau = 1/n$ units of time. We derive the formula for the probability that a particle starting at $x = 0$ at $t = 0$ will be at the position x at the time t. Thus, we must have

$$k\tau = t, \quad X_k = x.$$

We define

$$v(x,t) = P(X_k = x)$$

at time t.

Then, v satisfies the difference equation

$$v(x, t+\tau) = pv(x-\delta, t) + qv(x+\delta, t). \tag{20}$$

If we assume that v is differentiable, we can expand it in the Taylor series

$$\begin{aligned} v(x, t+\tau) &= v(x,t) + \tau v_t(x,t) + O(\tau^2), \\ v(x \pm \delta, t) &= v(x,t) \pm \delta v_x(x,t) + \frac{1}{2}\delta^2 v_{xx}(x,t) + O(\delta^3). \end{aligned} \tag{21}$$

Substituting into (20), we obtain

$$v_t = (q-p)\frac{\delta}{\tau}v_x + \frac{1}{2}\frac{\delta^2}{\tau}v_{xx} + \tau^{-1}O(\tau^2) + \tau^{-1}O(\delta^3)$$

Now, since $\delta^2 n = \delta^2/\tau = O(1)$, we have

$\tau^{-1}O(\delta^3) = O(1)\delta^{-2}O(\delta^3)$ and we can re-write the above as

$$v_t = (q-p)\frac{\delta}{\tau}v_x + \frac{1}{2}\frac{\delta^2}{\tau}v_{xx} + O(\tau) + O(\delta)$$

and passing to the limit as $\delta \to 0, \tau \to 0$ in such a way that (18) (with $\tau = 1/n$) holds

$$v_t = -cv_x + \frac{1}{2}Dv_{xx},$$

where we used (19) and (18).

In this equation, v must be interpreted as the probability density, that is, at time t,

$$P(a \leq x \leq b) = \int_a^b v(x,t)ds.$$

Boundary condition. The derivation above assumed that the random walks occurs on $\mathbb{R}$. If the movement is restricted to, say, $(-\infty, l)$, then we have to specify what happens at $x = l$.

- *Absorbing boundary.* A particle reaching $x = l$ stays there,

$$v(t + \tau, l) = pv(t, l - \delta).$$

Expanding

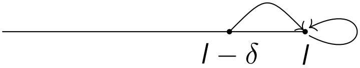
$$v(t, l) + v_t(t, l)\tau + o(\tau) = pv(t, l) - p\delta v_x(t, l) + o(\delta).$$

Letting $\delta, \tau \to 0$,

$$v(t, l) = 0;$$

Dirichlet boundary condition.

- *Reflecting boundary.*



$$v(t+\tau, l) = pv(l-\delta, t) + pv(l, t)$$

Expanding,

$$v(t, l) + v_t(t, l)\tau + o(\tau) = 2pv(t, l) - p\delta v_x(t, l) + o(\delta).$$

Using $2p - 1 = p - q$, multiplying by δ/τ,

$$v_t(t, l)\delta = (p-q)v(t, l)\frac{\delta}{\tau} - p\delta v_x(t, l)\delta\tau + o(\tau)\frac{\delta}{\tau} + \frac{\delta}{\tau}o(\delta).$$

With $\frac{\delta^2}{\tau} \to D$ and $(p-q)\frac{\delta}{\tau} \to c$, we obtain

$$\frac{D}{2} v_x(l,t) - cv(l,t) = 0,$$

often called the Robin condition.

In particular, if $p = q$, then $c = 0$ and

$$v_x(l,t) = 0,$$

called the (homogeneous) Neumann boundary condition.

We note that the derivation of the diffusion limit required $\delta^2/\tau \to D$ which, in turn, implies $\delta/\tau \to \infty$ since $\delta \to 0$. In other words, for the finite diffusion coefficient (variance) the velocity of the particle must be infinite. While certainly nonphysical, it is in agreement with the properties of the diffusion equation which predicts instantaneous transmission of signals. This drawback can be removed by considering the correlated random walk.

Correlated random walk. Similarly, we consider a particle which can jump by δ to the left or to the right. Each jump is executed in time τ. However, here p and q are the probabilities that the particle will, respectively, persist moving in the same direction and reverse the direction. Thus, let $\alpha(x, t)$ be the probability that a particle is at the point x and arrived there from the left, whereas $\beta(x, t)$ is the probability that a particle is at x and arrived there from the right.

Thus, using the total probability, we have

$$\begin{aligned}\alpha(x, t+\tau) &= p\alpha(x-\delta, t) + q\beta(x-\delta, t), \\ \beta(x, t+\tau) &= q\alpha(x+\delta, t) + p\beta(x+\delta, t). \end{aligned} \tag{22}$$

We assume that the shorter the time τ, the greater probability of persistence; that is, $p \to 1$ as $\tau \to 0$, so $q \to 0$. Assuming that both p and q are differentiable functions of τ, we can write

$$p = 1 - \lambda\tau + o(\tau),$$

$$q = \lambda\tau + o(\tau) \tag{23}$$

where λ is the rate of reversal of direction as $\tau \to 0$.

If we expand α and β, we get

$$\alpha_t(x,t)\tau + \alpha(x,t) + o(\tau) = -p\alpha_x(x,t)\delta + p\alpha(x,t) - q\beta_x(x,t)\delta + q\beta(x,t) + o(\delta),$$

$$\beta_t(x,t)\tau + \beta(x,t) + o(\tau) = p\beta_x(x,t)\delta + p\beta(x,t) + q\alpha_x(x,t)\delta + q\tau\alpha(x,t) + o(\delta)$$

or, using (23),

$$\alpha_t(x,t) = -(1-\lambda\tau)\alpha_x(x,t)\frac{\delta}{\tau} - \lambda\alpha(x,t) - \lambda\tau\beta_x(x,t)\frac{\delta}{\tau} + \lambda\beta(x,t) + o(1),$$

$$\beta_t(x,t) = (1-\lambda\tau)\beta_x(x,t)\frac{\delta}{\tau} - \lambda\beta(x,t) + \lambda\tau\alpha_x(x,t)\frac{\delta}{\tau} + \lambda\alpha(x,t) + o(1).$$

Next, assuming that $\delta/\tau \to \gamma$, the speed of motion, as $\delta, \tau \to 0$, we obtain the following coupled system of partial differential equation

$$\begin{aligned} \alpha_t(x,t) &= -\gamma\alpha_x(x,t) - \lambda\alpha(x,t) + \lambda\beta(x,t), \\ \beta_t(x,t) &= \gamma\beta_x(x,t) + \lambda\alpha(x,t) - \lambda\beta(x,t), \end{aligned} \tag{24}$$

where, as before α and β are to be interpreted as the probability densities.

Since at the point x the particle must have arrived either from the left or from the right, the function

$$v(x,t) = \alpha(x,t) + \beta(x,t) \tag{25}$$

is the probability density that a particle is at the point x at the time t.

If we introduce the net flux density to the right

$$w(x,t) = \alpha(x,t) - \beta(x,t),$$

then (24) can be transformed, by adding and subtracting, into

$$\begin{aligned} v_t(x,t) + \gamma w_x(x,t) &= 0, \\ w_t(x,t) + \gamma v_x(x,t) &= -2\lambda w. \end{aligned} \qquad (26)$$

In particular, (26) can be reduced to

$$v_{tt} - \gamma^2 v_{xx} + 2\lambda v_t = 0 \qquad (27)$$

which is the damped wave equation with waves moving with the speed γ, as follows from the microscopic description.

For λ close to 0, we have strong correlations resulting in a wave motion which does not display any stochasticity.

On the other hand, when we divide both sides of Eq. (27) by 2λ and let $\lambda \to \infty$ (which corresponds to very weak correlations) in such a way that $\gamma^2/\lambda \to D$, formally the equation becomes

$$-\frac{D}{2}v_{xx} + v_t = 0 \tag{28}$$

which is the diffusion equation of the uncorrelated random walk. This agrees well with intuition.

However, it is important to remember that the above reasoning does not constitute a proof that a correlated random walk tends to an uncorrelated random walk if both the reversal rate and the speed tend to infinity. For instance, (27) is second order in time and requires two initial conditions whereas (28) is first order in time, so imposing these initial conditions would render the problem unsolvable. Problems of this type are called *singularly perturbed* and require a delicate analysis. It can be proved, however, that solutions of (27) tend to a solution of (28) for large t.

Variations of the diffusion equation. In many cases the evolution is governed by more then one process.

• Simultaneous transport and diffusion – *drift-diffusion equation*. By combining (10) and (15), we get

$$u_t + \mathrm{div}\,(\boldsymbol{v} u) = D\Delta u. \tag{29}$$

• When the sources are present and the constitutive relation is given by Fick's law, then the resulting equation

$$u_t - D\Delta u = f(\boldsymbol{x}, t, u) \tag{30}$$

is called the *reaction-diffusion equation*.

• If f describes spontaneous decay (or creation) of the substance at the exponential rate,

$$u_t - D\Delta u = ru. \tag{31}$$

• The *Fisher equation*, where we assume that the population obeys the logistic law. Introduce the density of the population $u(\boldsymbol{x}, t)$; the conservation law takes the form

$$u_t + \mathrm{div}\,\boldsymbol{\Phi} = ru\left(1 - \frac{u}{N}\right). \tag{32}$$

With Fick's law, individuals migrate from the regions of higher density to regions of lower density. Then

$$u_t - D\Delta u = ru\left(1 - \frac{u}{N}\right). \tag{33}$$

Systems.

Epidemiology with age structure. In epidemiological problems, the rate of infection often significantly varies with age and thus it is important to consider the age structure of the population. We assume that in the absence of the disease, the age-dependent density of the population $n(a,t)$ would be the solution to (11). Due to the epidemics, we partition the population susceptibles, infectives and removed, represented by their respective age densities $s(a,t), i(a,t)$ and $r(a,t)$.

The SIRS model becomes

$$
\begin{aligned}
\partial_t s(a,t) + \partial_a s(a,t) + \mu(a)s(a,t) &= -\lambda(a,t)s(a,t) \\
&\quad + \gamma(a)i(a,t), \\
\partial_t i(a,t) + \partial_a i(a,t) + \mu(a)i(a,t) &= \lambda(a,t)s(a,t) \\
&\quad - (\nu(a) + \gamma(a))i(a,t), \\
\partial_t r(a,t) + \partial_a r(a,t) + \mu(a)r(a,t) &= \nu(a)i(a,t),
\end{aligned}
\tag{34}
$$

with boundary conditions

$$
\begin{aligned}
s(0,t) &= \int_0^{\omega} \beta(a)(s(a,t) + (1-q)i(a,t) \\
&\qquad + (1-w)r(a,t))da, \\
i(0,t) &= q\int_0^{\omega} \beta(a)i(a,t)da, \\
r(0,t) &= w\int_0^{\omega} \beta(a)r(a,t)da,
\end{aligned}
\tag{35}
$$

where $q \in [0,1]$ and $w \in [0,1]$ are the vertical transmission coefficients of infectiveness and immunity, respectively. The system is complemented by initial conditions $s(a,0) = s_0(a), i(a,0) = i_0(a)$ and $r(a,0) = r_0(a)$.

We assumed that the death and birth coefficients are not affected by the disease so that (34) is a conservation law, that is, for regular solutions, the total population density $n(a,t) = s(a,t) + i(a,t) + r(a,t)$ satisfies

$$\partial_t n(a,t) + \partial_a n(a,t) + \mu(a)n(a,t) = 0,$$

$$n(0,t) = \int_0^\omega \beta(a)n(a,t)da, \quad n(a,0) = s_0(a) + i_0(a) + r_0(a).$$

The force of infection λ is given by

$$\lambda(a,t) = K_0(a)i(a,t) + \int_0^\omega K(a,s)i(s,t)ds, \qquad (36)$$

where K_0 gives the intracohort infection (same age) and K gives the intercohort infections.

Spread of rabies in a fox population. Consider a population of foxes divided into subpopulations susceptible, infectious and removed individuals. Foxes are territorial, so we assume that susceptibles are stationary. Infectives lose orientation and move in a random way. Removed are the foxes that died of the disease. If we ignore vital processes, we get

$$\begin{aligned}\frac{\partial S}{\partial t} &= -\beta IS,\\ \frac{\partial I}{\partial t} &= \beta IS - \mu I + D\frac{\partial^2 I}{\partial x^2},\\ \frac{\partial R}{\partial t} &= \mu I,\end{aligned} \tag{37}$$

where β is the infection rate, μ–disease-induced death rate and D is the diffusion coefficient.

First-order equations – methods. We start with the simplest transport equation

$$au_t + bu_x = 0, \quad t, x \in \mathbb{R} \tag{38}$$

where a and b are constants. This equation can be written as

$$(a, b) \cdot (u_t, u_x) = \boldsymbol{v} \nabla u = D_{\boldsymbol{v}} u = 0,$$

where $\boldsymbol{v} = (a, b)$, and $D_{\boldsymbol{v}}$ is the directional derivative. Thus u is constant along each line with direction $\boldsymbol{v}$, that is, having the equation $bt - ax = \xi$, called characteristics. The solution can change from one line to another, hence

$$u(t, x) = f(\xi) = f(bt - ax), \tag{39}$$

where f is an arbitrary differentiable function.

The Cauchy problem. Let us assume that $a \neq 0$. To obtain a unique solution, we specify the initial value for u. Hence, let us consider the initial value problem for (38): find u satisfying

$$\begin{aligned} au_t + bu_x = 0 \quad x \in \mathbb{R}, t > 0, \\ u(0,x) = g(x), \quad x \in \mathbb{R}, \end{aligned} \tag{40}$$

where g is an arbitrary given function. From (39) we find that

$$u(t,x) = g\left(-\frac{bt - ax}{a}\right). \tag{41}$$

The initial shape propagates without any change along the characteristic lines

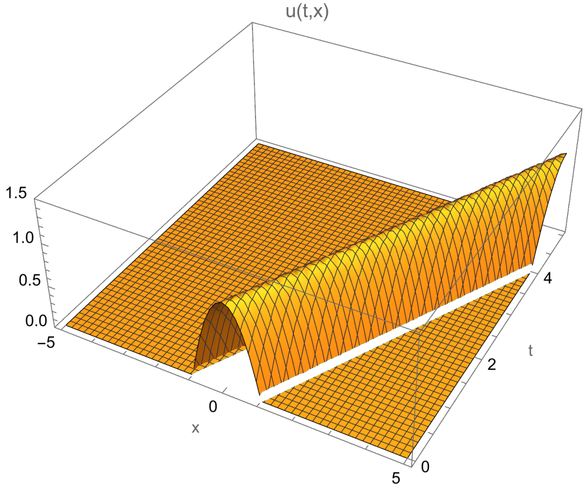


Figure: The graph of the solution to (40) where $g = 1 - x^2$ for $|x| < 1$ and zero elsewhere, and $b/a = 1$.

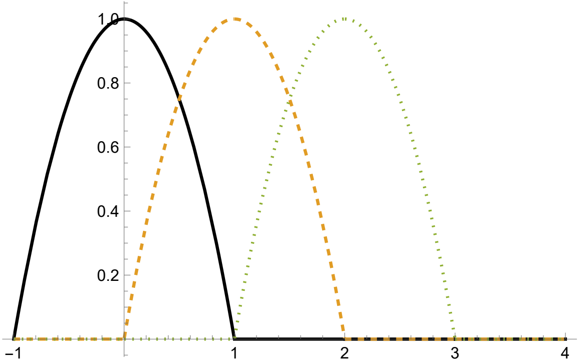


Figure: Travelling wave profile of the solution (40) for $t = 0$ (solid), $t = 1$ (dashed), $t = 2$ (dotted).

Initial–boundary value problems Let us consider a variation of this problem and solve the initial–boundary value problem

$$\begin{aligned} au_t + bu_x &= 0, \quad x, t > 0, \\ u(x,0) &= g(x), \quad x > 0, \\ u(t,0) &= h(t), \quad t > 0, \end{aligned} \tag{42}$$

for $a, b > 0$. The general solution of the equation is

$$u(t,x) = f(bt - ax).$$

• Setting $t = 0$ gives $f(-ax) = g(x)$ for $x > 0$, hence $f(x) = g(-x/a)$ for $x < 0$.

• Setting $x = 0$ gives $f(bt) = h(t)$ for $t > 0$, hence $f(x) = h(x/b)$ for $x > 0$.

Combining

$$u(t,x) = \begin{cases} g(-\frac{bt-ax}{a}) & \text{for} \quad x > bt/a \\ h(\frac{bt-ax}{b}) & \text{for} \quad x < bt/a \end{cases}$$

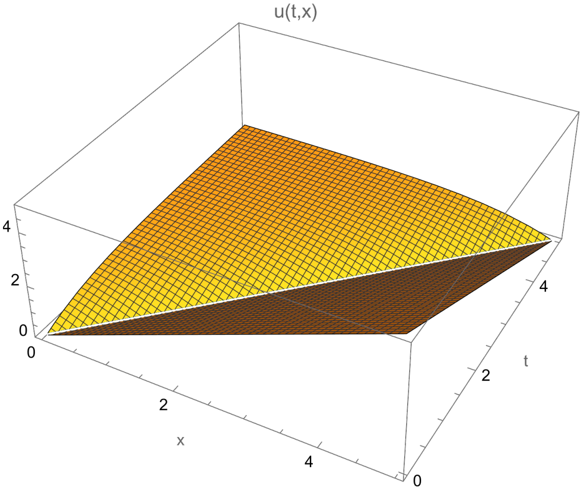


Figure: The solution of the initial–boundary problem (42) with $a = b$, $g(x) = x$ and $h(t) = 1 - e^{-t}$.

Incorrect boundary condition. Assume $a = 1 > 0$, $b = -1 < 0$. Then, the initial condition defines $f(x) = g(-x)$ for $x < 0$, and the boundary condition gives $f(x) = h(-x)$ also for $x < 0$! Hence, we cannot specify both initial and boundary conditions in an arbitrary way as this could make the problem ill-posed.

The physical explanation of this comes from the observation that since the characteristics are given by $\xi = x + t$, the flow occurs in the negative direction, and therefore, the values at $x = 0$ for any t are uniquely determined by the initial condition. Therefore, to have a well-posed problem, we must specify the boundary conditions at the point where the medium flows into the region.

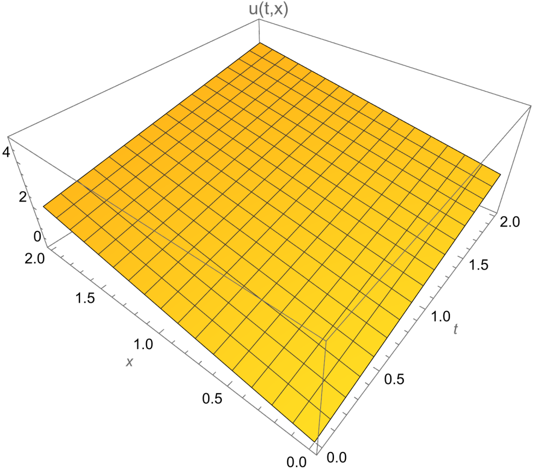


Figure: The solution of $u_t - u_x = 0$ and $u(0,x) = x$ given by $u(t,x) = x + t$. The boundary condition $u(t,0) = t$ is determined by the initial condition, and cannot be imposed.

Characteristic coordinates. We reformulate the above in a more analytic language, allowing for treatment of inhomogeneous problems. Let us introduce the change of variables according to $\xi = \xi(t,x),\ \eta = \eta(t,x)$ and $v(\xi,\eta) = u(t,x)$, then

$$u_t = v_\xi \xi_t + v_\eta \eta_t, \quad u_x = v_\xi \xi_x + v_\eta \eta_x,$$

and the equation can be written as

$$a(v_\xi \xi_t + v_\eta \eta_t) + b(v_\xi \xi_x + v_\eta \eta_x) = v_\xi(a\xi_t + b\xi_x) + v_\eta(a\eta_t + b\eta_x) = 0.$$

If we require the coefficient at u_η to be zero, the easiest way is to introduce $\eta_t = b,\ \eta_x = -a$, that is $\eta = bt - ax$.

Note that this is exactly the characteristic direction! This is an incomplete change of variables as knowing η alone, we are not able to determine the values of x and t. We need another variable $\xi = \xi(x,t)$ such that the system

$$\eta = bt - ax, \quad \xi = \xi(x,t)$$

is uniquely solvable. E.g., if $a \neq 0$, the easiest choice is $\xi = t$. However, sometimes, it is more convenient to use the orthogonal lines given by $\xi = at + bx$.

Inhomogeneous equations. Find the solution to the following equation

$$u_t + 2u_x - (x+t)u = -(x+t),$$

which satisfies the initial condition

$$u(0,x) = h(x), \quad x > 0,$$

and

$$u(t,0) = g(t), \quad t > 0.$$

Introducing new variables according to $\xi = t,\ \eta = 2t - x$ we transform the equation into

$$v_\xi - (3\xi - \eta)v = -(3\xi - \eta).$$

This equation can be regarded as a linear first-order ordinary

The integrating factor is given by

$$\mu(\xi,\eta) = e^{-\left(\frac{3}{2}\xi^2-\eta\xi\right)}.$$

Multiplying both sides of the equation by μ and rearranging the terms we obtain

$$\left(e^{-\left(\frac{3}{2}\xi^2-\eta\xi\right)}v(\xi,\eta)\right)_\xi = -(3\xi-\eta)e^{-\left(\frac{3}{2}\xi^2-\eta\xi\right)},$$

hence, the general solution is given by

$$v(\xi,\eta) = 1 + C(\eta)e^{\left(\frac{3}{2}\xi^2-\eta\xi\right)},$$

where C is an arbitrary differentiable function of one variable.

In the original variables, we obtain

$$u(t,x) = 1 + C(2t-x)e^{\left(-\frac{t^2}{2}+tx\right)},$$

where C is an arbitrary differentiable function. Next,

$$h(x) = u(0,x) = 1 + C(-x)$$

for $x > 0$. Thus

$$C(s) = h(-s) - 1, \quad s < 0.$$

On the other hand,

$$g(t) = u(t,0) = 1 + C(2t)e^{-\frac{t^2}{2}}, \quad t > 0,$$

hence

$$C(s) = e^{\frac{s^2}{8}}\left(g\left(\frac{s}{2}\right) - 1\right), \quad s > 0.$$

Thus, we have defined C for all values of the argument by

$$C(s) = \begin{cases} h(-s) - 1 & \text{for} \quad s < 0, \\ e^{\frac{s^2}{8}} \left(g\left(\frac{s}{2}\right) - 1\right) & \text{for} \quad s > 0. \end{cases}$$

Therefore,

$$u(t,x) = \begin{cases} 1 + (h(-2t+x) - 1)e^{\left(-\frac{t^2}{2}+tx\right)} & \text{for} \quad x > 2t, \\ 1 + e^{\left(\frac{x^2+tx}{8}\right)} \left(g\left(t - \frac{x}{2}\right) - 1\right) & \text{for} \quad x < 2t. \end{cases}$$

If $h(x) = x$ and $g(t) = \sin t$, then

$$u(t,x) = \begin{cases} 1 + (-2t + x - 1)e^{\left(-\frac{t^2}{2} + tx\right)} & \text{for} \quad x > 2t, \\ 1 + e^{\left(\frac{x^2+tx}{8}\right)} \left(\sin\left(t - \frac{x}{2}\right) - 1\right) & \text{for} \quad x < 2t. \end{cases} \quad (43)$$

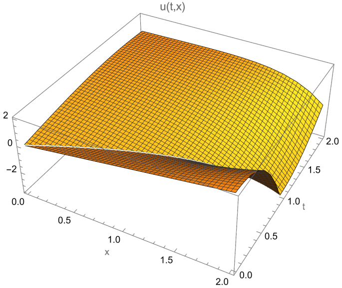


Figure: The graph of the solution (43).

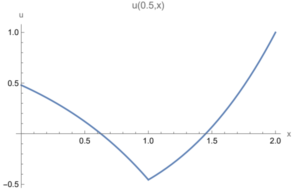


Figure: Snapshot of the solution (43) at $t = 0.5$.

We use the same principle to solve variable coefficient equations

$$a(t,x)u_t + b(t,x)u_x = 0. \tag{44}$$

This equation states that the derivative of u in the direction of the vector $(b(t,x), a(t,x))$ is equal to zero. Consider a family of curves $(t(s), x(s)), 0 \leq s < \infty$, which are tangent to these vectors, that is,

$$\begin{aligned} x'(s) &= b(t(s), x(s)), \quad x(0) = \xi \\ t'(s) &= a(t(s), x(s)), \quad t(0) = 0, \end{aligned} \tag{45}$$

where $\xi \in \mathbb{R}$. Then, for each ξ, we denote $v_\xi(s) := u(t(s), x(s))$ for $(x(s), t(s))$ satisfying (45), obtaining

$$v'_\xi = u_t t'(s) + u_x x'(s) = u_t a + u_x b = 0. \tag{46}$$

Thus, v_ξ is constant for each ξ; hence, $u(t,x)$ is constant along each trajectory defined by ξ.

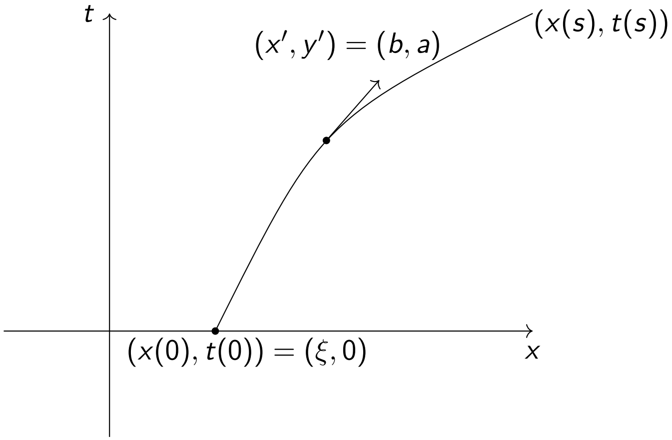


Figure: A characteristic line determined by the initial value $(\xi, 0)$.

Summarizing, for the inhomogeneous problem

$$u_t + c(t,x)u_x = f(t,x,u),$$
$$u(x,0) = u_0(x). \tag{47}$$

we form the so-called characteristic system

$$\frac{dv}{dt} = f(x,t,v), \quad u(0) = u_0(\xi),$$
$$\frac{dx}{dt} = c(x,t), \quad x(0) = \xi, \tag{48}$$

$\xi \in \mathbb{R}$. Here, as above, $v(t) = u(t, x(t))$, so $v(t)$ is $u(t,x)$ evaluated along a given characteristic.

The second equation is independent of the first and it determines the equation of characteristics $x = x(t, \xi)$. This solution can be substituted into the first equation, the solution of which gives the values of u as a function of t that is a parameter along a characteristic, and the parameter ξ. Assuming that only one characteristic passes through each (x, t) and determines a unique $\xi = \xi(x, t)$ at which it crosses the x-axis, we can eliminate ξ and produce the solution u in terms of x and t only.

Example. Find the solution to the following initial value problem

$$\begin{aligned} u_t + xu_x + u &= 0, \quad t > 0, x \in \mathbb{R}, \\ u(x,0) &= u_0(x), \end{aligned} \tag{49}$$

where $u_0(x) = 1 - x^2$ for $|x| < 1$, and $u_0(x) = 0$ for $|x| \geq 1$.

The differential equation for characteristic curves is

$$\frac{dx}{dt} = x$$

which gives $x = \eta e^t$, thus $\eta = xe^{-t} = \phi(t,x)$.

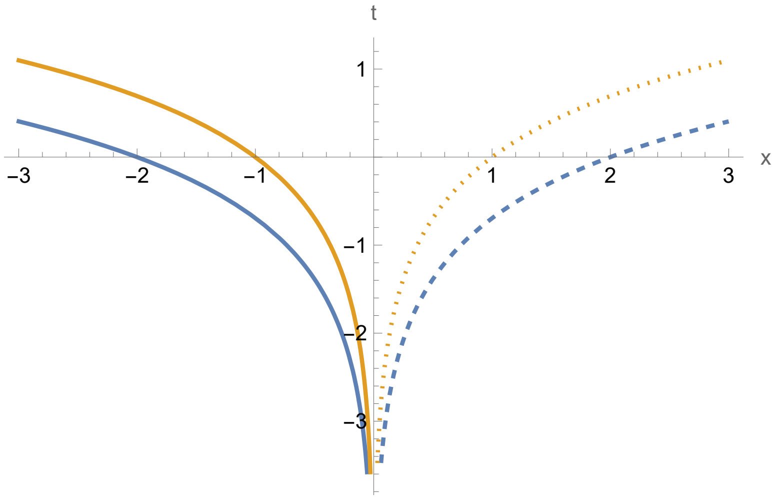


Fig 4.4. Characteristics of (49).

If the equation hadn't contained the zero-order term u, that is, if it had been in the form

$$u_t + xu_x = 0,$$

then the general solution would have had the form

$$u(t, x) = f(xe^{-t}),$$

for arbitrary function f.

The characteristic system is

$$\begin{aligned} v' &= -v, \\ x' &= x, \end{aligned} \tag{50}$$

with the initial conditions $v(0) = u_0(\xi), \quad x(0) = \xi$. Hence, with integration constants a, b, Using the initial conditions we get

$$v(0) = a = u_0(\xi), \qquad x(0) = b = \xi.$$

Thus, $x(t) = \xi e^t$ and, eliminating ξ, we get

$$u(x,t) = u_0(xe^{-t})e^{-t}.$$

For the chosen initial condition, we obtain

$$u(t,x) = \begin{cases} (1 - x^2 e^{-2t})e^{-t} & \text{for} \quad |x| \le e^t, \\ 0 & \text{for} \quad |x| > e^t. \end{cases} \tag{51}$$

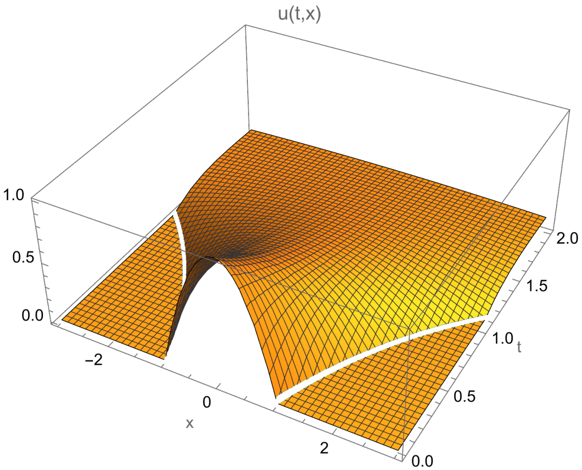


Figure: The graph of the solution (51).

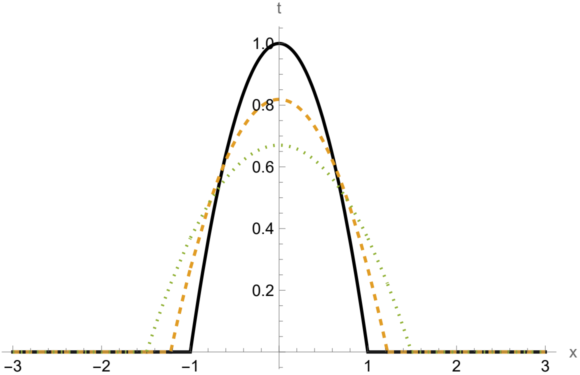


Figure: Snapshots of the solution (51) times $t = 0$ (solid), $t = 0.2$ (dashed) and $t = 0.4$ (dotted).

Multidimensional problems. The described procedure can be applied in multidimensional problems. Consider

$$\begin{aligned} u_t + 2u_x + 3u_y + u = 0, \quad (x,y) \in \mathbb{R}^2, t > 0, \\ u(0,x,y) = u_0(x,y) = e^{-x^2-y^2}. \end{aligned} \tag{52}$$

The characteristic system for this equation is of the form

$$\begin{aligned} v' &= -v, \\ x' &= 2, \\ y' &= 3, \end{aligned}$$

with initial conditions $v(0) = u_0(\xi,\eta), x(0) = \xi, y(0) = \eta$.

Then, we obtain

$$v(t) = u_0(\xi, \eta)e^{-t}, \quad x = 2t + \xi, \quad y = 3t + \eta,$$

where ξ, η are constants. Eliminating ξ and η, we obtain

$$u(t, x, y) = u_0(x - 2t, y - 3t)e^{-t} = e^{-(x-2t)^2-(y-3t)^2}e^{-t}. \quad (53)$$

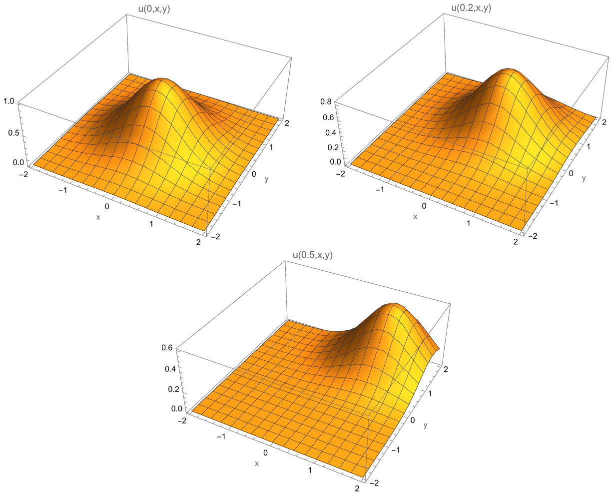


Figure: Snapshots of the solution (53) for $t = 0, 0.2, 0.5$.

Applications of first-order equations.

Birth-and-death type problem. Consider a population of $N(t)$ individuals at time t. We allow stochasticity to intervene in the process so that $N(t)$ becomes a random variable. Accordingly, we denote by

$$p_n(t) = P\{N(t) = n\}, \quad n = 1, 2, \ldots, \tag{54}$$

the probability that the population has n individuals at time t. Assume that, for a single individual,

$$P\{1 \text{ birth in } (t, t + \Delta t]|N(t) = 1\} = \beta\Delta t + o(\Delta t),$$

$$P\{1 \text{ death in } (t, t + \Delta t]|N(t) = 1\} = \delta\Delta t + o(\Delta t),$$

$$P\{\text{no change in } (t, t + \Delta t]|N(t) = 1\} = 1 - (\beta + \delta)\Delta t + o(\Delta t).$$

Possibility of more than one birth or death occurring in $(t, t + \Delta t]$ is assumed to be of order $o(\Delta t)$ and will be omitted in the discussion. Further, assume that in the population of n individuals births and deaths occur independently. Then, the probability of one birth is given by

$$
\begin{aligned}
&P\{1 \text{ birth in } (t, t + \Delta t] | N(t) = n\} \\
&= n(\beta \Delta t + o(\Delta t))(1 - (\beta + \delta)\Delta t + o(\Delta t))^{n-1} \qquad (55) \\
&= n\beta \Delta t + o(\Delta t).
\end{aligned}
$$

Similarly, the probability of one (net) death in the population is

$$P\{1 \text{ death in } (t, t+\Delta t] | N(t) = n\}$$
$$= n(\delta\Delta t + o(\Delta t))(1 - (\beta+\delta)\Delta t + o(\Delta t))^{n-1} \qquad (56)$$
$$= n\delta\Delta t + o(\Delta t),$$

and, finally,

$$P\{\text{no change in } (t, t+\Delta t] | N(t) = n\}$$
$$= (1 - (\beta+\delta)\Delta t + o(\Delta t))^{n} \qquad (57)$$
$$= 1 - n(\beta+\delta)\Delta t + o(\Delta t).$$

Now, using the law of total probability,

$$p_n(t) = (n-1)\beta\Delta t p_{n-1} + (n+1)\delta\Delta t p_{n+1} + (1-n(\beta+\delta)\Delta t)p_n(t) + o(\Delta t) \tag{58}$$

and, finally,

$$\frac{dp_n(t)}{dt} = -n(\beta+\delta)p_n(t) + (n-1)\beta p_{n-1}(t) + (n+1)\delta p_{n+1}(t). \tag{59}$$

This system has to be supplemented by the initial condition

$$p_n(0) = \begin{cases} 1 & \text{for} \quad n = n_0, \\ 0 & \text{for} \quad n \neq n_0. \end{cases} \tag{60}$$

System (59) difficult to solve. Even proving that there is a solution to it is a highly nontrivial exercise. In what follows, we assume that $(p_0(t), p_1(t), \dots,)$ exists and describes a probability, that is, for all $t \geq 0$, $p_n(t) \geq 0$ and

$$\sum_{n=0}^{\infty} p_n(t) = 1. \tag{61}$$

Then, we will be able to find formulae for p_n by the generating function method.

We define

$$F(t,x) = \sum_{n=0}^{\infty} p_n(t)x^n$$

By (61), the generating function is defined in the closed circle $|x| \leq 1$ and analytic in $|x| < 1$. The generating function has the following interpretation in this context.

(1) The probability of extinction at time t, $p_0(t)$, is given by

$$p_0(t) = F(t,0). \quad (62)$$

(2) The probabilities $p_n(t)$ are given by

$$p_n(t) = \frac{1}{n!}\frac{\partial^n F}{\partial x^n}\bigg|_{x=0}. \quad (63)$$

If $F(t,x)$ is analytic in a slightly larger circle containing $x=1$, we can use F to find other useful quantities. The expected value of $N(t)$ at time t is defined by

$$E(N(t)) = \sum_{n=1}^{\infty} np_n(t)$$

On the other hand,

$$\frac{\partial F}{\partial x}(t,x) = \sum_{n=1}^{\infty} np_n(t)x^{n-1}$$

so that

$$E[N(t)] = \sum_{n=1}^{\infty} np_n(t) = \left.\frac{\partial F}{\partial x}\right|_{(t,x)=(t,1)} \qquad (64)$$

Similarly, the variance is defined by

$$Var[N(t)] = E[N^2(t)] - (E[N(t)])^2.$$

On the other hand,

$$\left.\frac{\partial^2 F}{\partial x^2}(t,x)\right|_{x=1} = \sum_{n=0}^{\infty} n(n-1)p_n(t) = E[N^2(t)] - E[N(t)].$$

Combining these formulae, we get

$$Var[N(t)] = \left.\left(\frac{\partial^2 F}{\partial x^2} + \frac{\partial F}{\partial x} - \left(\frac{\partial F}{\partial x}\right)^2\right)\right|_{(t,x)=(t,1)} \tag{65}$$

Let us find the equation satisfied by F. Using (59) and $p_{-1} = 0$, we have

$$\begin{aligned}\frac{\partial F}{\partial t}(t,x) &= \sum_{n=0}^{\infty} n\frac{dp_n}{dt}(t) = -(\beta+\delta)\sum_{n=0}^{\infty} np_n(t)x^n \\ &\quad +\beta\sum_{n=0}^{\infty}(n-1)p_{n-1}(t)x^n + \delta\sum_{n=0}^{\infty}(n+1)p_{n+1}(t)x^n \\ &= -(\beta+\delta)x\frac{\partial F}{\partial x}(t,x) + \beta x^2\frac{\partial F}{\partial x}(t,x) + \delta\frac{\partial F}{\partial x}(t,x).\end{aligned}$$

That is, to find F, we have to solve the equation

$$\frac{\partial F}{\partial t} = \left(\beta x^2 - (\beta+\delta)x + \delta\right)\frac{\partial F}{\partial x}, \tag{66}$$

supplemented by the initial condition

$$F(0,x) = x^{n_0}.$$

It is a homogeneous equation, so that F is constant along characteristics, which are given by

$$\frac{dx}{dt} = -(\beta x - \delta)(x-1),$$

that is,

$$\begin{aligned} -t + C &= \int \frac{dt}{(\beta x - \delta)(x-1)} = \frac{1}{\beta - \delta}\left(-\int \frac{dx}{x - \frac{\delta}{\beta}} + \int \frac{dx}{x-1}\right) \\ &= \frac{1}{\beta-\delta} \ln \left| \frac{x-1}{x - \frac{\delta}{\beta}} \right| \end{aligned}$$

provided $\beta \neq \delta$ and $x \neq 1, \frac{\delta}{\beta}$.

This gives

$$\left|\frac{\beta x - \delta}{x - 1}\right| = Ce^{rt}$$

where $r = \beta - \delta$. Thus, we have a general solution

$$F(t, x) = G\left(e^{-rt}\left|\frac{\beta x - \delta}{x - 1}\right|\right),$$

where G is an arbitrary function. Using the initial condition, we get

$$x^{n_0} = G\left(\left|\frac{\beta x - \delta}{x - 1}\right|\right).$$

Assume $x < min\{1, \delta/\beta\}$ or $x > \max\{1, \delta/\beta\}$ so that we can drop absolute value bars. Solving

$$s = \frac{\beta x - \delta}{x - 1},$$

we get

$$x = \frac{s - \delta}{s - \beta},$$

so that

$$G(s) = \left(\frac{s - \delta}{s - \beta}\right)^{n_0}.$$

Thus, the solution is given by

$$F(x,t) = \left(\frac{e^{-rt\frac{\beta x - \delta}{x-1}} - \delta}{e^{-rt\frac{\beta x - \delta}{x-1}} - \beta}\right)^{n_0} = \left(\frac{e^{rt}\delta(1-x) + (\beta x - \delta)}{e^{rt}\beta(1-x) + (\beta x - \delta)}\right)^{n_0}. \quad (67)$$

Consider the zeroes of the denominator:

$$x = \frac{e^{rt} - \frac{\delta}{\beta}}{e^{rt} - 1}$$

• If $\delta/\beta < 1$, then $r > 0$ and we see that $x > 0$ and, as $t \to \infty$, x moves from $+\infty$ to 1 and thus F is analytical in the circle stretching from the origin to the first singularity, which is bigger than 1 for any finite t.

• If $\delta/\beta > 1$, then $r < 0$ and x above is again positive and moves from infinity to $\delta/\beta > 1$ so again F is analytic in a circle with a radius bigger than 1.

Since we know that the generating function (defined by the series), coincides with F defined above for $|x| < \min\{1, \delta/\beta\}$, by the principle of analytic continuation, the generation function coincides with F in the whole domain of its analyticity (note that this is not necessarily solution of the equation (66) outside this region as we have removed the absolute value bars).

Consider now the case $\beta = \delta$. Then the characteristic equation is

$$\frac{dx}{dt} = -\beta(x-1)^2,$$

solving which we obtain

$$\frac{1}{x-1} = \beta t + \xi,$$

or

$$\xi = \frac{1 - x\beta t + \beta t}{x-1}.$$

Hence, the general solution is given by

$$F(t,x) = G\left(\frac{1 - x\beta t + \beta t}{x-1}\right)$$

for some function G.

Using the initial condition, we have

$$x^{n_0} = G\left(\frac{1}{x-1}\right).$$

Hence

$$G(s) = \left(1 + \frac{1}{s}\right)^{n_0}.$$

Therefore

$$F(t,x) = \left(1 + \frac{x-1}{1 - x\beta t + \beta t}\right)^{n_0} = \left(\frac{\beta t + (1-\beta t)x}{1 - x\beta t + \beta t}\right)^{n_0}.$$

Summarizing,

$$F(t,x) = \begin{cases} \left(\frac{e^{rt}\delta(1-x)+(\beta x-\delta)}{e^{rt}\beta(1-x)+(\beta x-\delta)}\right)^{n_0} & \text{if} \quad \beta \neq \delta \\ \left(\frac{\beta t+(1-\beta t)x}{1-x\beta t+\beta t}\right)^{n_0} & \text{if} \quad \beta = \delta. \end{cases} \qquad (68)$$

Let us complete this section by evaluating some essential parameters. The probability of extinction at time t is given by

$$p_0(t) = F(t,0) = \begin{cases} \left(\frac{\delta(e^{rt}-1}{e^{rt}\beta-\delta}\right)^{n_0} & \text{if} \quad \beta \neq \delta \\ \left(\frac{\beta t}{1+\beta t}\right)^{n_0} & \text{if} \quad \beta = \delta, \end{cases} \tag{69}$$

where, recall, $r = \beta - \delta$. Hence, the asymptotic probability of extinction is given by

$$\lim_{t\to\infty} p_0(t) = \begin{cases} \left(\frac{\delta}{\beta}\right)^{n_0} & \text{if} \quad \beta > \delta \\ 1 & \text{if} \quad \beta \leq \delta. \end{cases} \tag{70}$$

We note that even for positive net growth rates $\beta > \delta$ the probability of extinction is non-zero. Populations with small initial numbers are especially susceptible to extinction.

To derive the expected size of the population, we use (64). We have

$$E[N(t)] = \left.\frac{\partial F}{\partial x}\right|_{(t,x)=(t,1)} = n_0 \frac{(-e^{rt}\delta + \beta)(\beta - \delta) + \beta(e^{rt} - 1)(\beta - \delta)}{(\beta - \delta)^2}$$

$$= n_0 e^{rt}.$$

The variance is given by

$$Var[N(t)] = \frac{e^{rt}(-1 + e^{rt}) n_0 (\beta + \delta)}{\beta - \delta}$$

for $\beta \neq \delta$, while for $\beta = \delta$ we obtain

$$V(t) = 2n_0\beta t.$$

McKendrick–von Foerster model. Thus, we have the classical formulation of the McKendrick–von Foerster model

$$\begin{aligned}
\partial_t n(a,t) + \partial_a n(a,t) &= -\mu(a)n(a,t), \qquad a>0, t>0, & (71)\\
n(0,t) &= \int\limits_0^{\omega} \beta(\alpha)n(\alpha,t)d\alpha, t>0, & (72)\\
n(a,0) &= n_0(a), & (73)
\end{aligned}$$

where the last equation provides the initial distribution of the population.

If $\omega < +\infty$, then we have to ensure that $n(a,t) = 0$ for $t \geq 0, a \geq \omega$, which can be done either by imposing an additional boundary condition on n, or by introducing assumptions on the coefficients which ensure that no individual survives beyond ω. If $\omega = \infty$ then, instead of such an additional condition, we impose some requirements on the behaviour of the solution at ∞, e.g., that they are integrable over $[0, \infty)$.

Linear constant coefficient case. Before we embark on more advanced analysis of (71)–(73), let us get a taste of the structure of the problem by solving the simplest case with $\mu(a) = \mu$ and $\beta(a) = \beta$:

$$\partial_t n(a,t) + \partial_a n(a,t) = -\mu n(a,t). \tag{74}$$

coupled with the boundary condition

$$n(0,t) = \beta \int_0^\infty n(a,t)da,$$

and the initial condition

$$n(a,0) = \mathring{n}(a).$$

First, let us simplify the equation (74) by introducing the integrating factor

$$\partial_t(e^{\mu a} n(a,t)) = -\partial_a(e^{\mu a} n(a,t))$$

and denote $u(a,t) = e^{\mu a} n(a,t)$. Then

$$u(0,t) = n(0,t) = \beta \int_0^{\infty} e^{-\mu a} u(a,t) da$$

with $u(a,0) = e^{\mu a} \mathring{p}(a) =: \mathring{u}(a)$. Now, if we knew $\psi(t) = u(0,t)$, then

$$u(a,t) = \begin{cases} \mathring{u}(a-t), & t < a, \\ \psi(t-a), & a < t. \end{cases} \tag{75}$$

The boundary condition can be rewritten as

$$\begin{aligned}\psi(t) &= \beta\int\limits_0^\infty e^{-\mu a}u(a,t)da \\ &= \beta\int\limits_0^t e^{-\mu a}\psi(t-a)da + \beta\int\limits_t^\infty e^{-\mu a}\mathring{u}(a-t)da \\ &= \beta e^{-\mu t}\int\limits_0^t e^{\mu\sigma}\psi(\sigma)d\sigma + \beta e^{-\mu t}\int\limits_0^\infty e^{-\mu r}\mathring{u}(r)dr\end{aligned}$$

which, upon denoting $\phi(t) = \psi(t)e^{\mu t}$ and using the original initial value, can be written as

$$\phi(t) = \beta\int\limits_0^t \phi(\sigma)d\sigma + \beta\int\limits_0^\infty \mathring{n}(r)dr. \tag{76}$$

Now, if we differentiate both sides, we get

$$\phi' = \beta\phi,$$

which is just a first order linear equation. Letting $t = 0$ in (76), we obtain the initial value for ϕ: $\phi(0) = \beta \int\limits_0^\infty \mathring{p}(r)dr$. Then

$$\phi(t) = \beta e^{\beta t} \int\limits_0^\infty \mathring{n}(r)dr$$

and

$$\psi(t) = \beta e^{(\beta-\mu)t} \int\limits_0^\infty \mathring{n}(r)dr.$$

Then

$$n(a,t) = e^{-\mu a}u(a,t) = e^{-\mu t}\begin{cases} \mathring{n}(a-t), & t < a, \\ \beta e^{\beta(t-a)}\int\limits_0^\infty \mathring{n}(r)dr, & a < t. \end{cases}$$

Observe that

$$\lim_{a\to t^+} n(a,t) = \mathring{n}(0)$$

and

$$\lim_{a\to t^-} n(a,t) = \beta\int\limits_0^\infty \mathring{n}(r)dr,$$

so that the solution is continuous, let alone differentiable, only if the initial condition satisfies the following compatibility condition

$$\mathring{n}(0) = \beta\int\limits_0^\infty \mathring{n}(r)dr. \qquad (77)$$

Thus, as we noted earlier, we must be very careful with using (71)-(73) in the differential form and interpreting the solution.

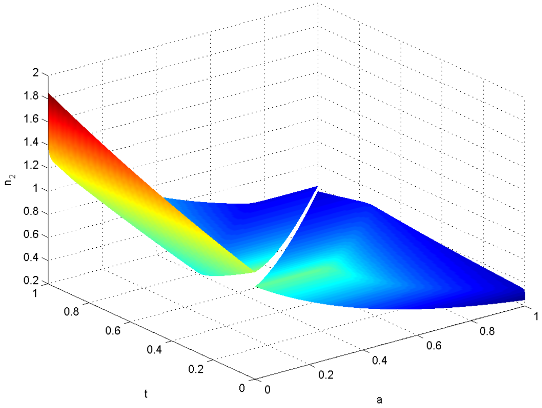


Figure: Discontinuity of the population density $n(a, t)$.

General linear McKendrick-von Foerster problem. The ideas used to solve the McKendrick-von Foerster case in the constant coefficient case can be also used in more general situations but, unfortunately, the resulting integral equation (76) cannot be explicitly solved. Before, however, we discuss the solvability of more general cases, let us introduce certain functions related to (71)-(73), which are relevant to the population dynamics.

Consider again the general McKendrick problem

$$\partial_t n(a,t) + \partial_a n(a,t) = -\mu(a) n(a,t)$$

$$n(0,t) = \int\limits_0^\omega \beta(\alpha) n(\alpha,t) d\alpha,$$

$$n(a,0) = \mathring{n}(a).$$

We recall that $\beta(a)$ is the *age specific fertility* which can be defined by saying that the number of newborns, in one time unit, coming from a single individual whose age is in the small time age interval $[a, a + da)$, is $\beta(a)da$. So, the number of births coming from all individuals in the population aged between a_1 and a_2 in a one time unit is

$$\int_{a_1}^{a_2} \beta(\alpha)n(\alpha, t)da$$

and we can define the *total birth rate* as

$$B(t) = \int_{0}^{\omega} \beta(\alpha)n(\alpha, t)da$$

which gives the total number of newborns in a unit time (ω is the maximum age in the population).

Let us consider the death rate $\mu(a)$, which is average number of deaths per unit of population aged a. We can relate $\mu(a)$ to a number of vital characteristics of the population. Similarly to the discrete case, we introduce the *survival probability* $S(a)$ as the proportion of the initial population surviving to age a. We can relate μ and S by the following argument.

Consider a population beginning with $\mathring{p}$ individuals of age 0. Then $\mathring{n}(a)S(a)(= n(a))$ is the average number of individuals that survived to age a. The decline in the population over a short age period $[a, a + da]$ is $\mathring{n}(a)S(a) - \mathring{n}(a)S(a + da)$. On the other hand, this decline can only be attributed to deaths: if the death rate is μ, then in this age interval we will have approximately $\mathring{n}(a)S(a)\mu(a)da$ deaths. Equating and passing to the limit as $da \to 0$ yields

$$\frac{dS}{da} = -S\mu$$

or

$$S(a) = S(0)e^{-\int\limits_0^a \mu(\sigma)d\sigma} = e^{-\int\limits_0^a \mu(\sigma)d\sigma},$$

since the probability of surviving to age 0 is 1.

We note that if no individuals can survive beyond ω, we must have $S(\omega) = 0$ or, equivalently,

$$\int\limits_0^\omega \mu(\sigma)d\sigma = \infty. \tag{78}$$

These considerations can be used to find the average life span of individuals in the population. In fact, the average life span is the mean value of the length of life in the population, which can be expressed as

$$L = \int_0^{\omega} a\mathfrak{n}(a)da,$$

where $\mathfrak{n}(a)$ is the probability (density) of an individual dying at age a. We approximate the integral as the Riemann sum

$$L \approx \sum_i a_i \mathfrak{n}(a_i)\Delta a_i,$$

where $\mathfrak{n}(a_i)$ is the probability that an individual survived till the age a_i and died at this age.

Thus

$$n(a_i) = S(a_i)\mu(a_i).$$

We note that $S(a)\mu(a)$ is, indeed, a probability density. Thus

$$L = \int_0^\omega a\mu(a)e^{-\int_0^a \mu(s)ds}da = -\int_0^\omega a\frac{d}{da}e^{-\int_0^a \mu(s)ds}da = \int_0^\omega S(a)da,$$

where we used integration by parts and $S(\omega) = 0$.

Further, we introduce

$$K(a) = \beta(a)S(a) \tag{79}$$

which is called the *maternity function* and describes the rate of birth relative to the surviving fraction of the population. Further, we define

$$R = \int_0^{\omega} \beta(a)S(a)da, \tag{80}$$

and call it the *net reproduction rate* of the population. It is the expected number of offspring produced by an individual during her life.

Solution of general McKendrick-von Foerster model. One of the easiest way of analysing the general model

$$\begin{aligned}
&\partial_t n(a,t) + \partial_a n(a,t) = -\mu(a) n(a,t) \\
&n(0,t) = \int_0^{\omega} \beta(a) n(a,t) da, \\
&n(a,0) = \mathring{n}(a), \qquad (81)
\end{aligned}$$

is to reduce it to an integral equation in the same way as for the constant coefficient case, though the technicalities are slightly more involved due to age dependence of the mortality and maternity functions.

First, we simplify (81) by introducing the integrating factor

$$\partial_t \left(e^{\int_0^a \mu(\sigma)d\sigma} n(a,t) \right) = -\partial_a \left(e^{\int_0^a \mu(\sigma)d\sigma} n(a,t) \right) \qquad (82)$$

and denote $u(a,t) = e^{\int_0^a \mu(\sigma)d\sigma} n(a,t)$. Then

$$u(0,t) = n(0,t) = \int_0^\omega \beta(a) e^{-\int_0^a \mu(\sigma)d\sigma} u(a,t)da = \int_0^\omega K(a)u(a,t)da,$$

where we recognized that the kernel in the integral above is the maternity function introduced in (79).

Further, $u(a,0) = e^{\int\limits_0^a \mu(s)ds} \mathring{n}(a) =: \mathring{u}(a)$. Also, the right hand side defines the total birth rate $B(t)$.

Now, if we knew $B(t) = u(0,t)$, then

$$u(a,t) = \begin{cases} \mathring{u}(a-t), & t < a, \\ B(t-a), & a < t. \end{cases} \tag{83}$$

The boundary condition can be rewritten as

$$\begin{aligned}
B(t) &= \int\limits_0^\infty \beta(a) e^{-\int\limits_0^a \mu(\sigma)d\sigma} u(a,t)da \\
&= \int\limits_0^t \beta(a) e^{-\int\limits_0^a \mu(\sigma)d\sigma} B(t-a)da + \int\limits_t^\infty \beta(a) e^{-\int\limits_0^a \mu(\sigma)d\sigma} \mathring{u}(a-t)da \\
&= \int\limits_0^t K(t-a)B(a)da + \int\limits_0^\infty \beta(a+t) e^{-\int\limits_0^{a+t} \mu(\sigma)d\sigma} e^{\int\limits_0^a \mu(s)ds} \mathring{n}(a)da,
\end{aligned}$$

where to shorten notation we extended coefficients by zero beyond $a = \omega$.

Summarizing, we arrived at the integral equation for the total birth rate

$$B(t) = \int\limits_0^t K(t-a)B(a)da + G(t) \tag{84}$$

where

$$G(t) = \int\limits_0^\infty \beta(a+t)\frac{S(a+t)}{S(a)}\mathring{n}(a)da, \tag{85}$$

is a known function.

Explicitly, we have

$$
\begin{aligned}
B(t) &= \int_0^t K(t-a)B(a)da + \int_0^{\omega-t} \beta(a+t)\frac{S(a+t)}{S(a)}\mathring{n}(a)da \\
&= \int_0^t K(t-a)B(a)da + \int_t^{\omega} \beta(a)\frac{S(a)}{S(a-t)}\mathring{n}(a-t)da, \quad (86)
\end{aligned}
$$

for $0 \leq t \leq \omega$, and

$$
B(t) = \int_0^{\omega} K(t-a)B(a)da \qquad (87)
$$

for $t > \omega$.

This equation cannot be solved explicitly and we have to use more abstract approach. For this we have to introduce a proper mathematical framework. As in the discrete case, the natural norm will be

$$\|p\|_1 = \int_0^\omega |n(\alpha)| d\alpha$$

which in the current context, with $p \geq 0$ being the density of the population distribution with respect to age, is the total population. Thus, the state space is the space $X_0 = L_1([0, \omega))$ of Lebesgue integrable functions on $[0, \omega)$.

Since we are dealing with functions of two variables, we often consider $(a, t) \to n(a, t)$ as a function $t \to u(t, \cdot)$, that is, for each t the value of this function is a function with argument a. For such functions, we consider the space $C([0, T], L_1([0, \omega]))$ of $L_1([0, \omega])$-valued continuous functions. For functions f bounded on $[0, \omega]$ we introduce $\|f\|_\infty = \sup_{0 \leq a \leq \omega} |f(a)|$.

We make the following assumptions.

(i)

$$\beta \geq 0 \text{ is bounded on } [0, \omega], \tag{88}$$

(ii)

$$0 \leq \mu \in L_1([0, \omega']) \text{ for any } \omega' < \omega \tag{89}$$

with

$$\int_0^{\omega} \mu(\alpha) d\alpha = \infty, \tag{90}$$

(iii)

$$0 \leq \mathring{p} \in L_1([0, \omega]). \tag{91}$$

Now, if (88)-(91) are satisfied, then we can show that K is a non-negative bounded function which is zero for $t \geq \omega$ and G is a continuous function which also is zero for $t \geq \omega$. If, additionally

$$\mathring{p} \in W^{1,1}([0,\omega]) \quad \text{and} \quad \mu\mathring{p} \in L_1([0,\omega]), \tag{92}$$

(here by W_1^1 we denote the Sobolev space of functions from L_1 with generalized derivatives in L_1), then G is differentiable with bounded derivative. Indeed, let us look at G for $t < \omega$

$$G(t) = \int\limits_t^\omega \beta(a)\frac{S(a)}{S(a-t)}\mathring{n}(a-t)da = \int\limits_t^\omega \beta(a)e^{-\int\limits_{a-t}^{a}\mu(s)ds}\mathring{n}(a-t)da$$

If we formally differentiate using the Leibniz rule, we get

$$
\begin{aligned}
G'(t) &= -\beta(t)S(t)\mathring{n}(0) + \int\limits_t^{\omega} \beta(a) e^{-\int\limits_{a-t}^{a} \mu(s)ds} \mu(a-t)\mathring{n}(a-t)da \\
&\quad + \int\limits_t^{\omega} \beta(a) e^{-\int\limits_{a-t}^{a} \mu(s)ds} \mu(a-t)\mathring{n}'(a-t)da
\end{aligned}
$$

so we see that for existence of the integrals we need integrability of $\mu\mathring{p}$ and differentiability of $\mathring{p}$. Then we can prove the main result

Theorem 5

If (88)-(91) are satisfied, then (84) has a unique continuous and nonnegative solution. If, additionally, (92) is satisfied that B then B is differentiable with B' bounded on bounded intervals.

Proof. We define iterates

$$B_0(t) = G(t),$$
$$B^{k+1}(t) = G(t) + \int_0^t K(t-s)B^k(s)ds. \qquad (93)$$

Take $T > 0$. Then, for any $t \in [0, T]$ we have

$$|B^1(t) - B^0(t)| = \int_0^t |K(t-s)F(s)|ds \leq tK_mF_m$$

where $K_m = \sup_{0\leq t\leq T} |K(s)|$ and $L_m = \sup_{0\leq t\leq T} |F(s)|$. Then

$$|B^2(t) - B^1(t)| \leq K_m \int_0^t |B^1(s) - B^0(s)|ds \leq \frac{K_m^2 F_m}{2} t^2$$

and, by induction,

$$|B^{k+1}(t) - B^k(t)| \leq K_m \int\limits_0^t |B^k(s) - B^{k-1}(s)| ds \leq \frac{K_m^{k+1} F_m}{(k+1)!} t^{k+1}. \tag{94}$$

Further

$$\lim_{k\to\infty} B^{k+1}(t) = G(t) + \lim_{k\to\infty} \sum_{i=0}^{k} (B^{i+1}(t) - B^i(t))$$

with

$$\begin{aligned}\sup_{0\leq t\leq T} \left| \sum_{i=0}^{k} (B^{i+1}(t) - B^i(t)) \right| &\leq \sum_{i=0}^{k} \sup_{0\leq t\leq T} \left| B^{i+1}(t) - B^i(t) \right| \\ &\leq F_m \sum_{i=0}^{k} \frac{(TK_m)^{k+1}}{(k+1)!}.\end{aligned}$$

The series on the right hand side converges to $F_m e^{TK_m}$ and thus $(B^k(t))_{k\geq 0}$ converges uniformly to a continuous solution B of (84). Uniqueness follows by the Gronwall inequality.

If, in addition, (92) is satisfied, then B^k can be differentiated with respect to t and the functions

$$V^k := \frac{d}{dt}B^k$$

satisfy the recurrence

$$V^{k+1}(t) = F'(t) + K(t)F(0) + \int\limits_0^t K(t-s)V^k(s)ds,$$

which converges uniformly to some continuous function V which, by the theorem of uniform convergence of derivatives, must be the derivative of B. □

Once we have B, we can recover p by (109) and back substitution

$$n(a,t) = e^{-\int\limits_0^a \mu(\sigma)d\sigma} u(a,t) = \begin{cases} \frac{S(a)}{S(a-t)}\mathring{n}(a-t), & t < a, \\ S(a)B(t-a), & a < t. \end{cases} \tag{95}$$

Thus, if (92) is satisfied in addition to (88)-(91), then it is easy to see that p defined above satisfies the equation (71) everywhere except the line $a = t$. Along this line we have, as before,

$$\lim_{a \to t^+} n(a,t) = S(0)\mathring{n}(0) = \mathring{n}(0)$$

and

$$\lim_{a \to t^-} n(a,t) = S(0)B(0) = \int\limits_0^\omega \beta(a)\mathring{n}(a)da.$$

To ensure at least continuity of the solution we need to assume the compatibility condition

$$\mathring{n}(0) = \int_0^{\omega} \beta(a)\mathring{n}(a)da. \tag{96}$$

We note that if a function is continuous at a point and differentiable in both one sided neighbourhoods, then it is a Lipschitz function and it is in fact differentiable almost everywhere (in the sense that the function can be recovered from its derivative). On the other hand, if a function has a jump at a point, then its derivative at this point is of a Dirac delta type. Thus, we can state that if (96) is satisfied, then the solution is continuous and satisfies (71) almost everywhere.

If we do not assume (96) then we can still claim that the solution satisfies

$$\begin{aligned} Dn(a,t) &= \lim_{h\to 0^+} \frac{n(a+h,t+h)-n(a,t)}{h} \\ &= -\mu(a)n(a,t),\ a>0, t>0. \end{aligned}$$

Furthermore, both the birth rate B and the solution p itself grow at most at an exponential rate. Consider again (84)

$$B(t) = \int\limits_0^t K(t-a)B(a)da + G(t),$$

with G given by (85),

$$S(a) = e^{-\int\limits_0^a \mu(\sigma)d\sigma},$$

and $K(a) = \beta(a)S(a)$, we see that $K(t) \leq \|\beta\|_\infty$ and $G(t) \leq \|\beta\|_\infty \|\mathring{p}\|_1$ so that

$$B(t) \leq \max_{0\leq a\leq\omega} \beta(a) \int\limits_0^t B(s)ds + \max_{0\leq a\leq\omega} \beta(a) \int\limits_0^\omega \mathring{n}(s)ds$$

$$=: \|\beta\|_\infty \int\limits_0^t B(s)ds + \|\beta\|_\infty \|\mathring{p}\|_1,$$

which, by Gronwall's inequality, yields

$$B(t) \leq \|\beta\|_\infty \|\mathring{p}\|_1 e^{t\|\beta\|_\infty}. \tag{97}$$

This gives the estimate for p:

$$\begin{aligned}\|n(\cdot,t)\|_1 &\leq \int\limits_0^t B(t-s)S(s)ds + \int\limits_t^\infty \frac{S(s)}{S(s-t)}\mathring{n}(s-t)ds\\ &\leq \|\beta\|_\infty\|\mathring{p}\|_1\left(\int\limits_0^t e^{(t-s)\|\beta\|_\infty}ds + 1\right),\end{aligned}$$

where we used $S(s)/S(s-t) \leq 1$. Then, by integration

$$\|n(\cdot,t)\|_1 \leq \|\mathring{p}\|_1 + \|\mathring{p}\|_1 e^{t\|\beta\|_\infty}(1-e^{-t\|\beta\|_\infty}) = \|\mathring{p}\|_1 e^{t\|\beta\|_\infty}. \quad (98)$$

Long-time behaviour of the solution. First, let us consider the eigenvalue problem for (74)

$$\lambda n(a) + n'(a) = -\mu n(a)$$
$$n(0) = \beta \int_0^\infty n(a)da. \tag{99}$$

The first equation is a linear equation with the general solution

$$n(a) = Ce^{-(\mu+\lambda)a}$$

while the nonlocal initial condition yields

$$1 = \beta \int_0^\infty e^{-(\mu+\lambda)a} da$$

where we cancelled the constant C.

This is an example of the Lotka renewal equation. In our case, we solve it explicitly. Integration gives

$$1 = \frac{\beta}{\mu + \lambda} \tag{100}$$

or

$$\lambda = \beta - \mu = r$$

and

$$p(a) = Ce^{-\beta a}.$$

So, the unique eigenvalue of (104) is (in this case) precisely the net growth rate. This eigenvalue is simple and the corresponding eigenvector is the stable age distribution. As we shall see, this is not a coincidence.

Long time behaviour–general case. By (97), we can apply the Laplace transform to analyse (84). The Laplace transform of an exponentially bounded integrable function f is defined by

$$\hat{f}(\lambda) = (\mathcal{L}f)(\lambda) = \int_0^\infty e^{-\lambda t} f(t)dt,$$

and $\hat{f}$ is defined and analytic in a right half-plane (determined by the rate of growth of f) of the complex plane $\mathbb{C}$. In the case of B, (97) shows that $\hat{B}(\lambda)$ is analytic in $\Re\lambda > \|m\|_\infty$. For our applications it is also important to note that if the f is only non-zero over a finite interval $[a, b]$, then its Laplace transform is defined and analytic everywhere in $\mathbb{C}$. Such functions are called entire.

Moreover, we also use $\hat{f}(\lambda) \to 0$ as $|\lambda| \to \infty$ in any closed strip contained in the domain of analyticity of $\hat{f}$.

We use the property of the Laplace transform that the convolution is transformed into the algebraic product of transforms; that is, for the convolution

$$(f * g)(t) = \int_0^t f(t-s)g(s)ds = \int_0^t f(s)g(t-s)ds,$$

using the definition of the Laplace transform and changing the order of integration, we obtain

$$[\mathcal{L}(f * g)](\lambda) = (\mathcal{L}f)(\lambda) \cdot (\mathcal{L}g)(\lambda). \qquad (101)$$

With this result, (84) yields

$$\hat{B}(\lambda) = \hat{B}(\lambda)\hat{K}(\lambda) + \hat{G}(\lambda). \tag{102}$$

Hence,

$$\hat{B}(\lambda) = \frac{\hat{G}(\lambda)}{1 - \hat{K}(\lambda)} = \hat{G}(\lambda) + \frac{\hat{G}(\lambda)\hat{K}(\lambda)}{1 - \hat{K}(\lambda)} \tag{103}$$

As we noted above, $\hat{G}$ is an entire function so the only singularities of $\hat{B}$ are due to zeroes of $1 - \hat{K}$. Since $\hat{K}$ is an entire function, these zeroes are isolated of finite order (thus giving rise to poles of $\hat{B}$ and with no finite accumulation point).

However, there may be infinitely many of them and this requires some care with handling the inverse. We know that if $\hat{f}$ is the Laplace transform of a continuous function f, then

$$f(t) = \frac{1}{2\pi i}\int\limits_{c-i\infty}^{c+i\infty} e^{\lambda t}\hat{f}(\lambda)d\lambda$$

where we integrate along a line in the domain of analyticity of $\hat{f}$.

Let us look closer at the equation

$$\hat{K}(\lambda) = 1, \tag{104}$$

or, explicitly,

$$\int\limits_0^\infty \beta(a)e^{-\lambda a - \int\limits_0^a \mu(\sigma)d\sigma} da = 1, \quad \lambda \in \mathbb{C}. \tag{105}$$

Theorem 6

Equation (104) has exactly one real root, $\lambda = \lambda_0$, of algebraic multiplicity 1. All other roots λ_j of (104) occur as complex conjugates (real root is its own conjugate). Moreover, $\Re\lambda_j < \lambda_0$ for any j, there could be only denumerable number of them and, in each strip $a < \Re\lambda < b$, there is at most a finite number of them.

Proof. We introduce the real function

$$\psi(\lambda) = \int_0^\infty e^{-\lambda a} K(a) da$$

for $\lambda \in \mathbb{R}$. We note that this function is well defined on $\mathbb{R}$ since K is non zero only on a finite interval. Also, because of this, it is continuous and differentiable, see Remark 1 below. Then

$$\begin{aligned} \lim_{\lambda \to -\infty} \psi(\lambda) &= \infty, \\ \lim_{\lambda \to \infty} \psi(\lambda) &= 0. \end{aligned}$$

Moreover,

$$
\begin{aligned}
\psi'(\lambda) &= -\int\limits_{\alpha}^{\beta} a e^{-\lambda a} K(a) da < 0, \\
\psi''(\lambda) &= \int\limits_{\alpha}^{\beta} a^2 e^{-\lambda a} K(a) da > 0,
\end{aligned}
$$

so that ψ is strictly decreasing and concave up function. Since it is continuous, it takes on every positive value exactly once. Thus, in particular, there is exactly one real value λ_* satisfying (104).

Suppose $\lambda = u + iv$ is a root of (104). Then

$$1 = \int_0^\infty e^{-va}(\cos(-ua) + i\sin(-ua))K(a)da$$

and, taking the real and imaginary part,

$$\begin{aligned}\int_0^\infty e^{-va}K(a)\cos ua\, da &= 1,\\ \int_0^\infty e^{-va}K(a)\sin ua\, da &= 0.\end{aligned}$$

We observe that these two equations are invariant under the change $v \to -v$ so that $\bar{\lambda} = u - iv$ also satisfies (104).

To prove the second part, we note that, since the variable a is continuous, there must be a range of a, say, $[\alpha, \beta]$ over which $\cos ua < 1$. Thus,

$$\int_0^\infty e^{-va}K(a)da > \int_0^\infty e^{-va}K(a)\cos ua\, da = 1.$$

However

$$\int_\alpha^\beta e^{-\lambda_* a}K(a)da = 1,$$

and direct comparison of these two integrals yields $\lambda_* > v = \Re\lambda$.

The last part follows from the fact that since $\hat{K} - 1$ is an entire function, in each bounded set there can be only finitely many zeros of it, by the principle of isolated zeros. Thus, there could be no more than denumerable amount of them in $\mathbb{C}$. Finally, since $\hat{K} \to 0$ as $|\lambda| \to \infty$ in any strip, we also see that there can be only finitely many of them in any vertical strip. $\square$

Remark 1

In the proof above, the continuity of ψ *is a consequence of the boundeness of the support of definition of* K*. In general, if we allow* K *to be nonzero on* $[0, \infty)$*, then the above statement is not true. Consider* $K(a) = c(1 + a^2)^{-1}$ *with* $c < 2/\pi$*. Then*

$$\psi(\lambda) = c \int\limits_0^\infty \frac{e^{-\lambda a}}{1 + a^2} da$$

then $\psi(\lambda) < 1$ *for* $\lambda \geq 0$ *but* $\psi(\lambda) = \infty$ *for* $\lambda < 0$ *and* $\psi(\lambda) < 1$ *for all* $\lambda \geq 0$ *and Eq. (104) has no real solution.*

Remark 2

In general, if $\omega = \infty$, one has to prove that the range of ψ contains 1. For instance, in the constant coefficient case, ψ is given by (100)

$$\psi(\lambda) = \frac{m}{\lambda + \mu}$$

and though it is discontinuous at $\lambda = -\mu$, its range for $\lambda \in (-\mu, \infty)$ is $\mathbb{R}$ and the argument holds.

Observe that the function ψ crosses the a axis at

$$R := \psi(0) = \int_0^\infty K(a)da \tag{106}$$

which is precisely the net reproductive rate. R must exceed 1 for λ_* to be positive, $R = 1$ if and only if $\lambda_* = 0$ and, finally, $R < 1$ if and only if $\lambda_* < 0$.

Next we shall show that the sign of λ_* indeed determines the long time behaviour of the population.

Let us consider the second term in the last formula of (103)

$$\hat{H}(\lambda) := \frac{\hat{G}(\lambda)\hat{K}(\lambda)}{1-\hat{K}(\lambda)}.$$

We noted that $\hat{G}(\lambda)$ and $\hat{K}(\lambda)$ tend to zero as $|\lambda| \to \infty$ in any half plane $\Re\lambda > \delta$, $\delta \in \mathbb{R}$. Furthermore, on any line $\{\sigma + iy;\ y \in \mathbb{R}\}$ which does not meet any root of (104), we have $\inf_{y\in\mathbb{R}} |1 - \hat{K}(\sigma + iy)| > 0$ and

$$\int\limits_{-\infty}^{\infty} \left|\frac{\hat{G}(\sigma+iy)\hat{K}(\sigma+iy)}{1-\hat{K}(\sigma+iy)}\right| dy < \infty. \tag{107}$$

This follows from the fact that any finitely supported function, multiplied by $e^{-\sigma t}$ is an L_2 function and thus its Laplace transform, treated as the Fourier transform, is in L_2 with respect to y. Then the result follows from the Plancherel theorem.

Inverting $\hat{H}(\lambda)$ we have

$$H(t) = \frac{1}{2\pi i} \int\limits_{\sigma - i\infty}^{\sigma + i\infty} \frac{\hat{G}(\sigma + iy)\hat{K}(\sigma + iy)}{1 - \hat{K}(\sigma + iy)} e^{(\sigma + iy)t} dy$$

for any $\sigma > \lambda_*$. Hence

$$B(t) = G(t) + H(t).$$

To estimate $H(t)$ we note that, by properties of $\hat{H}$, we can shift the line of integration to $\{\sigma_1 + iy;\ y \in \mathbb{R}\}$ where $\Re\lambda_1 < \sigma_1 < \lambda_*$ and λ_1 is the eigenvalue with the largest real part less than λ_*.

Then the Cauchy theorem gives

$$H(t) = H_1(t) + H_2(t)$$

where

$$H_1(t) = res_{\lambda=\lambda_*} \frac{e^{\lambda t}\hat{G}(\lambda)\hat{K}(\lambda)}{1-\hat{K}(\lambda)} = \lim_{\lambda\to\lambda_*}(\lambda-\lambda_*)\frac{e^{\lambda t}\hat{G}(\lambda)\hat{K}(\lambda)}{1-\hat{K}(\lambda)} = B_0 e^{\lambda_* t},$$

with

$$B_0 = \frac{\int\limits_0^\infty e^{-\lambda_* a}G(a)da}{\int\limits_0^\infty ae^{-\lambda_* a}K(a)da}$$

and

$$H_2(t) = \frac{1}{2\pi i}\int\limits_{\sigma_1-i\infty}^{\sigma_1+i\infty} \frac{\hat{G}(\sigma_1+iy)\hat{K}(\sigma_1+iy)}{1-\hat{K}(\sigma_1+iy)}e^{(\sigma_1+iy)t}dy.$$

The function H_2 satisfies the estimate

$$|H_2(t)| \leq \frac{e^{\sigma_1 t}}{2\pi} \int\limits_{\sigma_1 - i\infty}^{\sigma_1 + i\infty} \left| \frac{\hat{G}(\sigma_1 + iy)\hat{K}(\sigma_1 + iy)}{1 - \hat{K}(\sigma_1 + iy)} \right| dy = B_1 e^{\sigma_1 t}.$$

Here B_1 is a constant. Thus, we arrived at the representation

$$B(t) = e^{\lambda_* t} B_0 + G(t) + e^{\sigma_1 t} B_1.$$

However, remembering that $G(t) = 0$ for $t \geq 0$, we can write

$$B(t) = B_0 e^{\lambda_* t} \left(1 + \frac{e^{-\lambda_* t} G(t)}{B_0} + e^{(\sigma_1 - \lambda_*)t} \frac{B_1}{B_0} \right) = B_0 e^{\lambda_* t} (1 + \Omega(t)) \tag{108}$$

where $\Omega(t) \to 0$ as $t \to \infty$, provided $B_0 \neq 0$.

Now, $B_0 = 0$ if and only if $G(t) = 0$ for all $t \geq 0$ but then, from uniqueness, $B(t) = 0$ for all t.

Let us interpret this condition. We have

$$0 = G(t) = \int_0^{\infty} \beta(a+t)\frac{S(a+t)}{S(a)}\mathring{n}(a)da$$

which, by positivity of $\mathring{n}$, is possible only if

$$\beta(a+t)\mathring{n}(a) = 0$$

for $a \in [0, \omega]$ and $t \geq 0$. This occurs only if the support of β is to the left of the support of $\mathring{n}$ (as the support of $\beta(\cdot + t)$ moves to the left as t increases). In other words, this case occurs only if the original population is too old to become fertile.

In this case

$$n(a,t) = \begin{cases} \mathring{n}(a-t)\frac{S(a)}{S(a-t)}, & t < a, \\ 0, & a < t. \end{cases} \tag{109}$$

Otherwise, we can write

$$n(a,t) = \begin{cases} \mathring{n}(a-t)\frac{S(a)}{S(a-t)}, & t < a, \\ B_0 e^{\lambda_*(t-a)}\left(1+\Omega(t-a)\right)S(a), & a < t. \end{cases} \tag{110}$$

Now, in the case $\omega < +\infty$ we see that for $t \geq \omega$ we have

$$n(a,t) = B_0 e^{\lambda_*(t-a)}\left(1 + \Omega(t-a)\right) S(a)$$

and we identify the stable age distribution

$$n_\infty(a) = e^{-\lambda_* a - \int\limits_0^a \mu(s)ds}.$$

so that

$$\lim_{t\to\infty} e^{-\lambda_* t} n(a,t) = e^{-\lambda_* a - \int\limits_0^a \mu(s)ds}$$

on $[0,\omega]$ (provided the supports of $\mathring{n}$ and β meet).

Finally, we noted in (106) that $\lambda_* > 0$, $\lambda_* = 0$ and $\lambda_* < 0$ if and only if, respectively, $R > 1, R = 1$ and $R < 1$. Thus, the population is growing if $R > 1$, it is stable if $R = 1$ and it decays if $R < 1$ (again if supports of $\mathring{n}$ and β meet), in accordance with the interpretation of the parameter R.

Travelling wave solutions. A wave is a recognizable signal which is transferred from one part of the medium to another part at a determined speed of propagation. Wave propagation is of fundamental importance in

Fluid mechanics (water waves, aerodynamics);

Acoustics (sound waves in air and liquids);

Elasticity (stress waves, earthquakes);

Electromagnetic theory (optics, electromagnetism);

Biology (epizootic waves);

Chemistry (combustion and detonation waves).

The simplest form of a mathematical wave is a function of the form

$$u(x,t) = f(x \pm ct). \tag{111}$$

We adopt the convention that $c > 0$. The general solution of

$$u_t = \pm c u_x$$

is given by (111).

At $t = 0$ we have $u(x,0) = f(x)$, which is the initial wave profile. Then $f(x - ct)$ represents the profile at time t, that is just the initial profile translated to the right by ct spatial units. Thus c is the speed of the wave; (111) with $-c$ represents a wave travelling to the right with speed $c > 0$ and with $+c$ is a wave travelling to the left with the same speed.

Does a given PDE admit a wave solution? A *travelling wave solution* to a PDE is a solution of the form $u(x,t) = U(x \pm ct)$, with U being differentiable enough to be a solution to the PDE for all times and in the whole space and satisfying

$$U(-\infty) = u_{-\infty}, \qquad U(+\infty) = u_{+\infty}. \tag{112}$$

Mathematically, looking for travelling wave solutions is asking whether a given PDE has solutions invariant under a Galilean transformation; in such a case it can be reduced to an ODE.

For instance, for a second order PDE,

$$F(u, u_t, u_x, u_{xx}) = 0,$$

looking for solutions in the form $U(z) = U(x - ct) = u(x, t)$ leads to

$$F(U, -cU_z, U_z, U_{zz}) = G(U, U_z, U_{zz}) = 0.$$

Viscous Burgers equation

$$u_t + uu_x - \nu u_{xx} = 0, \qquad \nu > 0$$

with $z = x - ct$ transforms to

$$-cU_z + UU_z - \nu U_{zz} = 0.$$

Solving, we obtain travelling wave solutions

$$\begin{aligned} u(x,t) &= \frac{u_{-\infty} + u_{\infty} e^{K(x-ct)}}{1 + e^{K(x-ct)}} \\ &= \frac{1}{2}(u_{\infty} + u_{-\infty}) + \frac{1}{2}(u_{\infty} - u_{-\infty}) \tanh \frac{K}{2}(x - ct), \end{aligned}$$

where the speed of the wave is

$$c = \frac{1}{2}(u_{\infty} + u_{-\infty}); \quad K = \frac{1}{2\nu}(u_{\infty} - u_{-\infty}) > 0.$$

Korteweg-deVries equation – solitons

I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped – not so the mass of water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, which continued its course along the channel apparently without change of form or diminution of speed. I followed it on horseback, and overtook it still rolling on at a rate of some eight or nine miles an hour, preserving its original figure some thirty feet long and a foot in height.

$$u_t + uu_x + ku_{xxx} = 0.$$

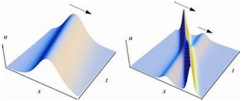
where $k > 0$ is a constant. Travelling wave ansatz gives

$$-cU_z + UU_z + kU_{zzz} = 0.$$

Solving, we obtain the solitary wave

$$u(x,t) = 3c\,\mathrm{sech}^2\left(\sqrt{\frac{c}{4k}}(x-ct)\right) = 3c\left(1 - \tanh^2\left(\sqrt{\frac{c}{4k}}(x-ct)\right)\right),$$

where c is given by tree times the amplitude of the wave.



Dynamical systems approach

As an example, we consider the Fisher equation

$$u_t - u_{xx} = u(1-u) \tag{113}$$

and, as before, we shall look for solutions of the form

$$u(x,t) = U(z), \qquad z = x - ct. \tag{114}$$

Substituting (114) into (113) yields a second order ordinary differential equation for U:

$$-cU_z - U_{zz} = U(1-U). \tag{115}$$

This equation cannot be solved in a closed form for arbitrary c.

We use a phase plane analysis. Defining $V = U'$, we obtain the first order system

$$\begin{aligned} U_z &= V, \\ V_z &= -cV - U(1-U). \end{aligned} \tag{116}$$

Remember: U is a wave profile if it solves (116) and

$$\lim_{z\to-\infty} U(z) = u_{-\infty}, \qquad \lim_{z\to\infty} U(z) = u_{+\infty}, \tag{117}$$

for some $u_{\pm\infty}$. Then, since $V = U_z$,

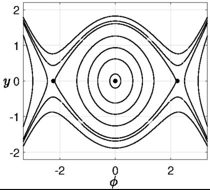
$$\lim_{z\to-\infty} V(z) = 0, \qquad \lim_{z\to\infty} V(z) = 0. \tag{118}$$

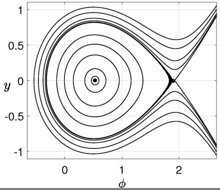
Crucial observation

Points $(u_{-\infty}, 0)$ and $(u_{\infty}, 0)$ satisfying (117) and (118) must be equilibria of (116).

Hence,

The existence of a travelling wave solution to (113) is equivalent to the existence of an orbit of (116) joining the equilibria. A wave front corresponds to a heteroclinic orbit (joining different equilibria), whereas a pulse corresponds to a homoclinic orbit.





The equilibria of (116) are $(0,0)$ and $(0,1)$. The eigenvalues of the Jacobi matrix at $(0,0)$ are

$$\lambda_{\pm}^{0,0} = \frac{-c \pm \sqrt{c^2-4}}{2},$$

and at $(1,0)$,

$$\lambda_{\pm}^{1,0} = \frac{-c \pm \sqrt{c^2+4}}{2}.$$

- For any c, $\lambda_{\pm}^{1,0}$ are real and of opposite sign and therefore $(1,0)$ is a saddle.
- $\lambda_{\pm}^{0,0}$ are both real and negative if $c \geq 2$ and in this case $(0,0)$ is a stable node, while for $0 < c < 2$ it is a stable focus.

First conclusion.

- No travelling waves for $0 < c < 2$.

Consider the triangle Ω, whose sides are defined by

Side I. $U = 1, \quad V < 0;$

Side II. $0 < U < 1, \quad V = 0;$

Side III. $V = -\alpha U, \quad 0 < U < 1, \;\; \alpha \in \mathbb{R}_+.$

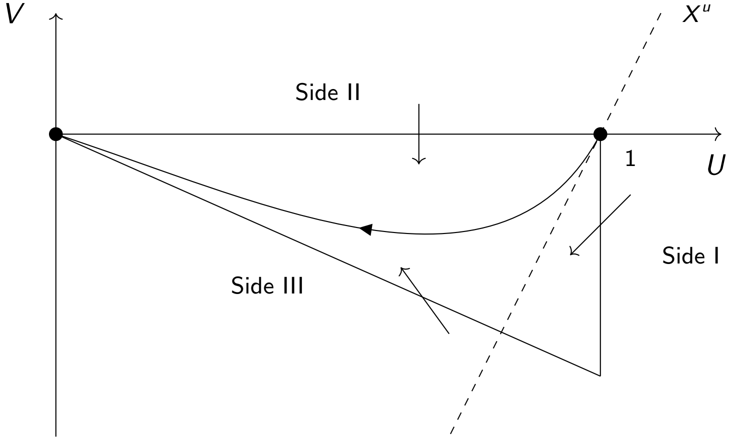


Figure: Invariant set for (116) with the heteroclinic orbit.

The equilibrium $(1,0)$ is hyperbolic independently of c. Hence there exist stable and unstable manifolds at this point. For the eigenvalues

$$\lambda_{\pm}^{1,0} = \frac{-c \pm \sqrt{c^2+4}}{2},$$

consider the corresponding eigenvectors $\boldsymbol{v}_{\pm}$,

$$\begin{pmatrix} -\lambda_{\pm}^{1,0} & 1 \\ 1 & -c-\lambda_{\pm}^{1,0} \end{pmatrix} \begin{pmatrix} v_{\pm}^{1} \\ v_{\pm}^{2} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

that is,

$$\boldsymbol{v}_{-} = \left(\frac{1}{\lambda_{-}^{0,1}}, 1\right), \quad \boldsymbol{v}_{+} = \left(\frac{1}{\lambda_{+}^{0,1}}, 1\right).$$

The trajectory emanating from $(1,0)$ is tangent to $\boldsymbol{v}_+$ at $(1,0)$.

We shall show that this trajectory is the heteroclinic trajectory joining $(1,0)$ with $(0,0)$.

Consider the vector field $\boldsymbol{f}$ of (116) on each side of Δ. We have

Side I. $U' = V < 0$ and $V' = -cU' < 0$, thus the vector field points invards;

Side II. $U' = 0$ and $V' = -U(1-U) < 0$, thus again the vector field points inwards;

Side III. Consider the dot product of $\boldsymbol{f}$ and the normal $(\alpha, 1)$ to Side III. We have

$$(\alpha, 1)\cdot(V, -cV - U(1-U)) = \alpha V - cV - U(1-U).$$

On Side III, $V = -\alpha\, U$, so

$$-\alpha^2 U + \alpha c U - U(1-U) = -U\left(\alpha^2 - c\alpha + 1 - U\right).$$

This product should be nonnegative, that is, we should have $\alpha^2 - c\alpha + 1 - U \leq 0$. By assumption, $0 < U < 1$, hence

$$\alpha^2 - c\alpha + 1 - U < \alpha^2 - c\alpha + 1.$$

We need α so that

$$\alpha^2 - c\alpha + 1 \leq 0. \tag{119}$$

It is sufficient that the quadratic polynomial on the left-hand side has two real roots. Thus we require $\Delta = c^2 - 4 \geq 0$. So, for any $c \geq 2$ we can select α for which $\boldsymbol{f}$ does not point outward Ω.

Thus, the trajectory emanating from $(1, 0)$ stays inside Δ. Using the Dulac citerion,

$$\mathrm{div}\boldsymbol{f} = -c < 0,$$

we find that there is no periodic orbit inside Δ. Using the Poincaré–Bendixson theory, the trajectory must enter $(0, 0)$.

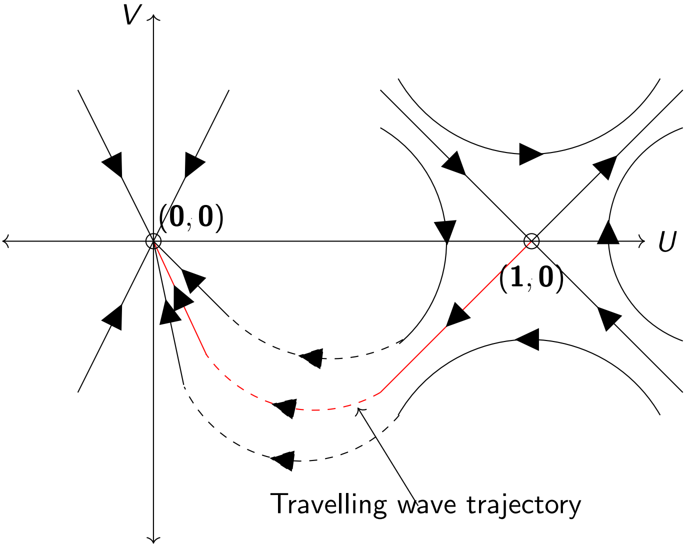


Figure: Trajectories of (116) for $c \geq 2$.

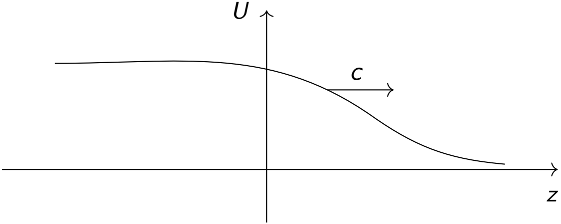


Figure: Travelling wave solution of the Fisher equation (116).

Explicit formula for a travelling wave for the Fisher equation

(W. Malfliet, W. Hereman, 1996)

$$u_t = u_{xx} + u(1-u). \tag{120}$$

For $u(x,t) = U(z) = U(k(x-ct))$ we obtain

$$-kcU_z = k^2 U_{zz} + U(1-U),$$

and we look for a solution in the form

$$U(z) = \sum_{i=0}^{N} a_i \tanh^i z;$$

here, $k, c, N, a_i, i = 0, \dots, N$, are to be determined.

Crucial property:

$$\frac{d}{dz}\tanh z = 1 - \tanh^2 z.$$

$$U_z(z) = \sum_{i=1}^{N} a_i i(1-\tanh^2 z)\tanh^{i-1} z = \sum_{i=0}^{N+1} \hat{a}_i \tanh^i z,$$

for some coefficients $\hat{a}_0, \hat{a}_1, \hat{a}_2$. Similarly,

$$U_{zz}(z) = \sum_{i=0}^{N+2} \tilde{a}_i \tanh^i z$$

and

$$U^2(z) = \sum_{i=0}^{2N} \bar{a}_i \tanh^i z.$$

We must have

$$-ck\sum_{i=0}^{N+1}\hat{a}_i Z^i = k^2\sum_{i=0}^{N+2}\tilde{a}_i Z^i + \sum_{i=0}^{N} a_i Z^i - \sum_{i=0}^{2N}\bar{a}_i Z^i, \qquad (121)$$

where $Z = \tanh z$.

Finding N - balancing the highest powers.

$$2N = N + 2 \quad \Rightarrow N = 2.$$

Hence, we postulate

$$U(z) = \sum_{i=0}^{2} a_i \tanh^i z.$$

Beyond the 'tanh' expansion.

The success of the method of expanding the solution in a series of functions $\{f_i(z)\}_{i\in\mathbb{N}_0} = \{\tanh^i z\}_{i\in\mathbb{N}_0}$ hinges upon the following facts:

1. The equation we solve

$$F(u, u_t, u_x, u_{tt}, u_{tx}, u_{xx}, \ldots) = 0, \tag{122}$$

does not depend on the independent variables and has polynomial nonlinearities.

2. In the postulated expansion of the solution in $z = k(x - ct)$ to

$$F(U, -ckU_z, kU_z, (ck)^2 U_{zz}, -ck^2 U_{zz}, k^2 U_{zz}, \dots) = 0,$$

given by

$$U(z) = \sum_{i=1}^{m} a_i f_i, \tag{123}$$

where $a_i = a_1, a_2, \cdots, a_m$, m, are constants to be determined, the family $\mathcal{F} := \{f_i\}_{i \in \mathbb{N}_0}$ satisfies

A1. $\mathcal{F}$ is linearly independent set.

A2. For any $f_i, f_j \in \mathcal{F}$, the product $f_i f_j \in \mathcal{F}$.

A3. For any $f_j \in \mathcal{F}$, $\frac{d}{dz} f_j \in \mathcal{L}in\mathcal{F}$.

Here $\mathcal{L}in\mathcal{F}$ denotes the set of all (finite) linear combinations of elements of $\mathcal{F}$.

Example 1. (Rodrigo & Mimura (2001)) For a given function differentiable function $f(z)$, consider

$$\mathcal{F} = \{1, f, f^2, \cdots\}, \tag{124}$$

so that the expansion is given by

$$U(z) = \sum_{i=0}^{N} a_i f^i(z). \tag{125}$$

Assumptions **A**1 and **A**2 are satisfied. For **A**3, take $i \geq 1$ so that

$$f_z^i = if^{i-1} f_z.$$

Hence, for $f_z^i \in \mathcal{L}in\mathcal{F}$, f must solve the differential equation

$$f_z = P(f), \tag{126}$$

where P is a polynomial in f.

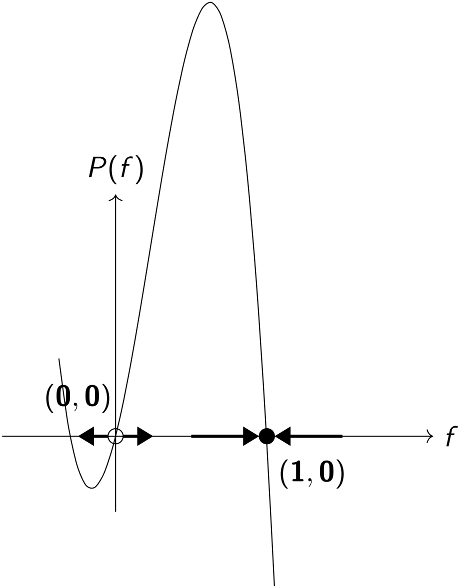


Figure: The graph of P.

For U given by (125) to have limits $u_{\pm\infty}$ as $z \to \pm\infty$, the solution f to (126) must have the same property:

$$\lim_{z\to\pm\infty} f(z) = f_{\pm\infty}.$$

Hence P must have at least two real roots and we can consider the solution f lying between two successive ones.

The most often used choice is a quadratic P, say,

$$f_z = \gamma(f-\beta)(f-\alpha), \tag{127}$$

with $\alpha < f < \beta$. Using $\phi = \frac{f-\alpha}{\beta-\alpha}$ and $\zeta = \frac{\gamma}{\beta-\alpha}z$, we obtain

$$\phi_\zeta = \phi(\phi-1), \tag{128}$$

with the solution

$$\phi(\zeta) = \frac{1}{1 + Ce^{\zeta}} \tag{129}$$

and we can consider the solution with $C = 1$ (as C introduces just a translation in ζ (and hence z, and x).

Then we observe that we can take $\frac{\gamma}{\beta-\alpha} = 1$ (so $z = \zeta$) as this coefficient plays the role of k (depending on the sign, it can reverse the direction of the wave.)

Finally, we observe that the transformation $\phi = \frac{1}{2}(\psi + 1)$ converts (128) into

$$\psi_\zeta = -\frac{1}{2}(1 - \psi^2) \tag{130}$$

whose solution is

$$\psi(\zeta) = \tanh\left(-\frac{\zeta}{2} + C'\right).$$

Hence, setting for convenience $C = 1$ and $C' = 0$, we obtain

$$\phi(\zeta) = \frac{1}{2} - \frac{1}{2}\tanh\left(\frac{\zeta}{2}\right). \tag{131}$$

Hence, any expansion in f solving (127) can be reduced to the expansion in tanh or ϕ. As we shall see, the latter is simpler.

Consider again the transformed Fisher equation

$$-kcU_z = k^2 U_{zz} + U(1-U),$$

with $k > 0, c > 0$.

Simplifying observation: if we look for

$$U(z) = b_0 + b_1\phi(z) + b_2\phi^2(z)$$

and we know that

$$\lim_{z\to\infty} U(z) = 0, \quad \lim_{z\to-\infty} U(z) = 1$$

and compare with (128), we obtain

$$0 = b_0,$$

$$1 = b_1 + b_2,$$

so that

$$U(z) = b_1\phi(z) + (1 - b_1)\phi^2(z)$$

and we obtain

$$\phi^4 : 6k^2 = 1 - b_1, \tag{132a}$$

$$\phi^3 : -2b_1(1 - b_1) + k^2(12b_1 - 10) + 2kc(1 - b_1) = 0, \tag{132b}$$

$$\phi^2 : -b_1^2 + k^2(4 - 5b_1) + kc(3b_1 - 2) + 1 - b_1 = 0, \tag{132c}$$

$$\phi : k^2 b_1 - kcb_1 + b_1 = 0. \tag{132d}$$

From (132d) we immediately obtain a solution $b_1 = 0$, which gives $b_2 = 1$, $c = \pm 5/\sqrt{6}$, with

$$k = \pm 1/\sqrt{6}. \tag{133}$$

Note. (132d) has solutions for $b_1 \neq 0$. However, adding together all equations to eliminate terms with kc, we find the equation

$$-2b_1 + b_1^2 + 8k^2 b_1 - 6k^2 + 1 = 0$$

from which, upon (132a), we obtain $b_1 = 1$, which is impossible as $k \neq 0$.

$$U(z) = a_0 + a_1 \tanh z + a_2 \tanh^2 z = b_0 + b_1 \phi(2z) + b_2 \phi^2(2z)$$
$$= \phi^2(2z) = \frac{1}{4}(1 - \tanh z)^2 = \frac{1}{4} - \frac{1}{2}\tanh z + \frac{1}{4}\tanh^2 z.$$

Travelling wave in SIR system. Consider (37). Let S_0 be the original number of susceptible foxes and

$$\overline{S} = \frac{S}{S_0}, \quad \overline{I} = \frac{I}{S_0}, \quad \overline{R} = \frac{R}{S_0}, \quad \overline{t} = rtS_0, \quad \overline{x} = x\sqrt{\frac{rS_0}{D}}, \quad m = \frac{a}{rS_0}.$$

We have $\overline{S}(\overline{x},\overline{t}) \in (0,1], \overline{I}(\overline{x},\overline{t}) \in [0,1]$, $\overline{R}(\overline{x},\overline{t}) \in [0,1)$ and (37) takes the form

$$\begin{aligned} \overline{S}_{\overline{t}} &= -\overline{IS} \\ \overline{I}_{\overline{t}} &= \overline{IS} - m\overline{I} + \frac{\partial^2 \overline{I}}{\partial \overline{x}^2}, \\ \overline{R}_{\overline{t}} &= m\overline{I}, \end{aligned} \tag{134}$$

where $\overline{t} > 0$, $S_0 > 0$, $m = \dfrac{a}{rS_0} > 0$.

We can drop the last equation for $\overline{R}$ and the bars in (134). Finally, we will consider

$$\begin{aligned} S_t &= -IS, \\ I_t &= IS - mI + \frac{\partial^2 I}{\partial x^2}, \end{aligned} \tag{135}$$

where $t > 0$, $S(x,t) \in (0,1]$, $I(x,t) \in [0,1]$, $S_0 > 0$, $m = \dfrac{a}{rS_0} > 0$. As before, a travelling wave is the solution of (135) of the form

$$S(x,t) = S(x-ct) =: S(z), \quad I(x,t) = I(x-ct) =: I(z),$$

where $c > 0$, $z \in \mathbb{R}$, with boundary conditions

$$S'(-\infty) = 0, \quad I(-\infty) = 0, \quad S(+\infty) = 1, \quad I(+\infty) = 0. \tag{136}$$

Observe

$$S_t(z) = -cS'(z), \quad I_t(z) = -cI'(z), \quad I_x = I'(z), \quad I_{zz} = I''(z).$$

Hence, (135) can be written as

$$\begin{aligned} cS' &= IS, \\ I'' + cI' + IS - mI &= 0, \end{aligned} \qquad (137)$$

so

$$\begin{aligned} cS' &= IS \\ I'' + cI' + cS' - mc\left(\frac{S'}{S}\right) &= 0. \end{aligned}$$

Finally,

$$cS' = IS$$
$$I' + cI + cS - mc\ln S = C, \quad C \in \mathbb{R} \tag{138}$$

where we used $S > 0$.

Using the the boundary conditions (136), we see that the second equation of (138) as $z \to +\infty$ takes the form

$$c - mc\ln 1 = C,$$

hence $C = c$.

Using now (136) at $-\infty$, the second equation of (138) takes the form

$$cS(-\infty) - mc \ln S(-\infty) = c,$$

thus, dividing by c, we get

$$m = \frac{S(-\infty) - 1}{\ln S(-\infty)}.$$

Denote $S(-\infty) =: b \in (0,1)$ and introduce additional assumptions: the death rate of infected foxes is smaller than the infection rate, $a < rS_0$. In other words, there will be more infections than deaths. Thus

$$m = \frac{a}{rS_0} = \frac{b-1}{\ln b} < 1.$$

Finally, we transformed (138) into

$$\begin{aligned} S' &= \frac{1}{c} IS, \\ I' &= -cI - cS + mc \ln S + c, \end{aligned} \tag{139}$$

where $c > 0$, $S(z) \in (0,1]$, $I(z) \in [0,1]$, $m = \dfrac{b-1}{\ln b} \in (0,1)$, $b \in (0,1)$.

We see that (139) has two equilibria: $(b, 0)$ i $(1, 0)$.

Let $\boldsymbol{f}$ be the vector field of (139):

$$\boldsymbol{f}(S, I) = (f_1(S, I), f_2(S, I)) = \left(\frac{1}{c} IS, -cI - cS + mc \ln S + c \right).$$

Then, the Jacobi matrix of $\boldsymbol{f}$ is of the form

$$\mathcal{J}_{\boldsymbol{f}}(S, I) = \begin{pmatrix} \dfrac{I}{c} & \dfrac{S}{c} \\ -c + \frac{mc}{S} & -c \end{pmatrix}.$$

- The equilibrium $(b,0)$ of (139) is a saddle. Indeed,

$$\mathcal{J}_{\boldsymbol{f}}(b,0) = \begin{bmatrix} 0 & \dfrac{b}{c} \\ -c+\frac{mc}{b} & -c \end{bmatrix},$$

and the eigenvalues of $\mathcal{J}_{\boldsymbol{f}}(b,0)$ are the roots of

$$\det\begin{pmatrix} 0-\lambda & \dfrac{b}{c} \\ -c+\frac{mc}{b} & -c-\lambda \end{pmatrix} = 0,$$

that is,

$$\lambda^2 + c\lambda + b - m = 0.$$

The discriminant is $\Delta = c^2 - 4(b-m)$. Since $b \in (0,1)$ and

$$b - m = b - \frac{b-1}{\ln b} = \frac{b\ln b - b + 1}{\ln b} =: h(b),$$

thus, graphing h, we get $b - m < 0$.

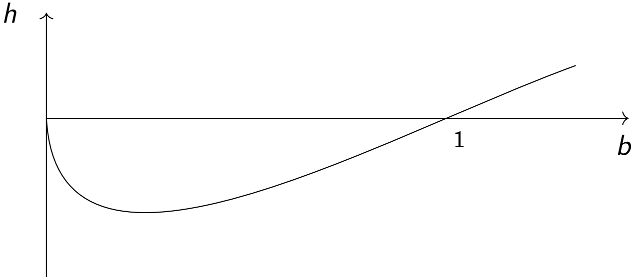


Figure: Graph of h

Thus, $\Delta > 0$ and Viete's formulae give

$$\lambda_1 \lambda_2 = \frac{b-m}{1} = b - m < 0.$$

Hence, there is a saddle at $(b, 0)$.

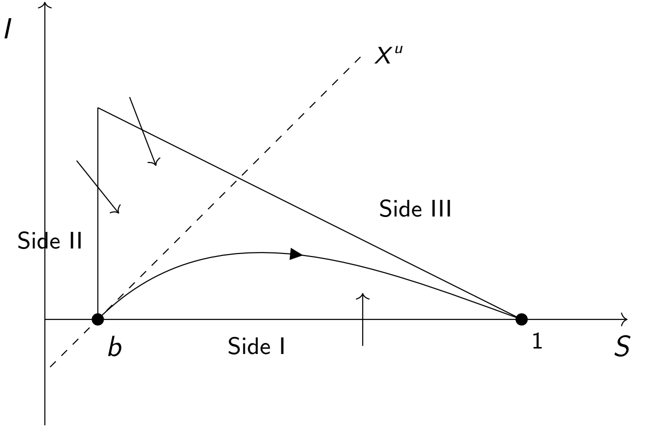
In a similar way we show that $(1, 0)$ is an asymptotically stable focus if $0 < c < 2\sqrt{1-m}$ and an asymptotically stable node if $c \geq 2\sqrt{1-m}$.

Consider the triangle Δ defined by

Side I. $I = 0$, $S \in (b, 1)$;

Side II. $I \in (0, \alpha(1-b))$, $S = b$, $\alpha \in \mathbb{R}_+$;

Side III. $I = \alpha(1-S)$, $S \in (b, 1)$, $\alpha \in \mathbb{R}_+$.



First, we show that there is no periodic orbit in Δ. Define $\rho(S, I) = \frac{1}{IS}$ and evaluate

$$\begin{aligned}\mathrm{div}(\rho f) &= \frac{\partial(\rho f_1)}{\partial S} + \frac{\partial(\rho f_2)}{\partial I} \\ &= 0 + \frac{c}{I^2} - \frac{mc \ln S}{I^2 S} - \frac{c}{I^2 S} \\ &= \frac{c}{I^2}\left(1 - \frac{m \ln S + 1}{S}\right).\end{aligned}$$

Since for $S > b$, by $m = \dfrac{b-1}{\ln b} \in (0, 1)$, we have $m \ln S + 1 > S$, $1 - \frac{m \ln S + 1}{S} < 0$, and, by $c > 0$,

$$\mathrm{div}(\rho f) < 0.$$

Thus, by Dulac's criterion, there is no periodic orbit in Δ.

Thus, there is a heteroclinic orbit joining $(b,0)$ i $(1,0)$ which is not oscillatory at $(1,0)$ if and only if $c \geq 2\sqrt{1-m}$.

Thus, model (135) has travelling wave solutions $(S(z), I(z))$ if and only if $c \geq 2\sqrt{1-m}$, where $S(z)$ is a wave front, and $I(z)$ jest a soliton.

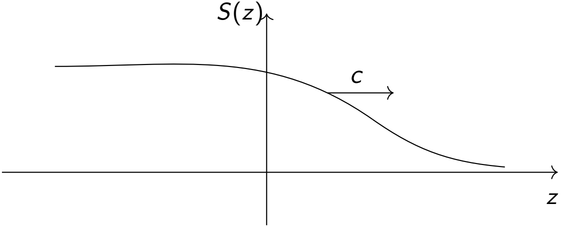


Figure: Front wave solution S.

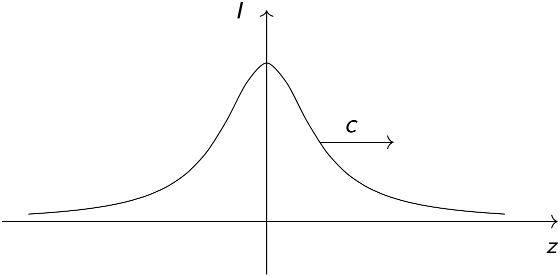


Figure: Soliton solution I.

Similarity method. This method can be applied to equations of arbitrary order. Here, we focus on second order partial differential equation in two independent variables

$$G(t, x, u, u_x, u_t, u_{xx}, u_{xt}, u_{tt}) = 0. \tag{140}$$

To shorten notation, we introduce

$$p = u_x, \quad q = u_t, \quad r = u_{xx}, \quad s = u_{xt}, \quad v = u_{tt}$$

and a family of stretching transformations, T_ϵ, by

$$\bar{x} = \epsilon^a x, \quad \bar{t} = \epsilon^b t, \quad \bar{u} = \epsilon^c u, \tag{141}$$

where a, b, c are reals and $\epsilon \in I = (1-\delta, 1+\delta)$.

T_ϵ induces a transformation of the derivatives as follows

$$\bar{p} = \frac{\partial \bar{u}}{\partial \bar{x}} = \epsilon^c \frac{\partial u}{\partial x}\frac{dx}{d\bar{x}} = \epsilon^{c-a} p, \tag{142}$$

and similarly for other derivatives,

$$\bar{q} = \epsilon^{c-b} q, \quad \bar{r} = \epsilon^{c-2a} r, \quad \bar{s} = \epsilon^{c-a-b} s, \quad \bar{v} = \epsilon^{c-2b} v. \tag{143}$$

We say that PDE (140) is invariant under the one parameter family T_ϵ of stretching transformations if there exists a smooth function $f(\epsilon)$ such that

$$G(\bar{t}, \bar{x}, \bar{u}, \bar{p}, \bar{q}, \bar{r}, \bar{s}, \bar{v}) = f(\epsilon) G(t, x, u, p, q, r, s, v) \tag{144}$$

for all $\epsilon \in I$, with $f(1) = 1$.

We have

Theorem 7

If the equation (140) is invariant under the family T_ϵ defined by (141), then the transformation

$$u = t^{c/b} y(z), \qquad z = \frac{x}{t^{a/b}} \tag{145}$$

reduces (140) to a second order ordinary differential equation in $y(z)$.

Proof. By invariance, we know that (144) holds for all ϵ in some open interval containing 1 thus we can differentiate (144) and set $\epsilon = 1$ after differentiation getting

$$axG_x + btG_t + cuG_u + (c-a)pG_p + (c-b)qG_q + (c-2a)rG_r$$
$$+ (c-a-b)sG_s + (c-2b)vG_v = f'(1)G,$$

where we used formulae like

$$\left.\frac{d\bar{x}}{d\epsilon}\right|_{\epsilon=1} = a\epsilon^{a-1}x\Big|_{\epsilon=1} = ax,$$

etc. The above equation is a first order equation so that we can integrate it using t as the parameter along characteristics.

The characteristic system will be then

$$\begin{aligned}
\frac{dG}{dt} &= \frac{f'(1)G}{bt}, \quad \frac{dx}{dt} = \frac{ax}{bt},\\
\frac{du}{dt} &= \frac{cu}{bt}, \quad \frac{dp}{dt} = \frac{(c-a)p}{bt},\\
\frac{dq}{dt} &= \frac{(c-b)q}{bt}, \quad \frac{dr}{dt} = \frac{(c-2a)r}{bt},\\
\frac{ds}{dt} &= \frac{(c-a-b)s}{bt}, \quad \frac{dv}{dt} = \frac{(c-2b)v}{bt}.
\end{aligned}$$

Thus, we obtain characteristics defined by

$$\begin{aligned}
xt^{-a/b} &= z, \quad ut^{-c/b} = \xi_1,\\
pt^{-(c-a)/b} &= \xi_2, \quad qt^{-(c-b)/b} = \xi_3,\\
rt^{-(c-2a)/b} &= \xi_4, \quad st^{-(c-a-b)/b} = \xi_5, \quad vt^{-(c-2b)/b} = \xi_6
\end{aligned}$$

and hence

$$G = t^{f'(1)} b\Psi(z, \xi_1, \xi_2, \xi_3, \xi_4, \xi_5, \xi_6), \tag{146}$$

where Ψ is an arbitrary function.

Now, we have $y = ut^{-c/b} = \xi_1$, $p = u_x = t^{c/b} y'_z z' x = y'_z t^{(c-a)/b}$, hence $\xi_2 = y'_z$. Further,

$$q = u_t = \frac{c}{b} t^{-1+c/b} y - \frac{a}{b} t^{c/b-a/b-1} x y'_z,$$

thus

$$\xi_3 = q t^{1-c/b} = \frac{c}{b} y - \frac{a}{b} z y'_z.$$

Similar calculations lead to

$$\Psi\left(z, y, y'_z, \frac{c}{b}y - \frac{a}{b}zy'_z, y''_{zz}, \frac{c-a}{b}y'_z - \frac{a}{b}y''_{zz}z,\right.$$
$$\left.\frac{c}{b}\left(\frac{c}{b}-1\right)y - \frac{a}{b}\left(2\frac{c}{b}-1-\frac{a}{b}\right)zy'_z + \frac{a^2}{b^2}z^2y''_{zz}\right) = 0,$$

which is a second order ordinary differential equation in z. ■

The diffusion equation. Consider the diffusion equation

$$u_t - Du_{xx}. \tag{147}$$

To find a stretching transformation under which this equation is invariant, with the simplified notation for derivatives, we have

$$\bar{q} - D\bar{r} = \epsilon^{c-b} q - D\epsilon^{c-2a} r.$$

Thus, we must have

$$b = 2a$$

with c and a at this moment arbitrary. Thus, (147) is invariant under the stretching transformation

$$\bar{x} = \epsilon^a x, \quad \bar{t} = \epsilon^{2a} t, \quad \bar{u} = \epsilon^c u. \tag{148}$$

Hence, the similarity transformation is given by

$$u = t^{c/2a} y(z), \quad z = \frac{x}{\sqrt{t}}. \tag{149}$$

We have $z'_x = \frac{1}{\sqrt{t}}$, $z'_t = -\frac{1}{2}xt^{-3/2} = -\frac{1}{2}zt^{-1}$, hence

$$u_t = -\frac{c}{2a}t^{-1+c/2a}y + t^{c/2a}y'_z z'_t = -t^{-1+c/2a}\left(\frac{c}{2a}y - \frac{z}{2}y'_z\right)$$

and

$$u_x = t^{c/2a-1/2}y'_z, \qquad u_{xx} = t^{c/2a-1}y''_{zz}.$$

Substituting the above relations into the diffusion equation yields

$$Dy''_{zz} + \frac{z}{2}y'_z - \frac{c}{2a}y = 0. \tag{150}$$

Constants c and a are in general arbitrary.

Though the diffusion equation has been reduced to an ordinary differential equation, the similarity approach by no means solves all diffusion problems, which involve initial and boundary conditions that, in general, cannot be translated into side conditions for (150). For instance, consider the initial value problem for the diffusion equation

$$\begin{aligned} u_t = Du_{xx}, \quad t > 0, -\infty < x < \infty, \\ u(0,x) = u_0(x). \end{aligned} \tag{151}$$

We convert the equation into an ODE for y defined as

$$u(t,x) = t^{c/2a} y(x/\sqrt{t}),$$

but then putting $t = 0$ in the preceding formula in general does not make any sense as, at best, we would have something like

$$\begin{aligned} y(\infty) &= \lim_{t\to 0^+} t^{-c/2a} u(t,x), \quad x > 0, \\ y(-\infty) &= \lim_{t\to 0^+} t^{-c/2a} u(t,x), \quad x < 0, \end{aligned} \tag{152}$$

with the right hand side equal to 0 if $c/2a < 0$, ∞ if $c/2a > 0$ or $u_0(x)$ if $c = 0$. In the first two cases all the information coming from the initial condition is lost and the last one imposes a strict condition on u_0: u_0 must be constant on each semi-axis.

Note that such a condition is also invariant under the transformation $z = x/\sqrt{t}$: $z \lesseqgtr 0$ if and only if $x \lesseqgtr 0$.

In general, the similarity method provides a full solution to the initial-boundary value problems only if the side conditions are also invariant under the same transformation or, in other words, can be expressed in terms of the similarity variable.

Hence, consider the initial value problem

$$u_t = Du_{xx}, \quad t > 0, -\infty < x < \infty,$$
$$u(0, x) = H(x),$$

where H is the Heaviside function: $H(x) = 1$ for $x \geq 0$ and $H(x) = 0$ for $x < 0$. According to the discussion above, this initial condition yields to the similarity method provided $c = 0$; in this case a is irrelevant and we put it equal to 1.

Thus, the initial value problem (153) is transformed into

$$y'' + \frac{z}{2D}y' = 0,$$
$$y(-\infty) = 0, \quad y(\infty) = 1.$$

Denoting $y' = h$, we reduce the equation to the first order equation

$$h' + \frac{z}{2D}h = 0,$$

yielding $y' = h = c_1 \exp(-z^2/4D)$, which gives

$$y(z) = c_1 \int\limits_0^z e^{-\frac{\eta^2}{4D}}\, d\eta + c_2. \tag{153}$$

The constants c_1 and c_2 can be obtained from the initial conditions

$$\begin{aligned}
1 &= \lim_{z\to+\infty} y(z) = c_1 \int_0^{\infty} e^{-\frac{\eta^2}{4D}} d\eta + c_2 = \sqrt{4D} c_1 \int_0^{\infty} e^{-s^2} ds + c_2 \\
&= c_1 \frac{\sqrt{4D\pi}}{2} + c_2, \\
0 &= \lim_{z\to-\infty} y(z) = c_1 \int_0^{-\infty} e^{-\frac{\eta^2}{4D}} d\eta + c_2 = \sqrt{4D} c_1 \int_0^{-\infty} e^{-s^2} ds + c_2 \\
&= -c_1 \frac{\sqrt{4D\pi}}{2} + c_2,
\end{aligned}$$

where we used $\int_0^{\infty} e^{-s^2} ds = \sqrt{\pi}/2$, so that

$$c_2 = \frac{1}{2}, \quad c_1 = \frac{1}{\sqrt{4\pi D}}.$$

Hence

$$u(t,x) = y\left(\frac{x}{\sqrt{t}}\right) = \frac{1}{2} + \frac{1}{\sqrt{4\pi D}}\int\limits_0^{\frac{x}{\sqrt{t}}} e^{-\frac{\eta^2}{4D}}\,d\eta \tag{154}$$

The fundamental role in the theory of the diffusion equation is played by the derivative of u with respect to x:

$$S(t,x) = \frac{\partial u}{\partial x}(t,x) = \frac{1}{\sqrt{4\pi Dt}}e^{-\frac{x^2}{4Dt}} \tag{155}$$

that is called source function or fundamental solution of the diffusion equation.

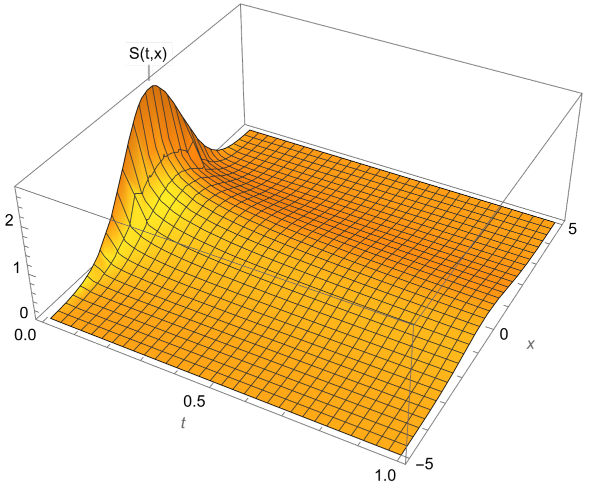


Figure: The graph of S with $D = 1$.

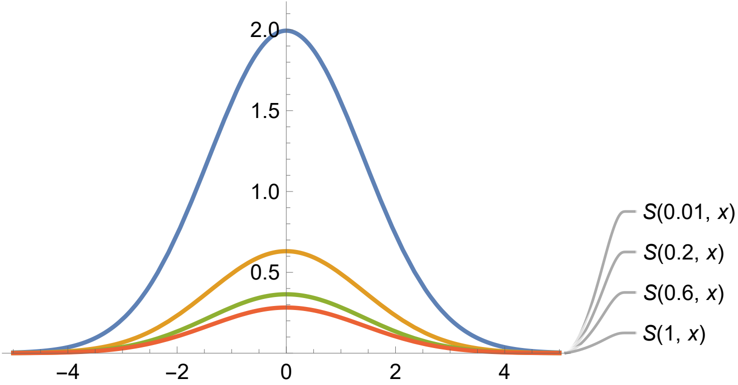


Figure: Snapshots of S.

One can check that $S(t,x)$ has the following properties:

$$\lim_{t\to 0^+} S(t,x) = 0, \quad x \neq 0, \tag{156}$$

$$\lim_{t\to 0^+} S(t,x) = \infty, \quad x = 0, \tag{157}$$

$$\lim_{x\to \pm\infty} S(t,x) = 0, \quad x > 0, \tag{158}$$

$$\int_{-\infty}^{\infty} S(t,x)dx = 1. \quad t > 0 \tag{159}$$

Mathematically, properties (156), (157) and (159) state

$$\lim_{t\to 0^+} S(t,x) = \delta(x),$$

where $\delta(x)$ is Dirac's delta concentrated at $x = 0$. It is a measure, or functional over the space of continuous functions defined by

$$\int_{-\infty}^{\infty} \phi(x)\delta(x)dx = \phi(0).$$

Stretching the idea of the principle of superposition, we define

$$u(t,x) = \int\limits_{-\infty}^{\infty} S(t, x-\xi)u_0(\xi)d\xi = \frac{1}{\sqrt{4\pi Dt}} \int\limits_{-\infty}^{\infty} e^{-\frac{(x-\xi)^2}{4Dt}} u_0(\xi)d\xi. \tag{160}$$

We can prove that $u(t,x)$ satisfies, at least formally,

$$\lim_{t\to 0^+} u(t,x) = \int_{-\infty}^{\infty} \delta(x-\xi)u_0(\xi)d\xi = u_0(\xi), \tag{161}$$

and thus $u(t,x)$ is a unique solution to

$$\begin{aligned} u_t &= u_{xx}, \quad t>0, -\infty<x<\infty, \\ u(0,x) &= u_0(x), \quad -\infty<x<\infty. \end{aligned} \tag{162}$$

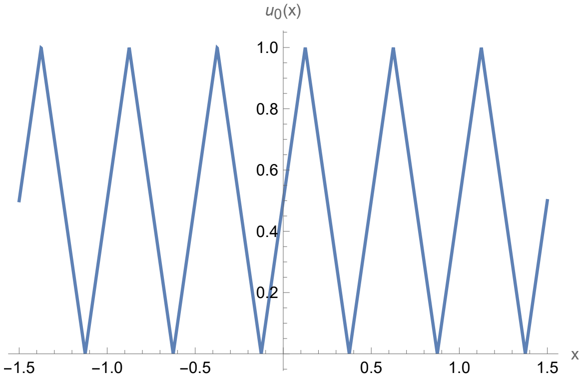
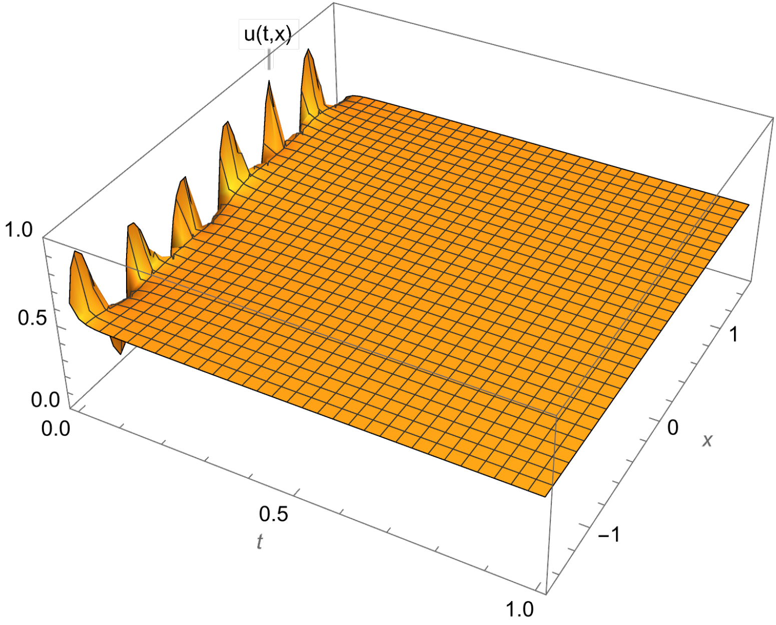


Figure: Irregular initial condition $u_0(x)$.


u(t,x)
1.0
0.5
0.0
0.0
0.5
t
1.0
1
0
x
−1

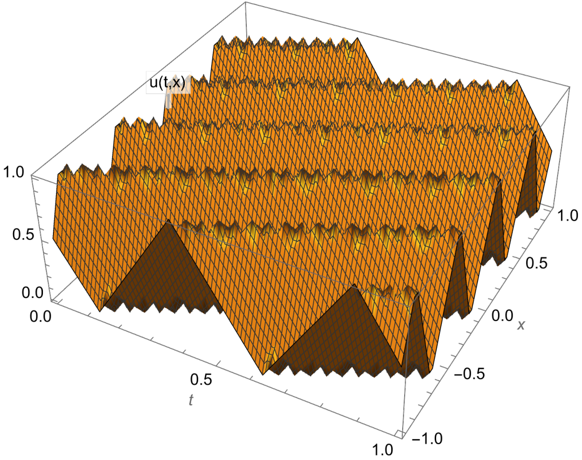


Figure: Comparison of the solution of the solution of the diffusion equation with the wave solution to $u_t + u_x = 0$ with the same initial condition.

Using (161) and since $S(t,x)$ satisfies the diffusion equation, we obtain, again formally,

$$\begin{aligned} u_t &= Du_{xx}, \\ u(x,0) &= \delta(x), \end{aligned}$$

which is the equation for the fundamental solution.

The reason for the importance of the fundamental solution is that it describes diffusion of a unit quantity of the medium concentrated at the origin and thus (160) expresses the superposition principle, that is, the solution determined by a continuously distributed sources is the sum of solutions originating from each source.

We emphasize that the above results hold if u_0 is sufficiently regular (e.g., bounded and continuous) and the proofs of them are not trivial.

We use this formula to derive the solution to inhomogeneous diffusion problems

$$\begin{aligned} u_t &= Du_{xx} + f(t,x), \\ u(0,x) &= u_0(x). \end{aligned} \tag{163}$$

Nonhomogeneous ordinary differential equations. Consider

$$u' = Au + f, \quad u(0) = u_0. \tag{164}$$

Let $u(t) = e^{tA}u_0$ be the solution to

$$\frac{du}{dt} = Au, \quad u(0) = u_0.$$

By the variation of constants formula, the solution to (164) is

$$u(t) = e^{tA}u_0 + e^{tA}\int_0^t e^{-sA}f(s)ds. \tag{165}$$

The first terms solves the homogeneous equation with given u_0 and the second the inhomogeneous equation with zero initial condition.

In analogy, it is convenient to introduce the so-called solution operator $\mathcal{S}(t)$ which acts on initial states producing the solutions to the differential equations, that is, for any given u_0, the function $u(t) = \mathcal{S}(t)u_0$ is the solution to the problem (164). In the ODE case,

$$u(t) = \mathcal{S}(t)u_0 = e^{tA}u_0, \tag{166}$$

and in the case of the diffusion equation,

$$u(t, x) = [\mathcal{S}(t)u_0](x) = \int_{-\infty}^{\infty} S(t, x-\xi)u_0(\xi)d\xi. \tag{167}$$

For the diffusion problem (163) we proceed as follows. We know that $[\mathcal{S}(t)u_0](x)$ solves the initial value problem with correct initial condition. Then, we expect the function

$$v(t,x) = \int_0^t \mathcal{S}(t-s)f(s,x)ds$$

to be the solution to the problem

$$v_t = Dv_{xx} + f(t,x), \quad v(0,x) = 0. \tag{168}$$

In this case the explicit expression of v is as follows:

$$v(t,x) = \int_0^t \left(\frac{1}{\sqrt{4\pi D(t-s)}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4D(t-s)}} f(s,y)dy \right) ds. \tag{169}$$

It follows that our construction is correct. To prove this we shall use the formula for differentiation of integrals:

$$\frac{d}{dt}\int\limits_0^t g(t,s)ds = g(t,t) + \int\limits_0^t g_t(t,s)ds, \qquad (170)$$

which is valid if g and g_t are continuous.

In our case the integrand

$$G(t-s,x) = \frac{1}{\sqrt{4\pi D(t-s)}}\int\limits_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4D(t-s)}} f(s,y)dy$$

unfortunately is not as regular as required in the above formula so that the proof we provide is purely formal.

Let us observe that in order to use Eq. (170) we have to calculate G at $t = s$. This is possible as in (161), and we obtain

$$G(x, 0) = \lim_{t \to s} G(x, t - s) = f(x, t).$$

Also, for any fixed s, $G(x, t - s)$ is the shift of the solution of the homogeneous equation with the initial value given by $f(x, s)$, therefore it is itself a solution of the same diffusion equation. Therefore

$$G_t(x, t - s) = DG_{xx}(x, t - s).$$

Using these two facts we have formally

$$
\begin{aligned}
v_t(x,t) &= G(x,0) + \int_0^t G_t(x,t-s)ds \\
&= f(x,t) + D\int_0^t G_{xx}(x,t-s)ds \\
&= f(x,t) + D\frac{\partial^2}{\partial x^2}\int_0^t G(x,t-s)ds \\
&= Dv_{xx}(x,t) + f(x,t)
\end{aligned}
$$

and v is indeed a solution to the nonhomogeneous diffusion equation.

Next we have

$$v(x,0) = \lim_{t\to 0^+} v(x,t) = \lim_{t\to 0^+} \int\limits_0^t G(x,t-s)ds = 0,$$

where we used the fact that the integrals of a bounded function over intervals with length tending to 0 also tend to zero.

Thus the variation of constants formula (169) for the diffusion equation has been formally justified.

Thus, the full solution is given by

$$
\begin{aligned}
u(t,x) &= \frac{1}{\sqrt{4\pi Dt}} \int\limits_{-\infty}^{\infty} e^{-\frac{(x-\xi)^2}{4Dt}} u_0(\xi) d\xi \\
&+ \int\limits_0^t \left(\frac{1}{\sqrt{4\pi D(t-s)}} \int\limits_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4D(t-s)}} f(s,y) dy \right) ds.
\end{aligned} \tag{171}
$$

Example. Solve the following initial value problem

$$\begin{aligned} u_t &= u_{xx} + xt, \quad \text{for } t > 0, -\infty < x < \infty+, \\ u(0,x) &= 0, \quad \text{for } -\infty < x < \infty. \end{aligned}$$

We see that $f(x,t) = xt$, thus according to the formula (169) we obtain the solution in the form

$$u(t,x) = \int\limits_0^t \left(\frac{1}{\sqrt{4\pi(t-s)}} \int\limits_{-\infty}^{\infty} e^{-(x-y)^2/4(t-s)} ys\, dy \right) ds.$$

Using the substitution $y = 2p\sqrt{t-s} + x$ we obtain

$$\begin{aligned} u(t,x) &= \frac{1}{\sqrt{\pi}} \int\limits_0^t \left(s \int\limits_{-\infty}^{\infty} e^{-p^2} \left(2p\sqrt{t-s} + x\right) dp \right) ds \\ &= \frac{1}{\sqrt{\pi}} \left(\int\limits_0^t 2s\sqrt{t-s} \int\limits_{-\infty}^{\infty} e^{-p^2} p dp + sx \int\limits_{-\infty}^{\infty} e^{-p^2} dp \right) ds = \frac{xt^2}{2}, \end{aligned}$$

where we used that $\int\limits_{-\infty}^{\infty} e^{-p^2} p dp = 0$ (as $e^{-p^2} p$ is an odd function) and $\int\limits_{-\infty}^{\infty} e^{-p^2} dp = \sqrt{\pi}$.

Miscellaneous examples.

Drift–diffusion equation. Consider the drift–diffusion equation

$$u_t = Au + Bu_x + Cu_{xx}.$$

Define v by

$$u(t,x) = e^{ax+bt}v(t,x),$$

where a and b are coefficients to be determined. Differentiating, we obtain

$$\begin{aligned} u_t(t,x) &= be^{ax+bt}v(t,x) + e^{ax+bt}v_t(t,x), \\ u_x(t,x) &= ae^{ax+bt}v(t,x) + e^{ax+bt}v_x(t,x), \\ u_{xx}(t,x) &= a^2e^{ax+bt}v(t,x) + 2ae^{ax+bt}v_x(t,x) + e^{ax+bt}v_{xx}(t,x). \end{aligned}$$

Inserting the above expressions into the equation, collecting terms and dividing by e^{ax+bt}, we arrive at

$$v_t = (A + Ba + Ca^2 - b)v + (B + 2Ca)v_x + Cv_{xx}.$$

From the above equation we see that taking

$$a = -\frac{B}{2C}, \qquad b = A - \frac{B^2}{4C}$$

will make the coefficients multiplying v and v_x equal to zero, so that v will be the solution to

$$v_t = Cv_{xx}.$$

Boundary conditions. Consider

$$u_t = u_{xx}, \quad \text{for } t > 0, x > 0,$$
$$u(0,x) = \phi(x), \quad \text{for } x > 0, \tag{172}$$
$$u(t,0) = h(t), \quad \text{for } t > 0.$$

Firstly, we convert this problem to the one with zero boundary data. Define

$$U(t,x) = u(t,x) - h(t). \tag{173}$$

Then

$$U_t = U_{xx} - h_t(t), \quad \text{for } t > 0, x > 0,$$
$$U(0,x) = \phi(x) - h(0), \quad \text{for } x > 0, \tag{174}$$
$$U(t,0) = 0, \quad \text{for } t > 0.$$

Next, let us observe that if $\tilde{U}$ solves the initial value problem

$$\begin{aligned} \tilde{U}_t &= \tilde{U}_{xx}, \quad \text{for } t > 0, -\infty < x < \infty, \\ \tilde{U}(0,x) &= \tilde{\phi}(x), \quad \text{for } -\infty < x < \infty, \end{aligned} \tag{175}$$

where $\tilde{\phi}$ is an odd function of x, then $\tilde{U}$ is also an odd function of x. In fact, $v(x,t) = -\tilde{U}(y,t)$, where $y = -x$, satisfies

$$v_t - v_{xx} = -(\tilde{U}_t - \tilde{U}_{yy}) = 0,$$

and

$$v(0,x) = -\tilde{U}(0,y) = -\phi(y) = -\phi(-x) = \phi(x)$$

as also ϕ is odd.

Thus, by the uniqueness, $v(t,x) = \tilde{U}(t,x)$, and hence

$$\tilde{U}(t,x) = v(t,x) = -\tilde{U}(t,-x).$$

By continuity,

$$\tilde{U}(t,0) = 0, \quad t > 0.$$

Example. Solve

$$
\begin{aligned}
u_t &= u_{xx}, \quad \text{for } t > 0, x > 0,\\
u(0,x) &= \phi(x), \quad \text{for } x > 0, \\
u(t,0) &= 0, \quad \text{for } t > 0.
\end{aligned}
\tag{176}
$$

Define

$$
\tilde{\phi}(x) = \begin{cases} \phi(x) & \text{for} \quad x > 0 \\ -\phi(-x) & \text{for} \quad x < 0 \end{cases}
$$

and solve

$$
\begin{aligned}
U_t &= U_{xx}, \quad \text{for } t > 0, x \in \mathbb{R},\\
U(x,0) &= \tilde{\phi}(x), \quad \text{for } x \in \mathbb{R}.
\end{aligned}
\tag{177}
$$

We have

$$U(t,x) = \frac{1}{\sqrt{4\pi Dt}} \int\limits_{-\infty}^{\infty} e^{-\frac{(x-\xi)^2}{4Dt}} \tilde{\phi}(\xi) d\xi$$

$$= \frac{1}{\sqrt{4\pi Dt}} \left(- \int\limits_{-\infty}^{0} e^{-\frac{(x-\xi)^2}{4Dt}} \phi(-\xi) d\xi + \int\limits_{0}^{\infty} e^{-\frac{(x-\xi)^2}{4Dt}} \phi(\xi) d\xi \right)$$

$$= \frac{1}{\sqrt{4\pi Dt}} \int\limits_{0}^{\infty} \left(e^{-\frac{(x-\xi)^2}{4Dt}} - e^{-\frac{(x+\xi)^2}{4Dt}} \right) \phi(\xi) d\xi,$$

and

$$u(t,x) = U(t,x), \quad x > 0.$$

Note that if instead of the Dirichlet problem (172) we are required to solve the Neumann problem

$$
\begin{aligned}
u_t &= u_{xx}, \quad \text{for } t > 0, x > 0,\\
u(x,0) &= \phi(x), \quad \text{for } x > 0,\\
u_x(0,t) &= h(t), \quad \text{for } t > 0.
\end{aligned}
$$

can be solved in a similar way with $U(x,t) = u(x,t) - xh(t)$ and an even extension of the data.

Nonlinear problems. Consider

$$u_t = Du_{xx} + \mu u(1-u), \quad t > 0, -\infty < x < \infty,$$
$$u(0,x) = \mathring{u}(x), \quad -\infty < x < \infty. \tag{178}$$

Using (171),

$$u(t,x) = [\mathcal{S}(t)\mathring{u}](x) + \int_0^t [\mathcal{S}(t-s)u(1-u)](s,x)ds$$
$$= \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4Dt}} \mathring{u}(y)dy$$
$$+ \int_0^t \left(\frac{1}{\sqrt{4\pi D(t-s)}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4D(t-s)}} u(s,y)(1-u(s,y))dy \right) ds.$$

The resulting equation is solved by successive Picard's iterations

$$u_0(t) = \mathcal{S}(t)\mathring{u},$$

$$u_{n+1}(t) = \mathcal{S}(t)\mathring{u} + \int\limits_0^t \mathcal{S}(t-s)u_n(1-u_n)ds.$$

Critical domain size. Consider a population on a one-dimensional domain $[0, l]$ evolving according to the Fisher equation

$$u_t = Du_{xx} + \mu u(1 - u). \tag{179}$$

We impose either the so-called "island" boundary condition

$$u(t, 0) = 0, \quad u(t, l) = 0, \tag{180}$$

or "box" boundary condition

$$u_x(t, 0) = 0, \quad u_x(t, l) = 0. \tag{181}$$

How large should an island or box be to support a population? We reformulate this question by asking for what values of l there exist nontrivial stationary solutions $u(x) \neq 0$, that is, we will seek non-zero solutions to

$$u_{xx} = -\frac{\mu}{D}u(1-u). \tag{182}$$

As in the case of travelling waves, we perform a phase-plane analysis of the equivalent system

$$\begin{aligned} u_x &= v, \\ v_x &= -\frac{\mu}{D}u(1-u), \end{aligned} \tag{183}$$

supplemented with the Dirichlet conditions

$$u(0) = 0, \quad u(l) = 0$$

in the case of an island, or Neumann conditions

$$v(0) = 0, \quad v(l) = 0.$$

The equilibria of (183) are

$$P_1 = (0, 0), \quad P_2 = (1, 0).$$

The Jacobi matrix of (183) is

$$\mathcal{J}(u,v) = \begin{pmatrix} 0 & 1 \\ 2\frac{\mu}{D}u - \frac{\mu}{D} & 0 \end{pmatrix}$$

with the linearizations

$$\mathcal{J}(P_1) = \begin{pmatrix} 0 & 1 \\ -\frac{\mu}{D} & 0 \end{pmatrix}, \qquad \mathcal{J}(P_2) = \begin{pmatrix} 0 & 1 \\ \frac{\mu}{D} & 0 \end{pmatrix}.$$

The eigenvalues at P_1 are $\lambda_{1,2} = \pm i\sqrt{\frac{\mu}{D}}$ and so P_1 a (linear) centre, while at P_2 we get $\lambda_{1,2} = \pm\sqrt{\frac{\mu}{D}}$ and hence P_2 is a saddle.

P_1 is not hyperbolic, so we cannot draw any conclusion about its stability from linearization. However, we can find a first integral (Hamiltonian) of the system. Multiplying the second equation by v and using the first equation, we find

$$\frac{1}{2}(v^2)_x = vv_x = -\frac{\mu}{D}(vu - vu^2) = -\frac{\mu}{D}(u_x u - u_x u^2)$$
$$= -\frac{\mu}{D}\left(\frac{1}{2}(u)^2_x - \frac{1}{3}(u)^3_x\right).$$

Integrating,

$$V(u,v) = \frac{1}{2}v^2 + \frac{\mu}{D}\frac{u^2}{2} - \frac{\mu}{D}\frac{u^3}{3} + C = \frac{1}{2}v^2 + F(u) + C,$$

where C is a constant.

Any trajectory must lay on a level curve of V.

We are interested only in trajectories with $u \geq 0$.

For any $\alpha \in (0, 1]$, we have

$$V(\alpha, 0) = F(\alpha)$$

and hence $(\alpha, 0)$ is a zero of

$$V_\alpha(u, v) = \frac{1}{2}v^2 + F(u) - F(\alpha).$$

Consider the equation

$$v = \pm\sqrt{2(F(\alpha) - F(u))}, \tag{184}$$

which is satisfied at least by $v = 0, u = \alpha$. Now, define

$\phi(u) = F(\alpha) - F(u)$.

We have $\phi(\alpha) = 0$ and

$$\phi'(u) = \frac{\mu}{D} u(u-1),$$

hence ϕ is decreasing from $F(\alpha)$ to zero on $[0,1]$ and is monotonically increasing from $-\infty$ at $u = -\infty$ to $F(\alpha)$ at $u = 0$. Hence, for each α, there a unique point $u_\alpha < 0$ such that $\phi(u_\alpha) = 0$.

This shows that (184) represent two branches of a solution which would be a periodic trajectory if it was smooth. Consider

$$\frac{dv}{du} = \pm \frac{\phi'(u)}{\sqrt{2\phi(u)}}.$$

We see that the branches join smoothly at any α and x_α (that is, where $\phi = 0$) unless also $\phi' = 0$ there. This happens only at $\alpha = 1$ which consistent with the fact that $(\alpha, 0)$ is an equilibrium and the level curve of V_1 consists of a homoclinic trajectory and the equilibrium.

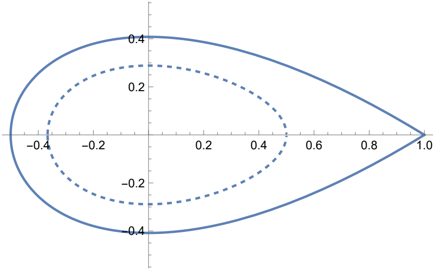


Figure: The homoclinic orbit from $(1, 0)$ (solid line) and a periodic orbit for $\alpha = 0.5$ (dashed line).

From the analysis, by symmetry, for any $0 < \alpha < 1$, there is a solution u such that $u(0) = 0$, $u\left(\frac{l_\alpha}{2}\right) = \alpha$, and $u(l_\alpha) = 0$. Since trajectories do not intersect, l_α is a uniquely determined by α and there is a positive solution u, and thus a nonzero population, as long as $\alpha > 0$. Thus, the minimum size above which an island is able to support a population is given by

$$l_{\min} = \inf_{\alpha \in (0,1)} l_\alpha.$$

We determine the relation between α and l_α. Having in mind that $v = u_x$, we write

$$u_x = \pm\sqrt{2(F(\alpha) - F(u))},$$

depending on which branch of the trajectory we are.

Integrating along the upper branch, we obtain

$$\int_{u(x)}^{\alpha} \frac{dz}{\sqrt{\frac{\mu}{D}\left(\alpha^2 - z^2 - \left(\frac{2\alpha^3}{3} - \frac{2z^3}{3}\right)\right)}} = \frac{l_\alpha}{2} - x,$$

so, by symmetry,

$$l_\alpha = 2\int_0^{\alpha} \frac{dz}{\sqrt{\frac{\mu}{D}\left(\alpha^2 - z^2 - \frac{2}{3}\left(\alpha^3 - z^3\right)\right)}}. \tag{185}$$

Now,

$$\alpha^2 - z^2 - \frac{2}{3}\left(\alpha^3 - z^3\right) = (\alpha^2 - z^2)\left(1 - \frac{2}{3}\frac{(\alpha^2 + \alpha z + z^2)}{\alpha + z}\right).$$

We evaluate

$$\frac{\partial}{\partial z}\frac{\left(\alpha^2+\alpha z+z^2\right)}{\alpha+z}=\frac{2\alpha z+z^2}{(\alpha+z)^2}>0,\quad z>0.$$

Hence,

$$\alpha\leq\frac{\left(\alpha^2+\alpha z+z^2\right)}{\alpha+z}\leq\frac{3\alpha}{2}$$

and therefore

$$\frac{2}{\sqrt{1-\alpha}}\int_0^\alpha\frac{dz}{\sqrt{\frac{\mu}{D}\left(\alpha^2-z^2\right)}}\leq 2\int_0^\alpha\frac{dz}{\sqrt{\frac{\mu}{D}\left(\alpha^2-z^2-\frac{2}{3}\left(\alpha^3-z^3\right)\right)}}$$

$$\leq\frac{2}{\sqrt{1-\frac{3}{2}\alpha}}\int_0^\alpha\frac{dz}{\sqrt{\frac{\mu}{D}\left(\alpha^2-z^2\right)}}.$$

Since

$$\int_0^\alpha \frac{dz}{\sqrt{\alpha^2 - z^2}} = \sin^{-1} 1 - \sin^{-1} 0 = \frac{\pi}{2},$$

we have

$$\lim_{\alpha \to 0^+} l_\alpha = \lim_{\alpha \to 0^+} 2 \int_0^\alpha \frac{dz}{\sqrt{\frac{\mu}{D}\left(\alpha^2 - z^2 - \frac{2}{3}\left(\alpha^3 - z^3\right)\right)}} = \sqrt{\frac{D}{\mu}}\pi.$$

By (185), l_α is double the area under the trajectory $v = v(u), 0 \leq u \leq \alpha$. Since the trajectories do not intersect, l_α is a strictly growing function of α and hence

$$l_{\min} = \inf_{\alpha \in (0,1)} l_\alpha = \lim_{\alpha \to 0^+} l_\alpha = \sqrt{\frac{D}{\mu}}\pi$$

gives the smallest size above which an island able to support the population.