A Brief Introduction to Delay Differential Equations and Applications

Prof. R. Ouifki
North-West University
email: Rachid.Ouifki@nwu.ac.za, ouifkir@gmail.com

March 2021

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Introduction

- Delay Differential Equations (DDEs) are a class of differential equations that involve delays or memory effects in their formulations. Something like

$$x'(t) = f(x(t), x(t-\tau_1), \cdots, , x(t-\tau_m))$$

  with all τ_j being positive.
- The presence of the terms $x(t-\tau_j)$ indicates that the state of the system at time t depends on its state at some previous times $t-\tau_j$.

- τ_j is the delay and it can be constant, we call such equation DDE with **discrete delays**.
- If τ_j depends on time, $\tau_j = \tau_j(t)$, we are talking abut DDEs with **time-dependent delays**.
- If τ_j depends on $x(t)$, $\tau_j = \tau_j(x(t))$, we are talking abut DDEs with **state-dependent delays**.
- There are other types of DDEs (such DDEs with **distributed delays** etc, DDEs of **neutral type**, etc...)

- DDEs have been widely used in modeling physical and biological phenomena that exhibit time delays in their dynamics.
- For instance, DDEs are commonly used to model the dynamics of populations with time delays in their reproduction, the spread of infectious diseases with incubation periods, and the synchronization of coupled oscillators with delayed interactions.

Early Developments

- In 1908, during the international conference of mathematicians, Picard emphasized the significance of accounting for hereditary effects when constructing models of physical systems with the following statement:

  > *Les équations différentielles de la mécanique classique sont telles qu'il en résulte que le mouvement est déterminé par la simple connaissance des positions et des vitesses, c'est-a-dire par l'état à un instant donné et à l'instant infiniment voison. Les états antérieurs n'y intervenant pas, l'hérédité y est un vain mot. L'application de ces équations ou le passé ne se distingue pas de l'avenir,... sont donc inapplicables aux êtres vivants. Nous pouvons rêver d'équations fonctionnelles plus compliquées que les équations classiques parce qu'elles renfermeront en outre des intégrales prises entre un temps passe très éloigné et le temps actuel, qui apporteront la part de l'hérédité.*

Early Developments

- In 1931, Volterra wrote a fundamental book on the role of hereditary effects on models for the interaction of species.
- DDEs gained much traction after 1940 driven by problems in engineering and control.
- During the 1950's, there was considerable activity in the subject which led to important publications by Myshkis (1951), Krasovskii (1959),
- In the 1960's, Bellman and Cooke (1963), Halanay (1966). These books give a clear picture of the subject up to the early 1960's.

Modern Developments

- In the 1970s and 1980s, DDEs were used to model the immune response to infections and the spread of epidemics. They were also used to describe the interaction between HIV and the immune system.
- In the 1990s and 2000s, DDEs were applied to a wide range of biological systems, including gene regulatory networks, cell signaling pathways, and neuronal networks.
- At the present time, much of the subject can be considered as well developed as ordinary differential equations (ODE).

Applications of DDEs

- DDEs have been used to model various biological systems, such as the regulation of hormone levels, the dynamics of populations, and the spread of diseases
- In economics, DDEs have been used to model the dynamics of markets, the effects of taxation, and the behavior of consumers and firms
- In engineering, DDEs have been used to model the dynamics of control systems, the stability of mechanical structures, and the behavior of materials
- DDEs have also been used in other fields, such as physics, chemistry, and neuroscience

Conclusion

- Delay differential equations have a rich history and have found numerous applications in various fields
- The development of numerical methods for solving DDEs has made it possible to simulate and analyze complex systems that exhibit time delays
- DDEs continue to be an active area of research, with new applications and theoretical results being discovered all the time

Delayed Malthusian Models

The familiar Malthusian Model describing the growth of a single population is given by

$$\frac{dN}{dt} = bN(t) - \mu N(t) = rN(t) \tag{1}$$

where $r > 0$ is the growth rate.
This model predicts exponential growth or exponential decline.
We will present here two delayed version of this model:

Delayed Malthusian Model 1:

To account for the influence of the past on the present population, we consider the following DDE:

$$\frac{dN}{dt} = rN(t - \tau) \tag{2}$$

1. The term $rN(t - \tau)$ represents the population growth rate at time t, which depends on the population size $N(t - \tau)$ at a previous time $t - \tau$.
2. Here τ is the delay and it accounts for the time it takes for changes in resource availability to affect population growth.

Question:

What would be a suitable Initial Value Problem (IVP) associated with this equation?

To answer this question, we first note that knowing the value of N at $t = 0$ is not enough to calculate the values of $N(t)$ for $t > 0$.

1. This can be seen by integrating (2), we obtain the following integral equation:
$$\begin{aligned} N(t) &= N(t_0) + \int_{t_0}^{t} rN(s-\tau)ds \\ &= N(t_0) + \int_{t_0-\tau}^{t-\tau} rN(s)ds. \end{aligned} \tag{3}$$
2. This integral representation suggests that if, in addition to $N(0)$, the values of $N(\theta)$ for $\theta \in [-\tau, 0]$ are also know, then one can find the values $N(t)$ for all $t \geq 0$ using the following method of steps:

1. Assume that $N(\theta) = \varphi(\theta)$ for $\theta \in [-\tau, 0]$.
2. For all $t \in [0, \tau]$, we integrate (2) on the interval $[0, t]$ leading to

$$\begin{aligned} N(t) &= N(0) + \int_{-\tau}^{t-\tau} rN(s)ds \\ &= \varphi(0) + \int_{-\tau}^{t-\tau} r\varphi(s)ds \\ &:= N_1(t). \end{aligned} \tag{4}$$

3. We repeat this same step to find the values of $N(t)$ on the interval $[\tau, 2\tau]$.

$$\begin{aligned} N(t) &= N(\tau) + \int_{0}^{t-\tau} rN(s)ds \\ &= N_1(\tau) + \int_{0}^{t-\tau} rN_1(s)ds \\ &:= N_2(t). \end{aligned}$$

4. We repeat the steps above to find the values of $N(t)$ on the intervals $[(k-1)\tau, k\tau]$,

$$N_k(t) = N_{k-1}((k-1)\tau) + r\int_{(k-2)\tau}^{t-\tau} N_{k-1}(s)ds, k = 3, 4, \cdots .$$

For an illustration of the above process, we consider the following problem

$$\begin{cases} N'(t) = rN(t-\tau), \\ N(\theta) = 1, \theta \in [-\tau, 0]. \end{cases} \tag{5}$$

The method of steps gives the following:

1. For $t \in [0, \tau]$, $N_1(t) = \varphi(0) + \int_{-\tau}^{t-\tau} r\varphi(s)ds = 1 + r\int_{-\tau}^{t-\tau} ds = 1 + rt$.
2. For $t \in [\tau, 2\tau]$, $N_2(t) = N_1(\tau) + \int_0^{t-\tau} rN_1(s)ds =$
   $1 + r\tau + r\int_0^{t-\tau}(1 + rs)ds = 1 + rt + r^2\dfrac{(t-\tau)^2}{2}$.
3. $N_k(t)$ can be calculated in the same way on the intervals $[(k-1)\tau, k\tau], k = 3, \cdots$

The solutions obtained using the approach above are plotted in figure (1).

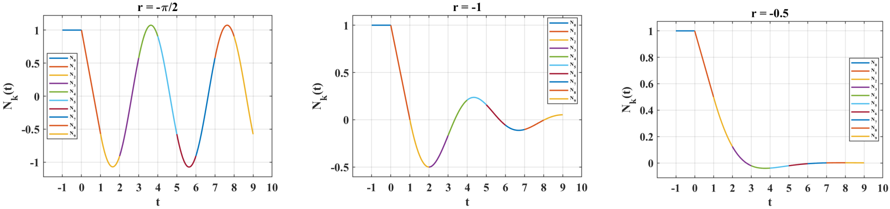


Figure: Plot of exact solution of (5) for $\tau = 1$ using the method of steps. When $r = -1$ the solution displays some damping oscillations that converge to 0. When $r = -\frac{\pi}{2}$, the solution is periodic. We note that the magnitude of the oscillations decrease with the values r. For small values of r, we observe no oscillations.

It is worth Noting that, unlike its ODE version, the delayed Malthusian model exhibits the following:

1. There is NO exponential Growth or Decline.
2. There is an oscillatory behavior.
3. The model has a periodic solution.

Delayed Malthusian Model 2:

Another of Malthus Model consists in categorizing the population into adults and juveniles. At time t, let $N(t)$ represent the density of adults. Assuming that each individual goes through a juvenile period of exact duration τ units of time, the model assumes that adults produce offspring at a rate of b per capita and die at a rate of d. Moreover, it is assumed that newborns survive to adulthood at a rate of p. With these assumptions, the dynamics of the adult population can be described using the following differential equation:

$$\frac{N(t)}{dt} = bpN(t-\tau) - \mu N(t)$$

The first term of this equation contains a delay term which represents the time needed for newborns to become adults provided that they survive the juvenile period.

Remark

1. *Ideally, it would be preferable to incorporate an equation that characterizes the population of juveniles. However, in cases where the main emphasis is on the adult population and there is insufficient data to validate our assumption regarding the juvenile population, a delay differential equation can be used to effectively model the behavior of the adult population. This would necessitate the inclusion of a single extra parameter τ, which can be directly or indirectly estimated using data on the adult population.*
2. *The TB modeling case, when to include or not a delay...*

The Delayed Logistic Model:

- The familiar logistic equation describing the growth of a single population is given by

$$\frac{dN}{dt} = rN(t)\left(1 - \frac{N(t)}{K}\right) \tag{6}$$

where r, K and τ are positive constants.

- To take into account the regulatory influence of the population from a preceding time, $t - \tau$, Hutchinson proposed the following delayed logistic equation

$$\frac{dN}{dt} = rN(t)\left(1 - \frac{N(t-\tau)}{K}\right) \tag{7}$$

The following points are worth noting:

1. Despite its simple look, the delayed logistic model is more complex. For instance, computing the solution for $t > 0$ requires the knowledge of $N(t)$ for all $t \in [-\tau, 0]$.
2. Nevertheless, the DDE shares certain characteristics with its ODE version. Notably, the constant functions $N(t) = 0$ and $N(t) = K$ continue to be equilibrium points.
3. We cannot generally obtain explicit expressions for the solutions of this DDE.
4. Fortunately, we can employ the method of steps to convert this equation into an ODE.
5. However, it is not always possible to construct explicit solutions for the resulting ODE.

Using the Method of Steps

1. Assume that $N(\theta) = \varphi(\theta)$ for $\theta \in [-\tau, 0]$.
2. For all $t \in [0, \tau]$, we have

$$\frac{dN}{dt} = rN(t)\left(1 - \frac{N(t-\tau)}{K}\right) = rN(t)\left(1 - \frac{\varphi(t-\tau)}{K}\right) \qquad (8)$$

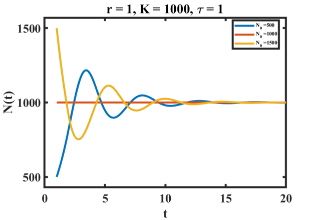
3. We denote by $N_1(t)$ the solution of the above ODE on the interval $[0, \tau]$.
4. We repeat this same step on the interval $[\tau, 2\tau]$ by solving

$$\frac{dN}{dt} = rN(t)\left(1 - \frac{N_1(t-\tau)}{K}\right) \qquad (9)$$

leading to the solution $N_2(t)$.

Remark

It is important to note that using the method of steps for this DDE is quite demanding in terms of algebraic manipulations.
One can use matlab solver dde23 to solve the logistic DDE numerically (see Figure 4).



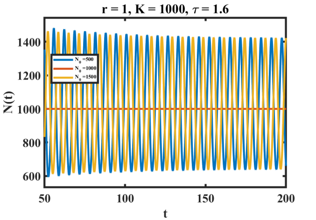


Figure: Plot of the solutions of the Delay Logistic Model (7). Solutions solutions converge to the equilibrium point K for $\tau < \pi/2$ (left) or to a periodic solution $\tau > \pi/2$ (right).

It is worth Noting that, unlike its ODE version, the delayed logistic model exhibits the following:

1. The model oscillates around K.
2. The model has a periodic solution.

SIR Model:

An SIR epidemic model is given by

$$\begin{cases} S'(t) = \lambda - \beta S(t)I(t) - \mu S \\ I'(t) = \beta SI - \gamma I(t) - \mu I \\ R'(t) = \gamma I(t) - \mu R \end{cases} \tag{10}$$

S denotes susceptible individuals, I denotes infectives, and R recovereds. In this model, it is assumed that individuals who get infected at time t become immediately infectious at rate β and move to the class I.

Delayed SIR Model:

Assume that individuals who get infected at time t become infectious at time $t + \tau$.

Which one of the following models is biologically correct:

-
$$\begin{cases} S'(t) = \lambda - \beta S(t+\tau) I(t+\tau) - \mu S \\ I'(t) = \beta S(t+\tau) I(t+\tau) - \gamma I(t) - \mu I \\ R'(t) = \gamma I(t) - \mu R \end{cases} \tag{11}$$

-
$$\begin{cases} S'(t) = \lambda - \beta S(t-\tau) I(t-\tau) - \mu S \\ I'(t) = \beta S(t-\tau) I(t-\tau) - \gamma I(t) - \mu I \\ R'(t) = \gamma I(t) - \mu R \end{cases} \tag{12}$$

Delayed SIR Model Contd:

- $$\begin{cases} S'(t) = \lambda - \beta S(t-\tau)I(t-\tau) - \mu S \\ I'(t) = \beta S(t)I(t) - \gamma I(t) - \mu I \\ R'(t) = \gamma I(t) - \mu R \end{cases} \tag{13}$$

- $$\begin{cases} S'(t) = \lambda - \beta S(t)I(t) - \mu S \\ I'(t) = \beta S(t-\tau)I(t-\tau) - \gamma I(t) - \mu I \\ R'(t) = \gamma I(t) - \mu R \end{cases} \tag{14}$$

Remark

- *The presence of the term $-\beta S(t)I(t)$ in the* **first** *equation means that individuals in class S move out of this class immediately, as they are no longer susceptible. This is the ne that makes biological sense. The others, NOOO*
- *If instead, we use the term $-\beta S(t+\tau)I(t+\tau)$ does not make any biological sense.*
- *Using the term $-\beta S(t-\tau)I(t-\tau)$ does not make any biological sense, and the model not well posed (the solutions may become negative).*
- *Regarding the second equation, the terms $\beta S(t-\tau)I(t-\tau)$ makes sense.*

Delayed SIRS Model:

An SIR epidemic model with fixed period of temporary immunity was proposed by Brauer and Castillo-Chavez, it is given by

$$\begin{cases} S'(t) = -\beta S(t)I(t) + \gamma I(t-\tau) \\ I'(t) = \beta S(t)I(t) - \gamma I(t) \\ R'(t) = \gamma I(t) - \gamma I(t-\tau) \end{cases} \tag{15}$$

S denotes susceptible individuals, I denotes infectives, and R recovereds. In this model, we assume that an individual remains in the R class precisely τ units of time.

One needs to handle this system with caution to avoid getting a negative value for R.
For example, since R does not interact with the other variables, one can remove its equation from the system. This leads to

$$\begin{cases} S'(t) = -\beta S(t)I(t) + \gamma I(t-\tau) \\ I'(t) = \beta S(t)I(t) - \gamma I(t) \end{cases} \tag{16}$$

where

$$R(t) = \int_{t-\tau}^{t} \gamma I(x)dx$$

SEI Model vs Delayed SI:

- Consider a population that we want to model by SEI or Delayed SI models.

SEI model:

$$\begin{cases} S' = -\beta SI \\ E' = -\beta SI - \rho E \\ I' = \rho E \end{cases}$$

Delayed SI model:

$$\begin{cases} S' = -\beta SI \\ I' = \beta S(t-\tau)I(t-\tau) \end{cases}$$

- The population is given by

$N = S + E + I$

$N = S + I + E$? If yes, what is E?

Remark

Mass-action force of infection: βSI

1. *The exposed class, is given by*

$$E(t) = \int_{t-\tau}^{t} \beta S(x)I(x)dx$$

2. *Note that differentiating $E(t)$ gives:*

$$E'(t) = \beta S(t)I(t) - \beta S(t-\tau)I(t-\tau)$$

3. *There is no need to include the expression or derivative of $E(t)$ in the model, since $E(t)$ does not affect the other variables.*

Remark

Standard force of infection:

$$\begin{cases} S' = -\dfrac{\beta SI}{N} \\ I' = \dfrac{\beta S(t-\tau)I(t-\tau)}{N(t-\tau)} \end{cases}$$

where $N = S + E + I$.

Remark

1. *The exposed class, is given by*

$$E(t) = \int_{t-\tau}^{t} \frac{\beta S(x) I(x)}{N(x)} dx$$

2. *Note that differentiating $E(t)$ gives:*

$$E'(t) = \beta S(t) I(t) - \frac{\beta S(t-\tau) I(t-\tau)}{N(t-\tau)}$$

3. *Here, one must include the expression or derivative of $E(t)$ in the model, as it appears in the model (in the variable N).*

Delayed SI with vital dynamics

The SIR epidemic model with vital dynamics is:

$$\begin{cases} S' = \lambda - \beta SI - \mu S \\ I' = \beta S(t-\tau)I(t-\tau) - \mu I \end{cases} \tag{17}$$

The exposed class for the delayed model with vital dynamics is:

$$E(t) = \int_{t-\tau}^{t} e^{-\mu(t-x)}\beta S(x)I(x)dx,$$

Differentiating w.r.t. t leads to

$$\begin{aligned} E' &= \beta S(t)I(t) - \beta S(t-\tau)I(t-\tau) - \mu \int_{t-\tau}^{t} e^{-\mu(t-x)}\beta S(x)I(x)dx \\ &= \beta S(t)I(t) - \beta S(t-\tau)I(t-\tau) - \mu E(t). \end{aligned} \tag{18}$$

Non-uniqueness of solutions of DDEs

The functions $\sin\left(\frac{\pi}{2}\left(t+\frac{1}{2}\right)\right)$ and $\cos\left(\frac{\pi}{2}\left(t+\frac{1}{2}\right)\right)$ both satisfy the following problem

$$\begin{cases} x'(t) = -\pi/2x(t-1) \\ x(0) = \sqrt(2)/2 \end{cases} \tag{19}$$

General Form of DDEs

- **Question:**
  Is there a way to group all the previous DDEs under the same class/notation that looks similar to the $x'(t) = f(t, x(t))$ for ODEs.
- "*It took considerable time to take an idea from ODE and to find the appropriate way to express this idea in FDE [DDE]* $\cdots$
  A new approach was necessary to obtain results which were difficult if not impossible to obtain in the classical way". J.K Hale, 2006.

Let $\tau \geq 0$ be a given constant, and denote by $\mathcal{C} = \mathcal{C}([-\tau, 0], \mathbb{R}^n)$ the ste of countinuous functions defined from $[-\tau, 0]$ to $\mathbb{R}^n$.

Definition (Shift function)

Let $\sigma \in \mathbb{R}$, $\delta > 0$ and let $x : [\sigma - \tau, \sigma + \delta] \to \mathbb{R}^n$, $\delta > 0$. For any $t \in [\sigma, \sigma + \delta]$, we denote by x_t the function defined by

$$x_t(\theta) = x(t + \theta), \theta \in [-\tau, 0]$$

.

One can prove that the function $t \to x_t$ a continuous on $[\sigma, \sigma + \delta]$

Definition (DDE IVP)

Let $f : \mathbb{R} \times \mathcal{C} \to \mathbb{R}^n$ be a given function

- A Delayed Differential Equation is defined by the relation

$$\frac{dx}{dt} = f(t, x_t) \tag{20}$$

- Given a function $\varphi \in \mathcal{C}$, the initial value problem (IVP) associated with this delay equation is

$$\begin{cases} \dfrac{dx}{dt} = f(t, x_t), \text{ for } t \geq \sigma \\ x_\sigma = \varphi, \text{ i.e. } x(\sigma + \theta) = \varphi(\theta) \end{cases} \tag{21}$$

.

1. This notation, referred to as **the modern notation**, was essentially due to Krasovskii in 1956, where he used the notation $f(x(t+\theta))$.
2. The notation $f(x_t)$ was introduced by Hale in 1963.

Definition

A solution of the IVP (21) is defined by a function $x(t)$ defined on $[\sigma - \tau, \sigma + \delta)$, for some $\delta > 0$, such that:

1. $x(t)$ has a continuous derivative on $(\sigma - \tau, \sigma + \delta)$, a right hand derivative at $t = \sigma$ and satisfies the DDE (20) for $t \in [\sigma, \sigma + \delta)$,
2. $x_\sigma = \varphi$, that is $x(\sigma + \theta) = \varphi(\theta), \theta \in [-\tau, 0]$.

Remark

1. *Results concerning existence, uniqueness and continuation of solutions, as well as the dependence on parameters, are essentially the same as for ODE.*
2. *However, there are a few additional technicalites due to the infinite dimensional character of the problem, due to f being a functional defined on a Banach space of continuous functions, $\mathcal{C} = \mathcal{C}([-\tau, 0], \mathbb{R}^n)$, instead of $\mathbb{R}^n$.*

Theorem

1. *If f is continuous, then there is (at least) one solution $x(t, \varphi)$ of the IVP (21) which exists on a maximal interval $[\sigma - \tau, \alpha_\varphi)$.*
2. *If, in addition, f is locally Lipschitzian w.r.t. φ, then the solution is unique.*
3. *If $\alpha_\varphi < \infty$, then the solution becomes unbounded as $t \to \infty$.*
4. *If f is $\mathcal{C}^k$, $t \mapsto x(t, \varphi))$ is $\mathcal{C}^{k+1}$ on $[0, \alpha_\varphi)$ and $\varphi \mapsto x(t, \varphi)$ is $\mathcal{C}^k$ on $\mathcal{C}$.*

Definition

Assume that x^* is an equilibrium point of (20), i.e. $f(t, x^*) = 0$ for all t. x^* is said to be:

1. Stable if, for any $\sigma \in \mathbb{R}, \varepsilon > 0$, there is a $\delta = \delta(\varepsilon, \sigma) > 0$ such that: for any $\varphi \in \mathcal{C}$ with $\|\varphi - x^*\| < \delta$, we have $\|x(t, \varphi) - x^*\| < \varepsilon$ for all $t \geq \sigma$.
2. Asymptotically stable if it is stable and there is $b > 0$ such that $\|\varphi - x^*\| < b$ implies that $\|x(t, \varphi) - x^*\| \to 0$ as $t \to \infty$.
3. A local attractor if there is a neighborhood U of x^* such that $\lim_{t\to\infty} dist(x(t, U), x^*) = 0$.

Frechet derivatives

Definition

Let $(X, \|.\|_X)$ and $(Y, \|.\|_Y)$ be Banach spaces and let $\mathcal{L}(X, Y)$ denote the space of continuous linear operators from X to Y.
Suppose $x \in X$ and let U be an open neighborhood of x.
A function $F : U \to Y$ is Frechet differentiable if there exists an operator $A \in \mathcal{L}(X, Y)$ such that

$$\lim_{\|h\|_X \to 0} \frac{\|F(x+h) - F(x) - Ah\|_Y}{\|h\|_X} = 0.$$

The operator A is called the Frechet derivative of F at x and is denoted by $DF(x)$.

Lemma

1. *If* $f(\phi) = \phi(a)$, *then*
$$Df_{\varphi}(\psi) = \psi(a).$$
2. *Moreover, if* $f(\phi) = g(\varphi(a_1), \cdots, \varphi(a_m))$, *then*
$$Df_{\varphi}(\psi) = \sum_{j=1}^{m} D_j g(\varphi(a_1), \cdots, \varphi(a_m))\, \psi(a_j).$$

Proof.

1. If $f(\phi) = \phi(a)$, then
   $f(\varphi + \psi) - f(\varphi) = (\varphi + \psi)(a) - \phi(a) = \psi(a)$, implying that
   $$\lim_{\|\psi\|_{\mathcal{C}} \to 0} \frac{\|f(\varphi + \psi) - f(\varphi) - \psi(a)\|_{\mathcal{C}}}{\|\psi\|_{\mathcal{C}}} = 0.$$
   Hence $Df(\varphi)\psi = \psi(a)$.
2. If $f(\phi) = g(\varphi(a_1), \cdots, \varphi(a_m))$, then
   $$f(\varphi) = (g \circ h)(\varphi)$$
   where
   $$\begin{aligned} h: \quad \mathcal{C}([-\tau, 0], \mathbb{R}^n) &\to \underbrace{\mathbb{R}^n \times \cdots \times \mathbb{R}^n}_{m \text{ times}} \\ \varphi &\to (\varphi(a_1), \cdots, \varphi(a_m)) \end{aligned} \tag{22}$$

□

Proof.

We have

- $Dg(x_1, \cdots, x_m)(u_1, \cdots, u_m) = \sum_{j=1}^{m} D_j g(x_1, \cdots, x_m) u_j$
- $Dh_\varphi(\psi) = (\psi(a_1), \cdots, \psi(a_m))$

Therefore, by using the chain rule, we have

$$\begin{aligned} Df_\varphi(\psi) &= Dg(h(\varphi)) Dh_\varphi(\psi) \\ &= Dg((\varphi(a_1), \cdots, \varphi(a_m)))(\psi(a_1), \cdots, \psi(a_m)) \\ &= \sum_{j=1}^{m} D_j g(\varphi(a_1), \cdots, \varphi(a_m)) \psi(a_j) \end{aligned}$$

□

Linearization of DDEs

Let $f : \mathcal{C} \to \mathbb{R}$ and consider the following DDE

$$x' = f(x_t)$$

Assume that f is C^1 and that there is x^* such that $f(x^*) = 0$.
The linear part of this DDE is given

$$x' = Lx_t$$

where $L = Df(x^*)$ is the Frechet derivative of the function f at the "constant" function x^*.

Examples: DDE with multiple discrete delays

Consider the following equation

$$x' = g\left(x\left(t\right), x\left(t-\tau_1\right), \cdots, x\left(t-\tau_m\right)\right)$$

where $g : \underbrace{\mathbb{R}^n \times \cdots \times \mathbb{R}^n}_{m+1 \text{ times}} \to \mathbb{R}^n$ is C^1 and has an equilibrium point $x^* = \left(x_0^*, x_1^* \cdots, x_m^*\right) \in \mathbb{R}^n \times \cdots \times \mathbb{R}^n$.
To linearize this equation around x^*, we written it as

$$x' = f\left(x_t\right)$$

where $f\left(\phi\right) = g\left(\phi\left(0\right), \varphi\left(-\tau_1\right), \cdots, \varphi\left(-\tau_m\right)\right)$.

Examples: Linear DDE with multiple discrete delays

We know that

$$
\begin{aligned}
Df_{\varphi}(\psi) = \quad & D_1 g\left(\phi(0), \varphi(-\tau_1), \cdots, \varphi(-\tau_m)\right)\psi(0) \\
& + \sum_{j=1}^{m} D_{j+1} g\left(\phi(0), \varphi(-\tau_1), \cdots, \varphi(-\tau_m)\right)\psi(-\tau_j).
\end{aligned} \tag{23}
$$

Then, the linear part of the DDE is

$$
Lx_t = Df_{x^*}(x_t) = D_1 g(x^*)\, x(t) + \sum_{j=1}^{m} D_{j+1} g(x^*)\, x(t-\tau_j). \tag{24}
$$

We can apply the above to the following examples:

1. Consider the following equation

$$x' = g\left(x\left(t-\tau\right)\right)$$

where $g : \mathbb{R}^n \to \mathbb{R}^n$ is C^1 and has an equilibrium point $x^* \in \mathbb{R}^n$. Then

$$Lx_t = Dg\left(x^*\right)x\left(t-\tau\right).$$

2. Consider the following equation

$$x' = g\left(x\left(t\right), x\left(t-\tau\right)\right)$$

where $g : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is C^1 and has an equilibrium point $x^* \in \mathbb{R}^n \times \mathbb{R}^n$.
Then

$$Lx_t = D_1 g\left(x^*\right) x\left(t\right) + D_2 g\left(x^*\right) x\left(t-\tau\right)$$

Stability Properties

For linear DDEs, there is a solution of the form $ce^{\lambda t}$ for some nonzero n-vector c if and only if λ satisfies the characteristic equation

$$det\left(\Delta(\lambda) - Le^{\lambda \cdot}\mathbf{I}_n\right) = 0 \tag{25}$$

λ is called an eigenvalue of the linear equation.

Theorem

1. *If all eigenvalues of (25) have negative real parts then* 0 *is asymptotically stable.*
2. *If at least one eigenvalue has a positive real part, then* 0 *is unstable.*

Stability and Bifurcation: Malthusian DDE

To account for the influence of the past on the present population, we consider the following DDE:

$$\frac{dN}{dt} = rN(t - \tau) \tag{26}$$

Stability: DDE Malthusian Model $N^* = 0$

The characteristic equation at $N^* = 0$ is $\det(\lambda I - re^{-\lambda\tau}) = 0$. that is

$$\lambda - re^{-\lambda\tau} = 0. \qquad (27)$$

When $\tau = 0$ equation (32) has only one root $\lambda = r$. Hence we discuss two cases:

1. $r < 0$
2. $r > 0$

DDE Malthusian: $r < 0$

- **Case** $r < 0$:
  In this case, $N^* = 0$ is locally asymptotically stable for $\tau = 0$.
  By continuity, as τ increases, the stability of $N^* = 0$ may change if the real part of one of the eigenvalues **crosses** the imaginary axis **from left to right**.
  This occurs if, for some value of τ, say τ_0, equation (32) has a pair of imaginary roots $\lambda = \pm i\omega$, i.e.

$$i\omega - re^{-i\tau\omega} = 0$$

Clearly $\omega \neq 0$ (otherwise $0 - re^{0} = 0$). Let us then assume, without loss of generality that $\omega > 0$. Thus

$$\begin{cases} r\cos(\tau\omega) = 0 \\ \omega + r\sin(\tau\omega) = 0. \end{cases} \tag{28}$$

Then $\omega = -r$ and we have the following values for τ

$$\tau_k = -\frac{\pi}{2r} - \frac{2k\pi}{r}, k = 0, 1, \cdots$$

In particular, when $\tau_0 = -\dfrac{\pi}{2r}$ equation (32) has a pair of purely imaginary roots $\pm ir$, which are simple and all other roots have negative real parts.

DDE Malthusian: $r < 0$

Therefore, when $0 < \tau < -\dfrac{\pi}{2r}$, all roots of (32) have strictly negative real parts. Denote $\lambda(\tau) = a(\tau) + ib(\tau)$ the root of equation (32) satisfying

$$a(\tau_k) = 0, b(\tau_k) = \omega, k = 0, 1, \cdots \tag{29}$$

To find out if the eigenvalue $\lambda(\tau)$ crosses the imaginary axis, we calculate the $\frac{d\,\mathrm{Re}\,\lambda(\tau)}{d\tau}$. For instance if $\frac{d\,\mathrm{Re}\,\lambda(\tau)}{d\tau} > 0$, then the eigenvalue $\lambda(\tau)$ crosses the imaginary axis from left to right.

DDE Malthusian: $r < 0$

Since $\lambda - re^{-\lambda\tau} = 0$, then $\frac{d\lambda}{d\tau} = -re^{-\lambda\tau}\left(\lambda + \tau\frac{d\lambda}{d\tau}\right)$, implying that

$$\frac{d\lambda}{d\tau} = -\frac{r\lambda}{r\tau + e^{\lambda\tau}}.$$

at $\tau = \tau_k$, we obtain

$$\left.\frac{d\lambda}{d\tau}\right|_{\tau=\tau_k} = -\frac{ri\omega}{r\tau_k + e^{i\omega\tau_k}}$$

which by $e^{i\tau_k\omega} = -\frac{ir}{\omega}$, implies that

$$\left.\frac{d\lambda}{d\tau}\right|_{\tau=\tau_k} = -\frac{ri\omega}{r\tau_k + -\frac{ir}{\omega}} = -\frac{i\omega\tau_k}{\tau_k^2 + \frac{1}{\omega^2}} + \frac{1}{\tau_k^2 + \frac{1}{\omega^2}}$$

DDE Malthusian: $r < 0$

Thus

$$\left.\frac{d\,\mathrm{Re}\,\lambda}{d\tau}\right|_{\tau=\tau_k} = \mathrm{Re}\left(\left.\frac{d\lambda}{d\tau}\right|_{\tau=\tau_k}\right) = \frac{1}{\tau_k^2 + \frac{1}{\omega^2}} > 0.$$

This implies that the branchs of eigenvalues $\lambda(\tau)$ **crosse,** at $\tau = \tau_k$, the imaginary axis and that the crossings are **from left to right**.
This also proves the existence of a branch of periodic solutions that emerge from $x^* = 0$ at $\tau = \tau_0$. The period of this solution is $T = \dfrac{2\pi}{\omega} = \dfrac{2\pi}{r}$.

- **Case** $r > 0$:
  In this case, $N^* = 0$ is unstable for $\tau = 0$.
  We saw in the previous case that if the eigenvalues cross the imaginary axis, then te crossing if from left to right. This means that the number of eigenvalues with positive real parts does not decrease implying that $N^* = 0$ remains unstable for $\tau \geq 0$.

Proposition

- *If* $r > 0$, *then* $N^* = 0$ *is unstable for all* $\tau \geq 0$.
- *If* $r < 0$, *then*
  - *If* $0 < \tau < -\dfrac{\pi}{2r}$, *then* $N^* = 0$ *is asymptotically stable.*
  - *If* $\tau > -\dfrac{\pi}{2r}$, *then* $N^* = 0$ *is unstable.*
  - *When* $\tau = \tau_0 := -\dfrac{\pi}{2r}$, *a Hopf bifurcation occurs at* τ_0*; that is, periodic solutions bifurcate from* 0. *The periodic solutions exist for* $\tau > -\dfrac{\pi}{2r}$ *and are stable.*

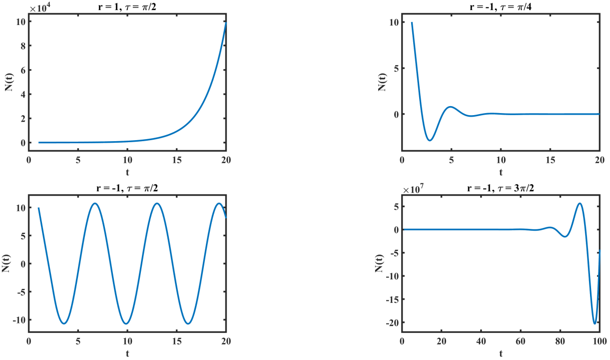


Figure: Plot of the solutions of the Delay Malthusian Model (26). Solutions blow up when $r > 0$ and converge to the equilibrium point 0 when $r < 0, \tau < \pi/2$. Periodic solutions exist when $r < 0, \tau = \pi/2$ with period of the periodic solution is $2\pi/r = 2\pi$. The solutions display a chaotic behaviour for $\tau \geq \pi/2$.

Stability and Bifurcation: Logistic DDE

Consider Hutchinson's model (7)

$$\frac{dN}{dt} = rN(t)\left(1 - \frac{N(t-\tau)}{K}\right) \tag{30}$$

This equation can be writtens as

$$x'(t) = f(x_t) \tag{31}$$

where

$$f(\phi) = r\phi(0)\left(1 - \frac{\phi(-\tau)}{K}\right) = g(\phi(0), \phi(-\tau))$$

with $g(x, y) = rx\left(1 - \frac{y}{K}\right)$.

We have

$$\begin{cases} D_1 g_{(x,y)}(u,v) &= r\left(1-\frac{y}{K}\right)u \\ D_2 g_{(x,y)}(u,v) &= -\frac{r}{K}xv. \end{cases}$$

Therefore, the Frechet differential of the function f at φ is given by

$$\begin{aligned} Df_{\varphi}(\psi) &= D_1 g\left(\left(\varphi(0), \varphi(-\tau)\right)\right)\psi(0) + D_2 g\left(\left(\varphi(0), \varphi(-\tau)\right)\right)\psi(-\tau) \\ &= r\left(1-\frac{\varphi(-\tau)}{K}\right)\psi(0) - \frac{r}{K}\varphi(0)\psi(-\tau). \end{aligned}$$

Replacing φ by N^* and ψ by N_t leads to

$$Df_{xN^*}(N_t) = r\left(1-\frac{N^*}{K}\right)N_t(0) - \frac{r}{K}N^* N_t(-\tau).$$

Thus

Lemma

1. *The linearized equation around* $N^* = 0$ *is*

$$N'(t) = rN(t)$$

2. *The linearized equation around* $N^* = K$ *is*

$$N'(t) = -rN(t-\tau)$$

Stability: $N^* = 0$

The characteristic equation at $N^* = 0$ is $det(\lambda I - Le^{\lambda \cdot}) = 0$. That is $det(\lambda I - re^{\lambda 0}) = \lambda - r = 0$. This implies that $\lambda = r$. Hence

Proposition

- *If $r < 0$, then 0 is locally asymptotically stable.*
- *If $r > 0$, then 0 is unstable.*

Stability: $N^* = K$

The characteristic equation at $N^* = K$ is $det(\lambda I + re^{-\lambda\tau}) = 0$. Thus

$$\lambda + re^{-\lambda\tau} = 0. \tag{32}$$

This equation is similar to the charecteristic euqation of the Malthusian model (32) provided that r is replaced by $-r$.
Thus we have the following results;

Proposition

- *If $r < 0$, then $N^* = 0$ is locally asymptotically stable and $N^* = K$ is unstable for all $\tau \geq 0$.*
- *If $r > 0$, then $N^* = 0$ becomes unstable and*
  - *If $0 < \tau < \dfrac{\pi}{2r}$, then $N^* = K$ is asymptotically stable.*
  - *If $\tau > \dfrac{\pi}{2r}$, then $N^* = K$ is unstable.*
  - *When $\tau = \dfrac{\pi}{2r}$, a Hopf bifurcation occurs at τ; that is, periodic solutions bifurcate from K.*
  - *The periodic solutions exist for $\tau > \dfrac{\pi}{2r}$ and are stable.*

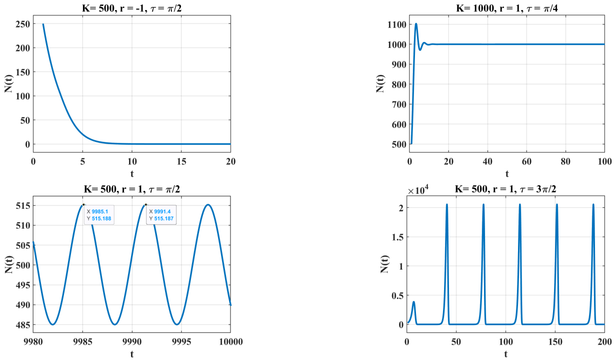


Figure: Plot of the solutions of the Delay Logistic Model (7). Solutions converge to the equilibrium point 0 for negative r, and converge to K for positive r and $\tau < \pi/2$ (UP). Periodic solutions exist for positive r and $\tau \geq \dfrac{\pi}{2}$ (Down). Note that when $\tau = \dfrac{\pi}{2}$, the period of the periodic solution is $\dfrac{2\pi}{r} = 2\pi$.

Remark

In the last two examples, we investigate the roots of the characteristic equation of the delayed Malthusian and Logistic Models.
However, such investigation has been performed in more detail for a more general class of equations, by H.I. Freedman & Y. Kuang, in 1991. This will be presented in the next slides.

First order Neutral Delay Equation

Consider the following Linear First order Neutral Delay Equation

$$x'(t) + ax'(t-\tau) + bx(t) + cx(t-\tau) = 0. \tag{33}$$

The coresponding characteristic equation is

$$\lambda + a\lambda e^{-\lambda\tau} + b + ce^{-\lambda\tau} = 0.$$

Theorem (H.I. Freedman & Y. Kuang, 1991)

Assume that $|a| \neq 1$, *then the following statements hold:*

1. *If* $|a| > 1$, *then* 0 *is unstable for all positive delay* τ.
2. *If* $|a| < 1$, $c^2 < b^2$ *or* $c = b \neq 0$, *then increasing* τ *does not change the stability of* 0.
3. *If* $|a| < 1$, $c^2 > b^2$ *and*
   1. $b + c < 0$, *then* 0 *is unstable for all positive delay* τ.
   2. $b + c > 0$, *then* 0 *is stable when* $\tau < \tau_0$ *and unstable when* $\tau > \tau_0$, *where* $\tau_0 = \frac{\theta}{\omega}$, *and*

$$\begin{cases} \omega = \sqrt{\frac{c^2 - b^2}{1 - a^2}} \\ \theta = \operatorname{arccot}\left(-\frac{a\omega^2 + bc}{\omega(c - ab)}\right) \end{cases}$$

Stability of the Delayed SI with vital dynamics

The SIR epidemic model with vital dynamics is:

$$\begin{cases} S' = \lambda - \beta SI - \mu S \\ I' = \beta S(t-\tau)I(t-\tau) - \mu I \end{cases} \tag{34}$$

To analyse the stability behaviour of this DDE, we write it as:

$$\begin{bmatrix} S' \\ I' \end{bmatrix} = g\left(\begin{bmatrix} S \\ I \end{bmatrix}\right) + h\left(\begin{bmatrix} S(t-\tau) \\ I(t-\tau) \end{bmatrix}\right)$$

where

$$\begin{cases} g\left(\begin{bmatrix} S \\ I \end{bmatrix}\right) = \begin{bmatrix} b - \beta SI - dS \\ -dI \end{bmatrix} \\ h\left(\begin{bmatrix} S \\ I \end{bmatrix}\right) = \begin{bmatrix} 0 \\ \beta SI \end{bmatrix} \end{cases}$$

We have

$$\begin{cases} Dg\left(\begin{bmatrix} S \\ I \end{bmatrix}\right) = \begin{bmatrix} \beta I - d & \beta S \\ 0 & -d \end{bmatrix} \\ Dh\left(\begin{bmatrix} S \\ I \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 \\ \beta I & \beta S \end{bmatrix} \end{cases}$$

Hence, the linearized system at the DFE $\mathcal{E} = \left(\frac{b}{d}, 0\right)$, is

$$\begin{aligned} \begin{bmatrix} S' \\ I' \end{bmatrix} &= Dg\left(\begin{bmatrix} \frac{b}{d} \\ 0 \end{bmatrix}\right)\begin{bmatrix} S \\ I \end{bmatrix} + Dh\left(\begin{bmatrix} \frac{b}{d} \\ 0 \end{bmatrix}\right)\begin{bmatrix} S(t-\tau) \\ I(t-\tau) \end{bmatrix} \\ &= \begin{bmatrix} -dS - \frac{\beta b}{d} I \\ -dI \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{\beta b}{d} I(t-\tau) \end{bmatrix} \end{aligned}$$

which we write as

$$\begin{cases} S' = -dS - \frac{\beta b}{d} I \\ I' = \frac{\beta b}{d} I(t-\tau) - dI \end{cases}$$

The characteristic equation can be obtained by finding solutions of this linear system that take the form $e^{\lambda t}\begin{bmatrix} S^0 \\ I^0 \end{bmatrix}$. This leads to

$$\begin{cases} \lambda e^{\lambda t} S^0 = -d e^{\lambda t} S^0 - \frac{\beta b}{d} e^{\lambda t} I^0 \\ \lambda e^{\lambda t} I^0 = \frac{\beta b}{d} e^{\lambda(t-\tau)} I^0 - d e^{\lambda t} I^0 \end{cases}$$

Therefore

$$\lambda \begin{bmatrix} S_0 \\ I_0 \end{bmatrix} = \begin{bmatrix} -d & -\frac{\beta b}{d} \\ 0 & \frac{\beta b}{d} e^{-\lambda \tau} - d \end{bmatrix} \begin{bmatrix} S_0 \\ I_0 \end{bmatrix}$$

Hence

$$\det\left(\lambda I - \begin{bmatrix} -d & -\frac{\beta b}{d} \\ 0 & \frac{\beta b}{d} e^{-\lambda \tau} - d \end{bmatrix}\right) = 0$$

that is

$$(d+\lambda)\left(d\lambda + d^2 - b\beta e^{-\lambda\tau}\right) = 0.$$

This leads to $\lambda = -d$, or

$$d^2 + d\lambda - b\beta e^{-\lambda\tau} = 0.$$

This equation can be analyzed using the Theorem above (H.I. Freedman & Y. Kuang, 1991).
The same can be done for the EEP.

Summary

In these notes, we covered the following:

1. A brief historical overview and applications of DDEs
2. Some Models that contain delay(s)
3. Some qualitative properties of solutions of DDEs such as damped and periodic oscillations.
4. The general form of DDEs $x' = f(t, x_t)$ and how useful it is in drawing parallels between the theory of DDEs and that of ODEs.
5. A very brief overview of some standard properties of DDEs
6. Stability and Bifurcation of the DDE Malthusian and DDE Logistic Models
7. Stability and Bifurcation of a General First order Linear Neutral DDE.

Softwares for Numerical Solutions, Stability and Bifurcation of DDEs

A list of softwares for numerically solving and studying DDEs can be found in http://twr.cs.kuleuven.be/research/software/delay/software.shtml.
Special attention to be made to the following ones:

1. The matlab solver dde23 which can be used to solve DDEs with discrete delays. For more details and examples on this solver, see http://www.runet.edu/ thompson/webddes/
2. Also, check the MATLAB package DDE–BIFTOOL for bifurcation analysis of delay differential equations. For documentation and tutorials on the tool, see https://twr.cs.kuleuven.be/research/software/delay/ddebiftool.shtml.
3. The sofware XPP can also be used to numerically solve DDEs with discrete delays. XPPAUT can be used to numericaly explore their stability and bifurcation. The link to both XPP and XPPAUT is: https://sites.pitt.edu/ phase/bard/bardware/xpp/xpp.html