Toolbox to analyse discrete time models

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Outline

Order of a difference equation

Definition

The *order* of a difference equation

$$f(x_{t+p}, x_{t+p-1}, \ldots, x_{t+1}, x_t, t) = 0$$

is the difference between the largest and the smallest arguments p appearing in it.

Definition

The difference equation is called *autonomous* if f does not depend explicitly on t and it is called *nonautonomous* otherwise.

Definition

Let

$$x_{t+p} + a_1 x_{t+p-1} + a_2 x_{t+p-2} + \cdots + a_{p-1} x_{t+1} = b_t.$$

If the coefficients a_j, $j = 1, \ldots, p$ are constant or depend on t but **do not** depend on the state variables, then the difference equation is said to be *linear*; otherwise, it is *nonlinear*.

Definition

If the difference equation is linear and $b_t = 0$ for all t, then it is said to be *homogeneous*; otherwise, it is said to be *nonhomogeneous*.

Definition

A solution of the difference equation

$$f(x_{t+k}, x_{t+k-1}, \dots, x_{t+1}, x_t, t) = 0$$

is a function x_t, $t = 0, 1, 2, \dots$ such that when substituted into the equation makes it a true statement.

Definition

A point x^* in the domain of f is said to be an **equilibrium point** (an equilibrium solution) of the first-order difference equation

$$x_{t+1} = f(x_t)$$

if it is a fixed point of f *i.e.* a constant solution that satisfies

$$f(x^*) = x^*.$$

Graphically, if $f : \mathbb{R} \to \mathbb{R}$, an equilibrium point is the $x-$coordinate of a point where the graph of $f(x)$ intersects the diagonal $y = x$ (since at such a point, $x = f(x)$).

Outline

First-order linear homogeneous equations with constant coefficients

Consider the first-order linear homogeneous difference equation with constant coefficients

$$x_{t+1} = ax_t$$

If an initial value x_0 is known, the solution is unique and is given by

$$x_t = a^t x_0.$$

The **asymptotic behavior** of the solution depends on the value of a:

- if $|a| < 1$, then $\lim_{t\to\infty} x_t = 0$, i.e., x_t converges to 0,
- if $a = 1$, then for all $t \geq 0$, $x_t = x_0$, i.e., x_t remains constant,
- if $a = -1$, then for all $t \geq 0$, $x_t = (-1)^t x_0$, i.e., x_t alternates,
- if $|a| > 1$ then x_t diverges (either approaches infinity if $a > 1$ or diverges with alternating signs if $a < -1$).

First-order linear homogeneous equations with non-constant coefficients

Proposition

Consider the first-order linear homogeneous difference equation defined for $t = 0, 1, 2, \ldots$ by

$$x_{t+1} = a_t x_t.$$

If an initial value x_0 is known, then the solution is unique and is given by

$$x_t = \left[\prod_{i=0}^{t-1} a_i\right] x_0.$$

Proposition

Consider the first-order linear nonhomogeneous difference equation defined for $t = 0, 1, 2, \ldots$ by

$$x_{t+1} = a_t x_t + b_t.$$

If an initial value x_0 is known, then the solution is unique and is given by

$$x_t = \left[\prod_{i=0}^{t-1} a_i\right] x_0 + b_{t-1} + \sum_{i=0}^{t-2} \left[\prod_{r=i+1}^{t-1} a_r\right] b_i.$$

In particular,

- If $x_{t+1} = a x_t + b_t$, then $x_t = a^t x_0 + \sum_{i=0}^{t-1} a^{t-i-1} b(i)$.
- If $x_{t+1} = a x_t + b$, then

$$x_t = \begin{cases} a^t x_0 + b\left[\dfrac{a^t - 1}{a - 1}\right] & a \neq 1 \\ x_0 + bt & a = 1. \end{cases}$$

Outline

Higher-order linear equations

Principle of superposition

If x_t^1, x_t^2,…, x_t^k are solutions of a k^{th}-order linear homogeneous difference equation, then

$$c_1 x_t^1 + c_2 x_t^2 + \ldots, c_k x_t^k$$

is also solution of the k^{th}-order linear homogeneous difference equation.

Definition

A set of k linearly independent solutions of a k^{th}-order linear homogeneous difference equation is called a *fundamental set of solutions*.

Higher-order linear equations

Theorem

If the Casoratian of x_t^1, x_t^2,..., x_t^k satifies

$$C(x_t^1, x_t^2, \dots, x_t^k) \neq 0, \quad \forall t,$$

then x_t^1, x_t^2,..., x_t^k are k linearly independent functions.

Definition

The *Casoratian* of k functions x_t^1, x_t^2,..., x_t^k is defined as

$$C(x_t^1, x_t^2, \dots, x_t^k) = \det \begin{pmatrix} x_t^1 & x_t^2 & \dots & x_t^k \\ x_{t+1}^1 & x_{t+1}^2 & \dots & x_{t+1}^k \\ x_{t+2}^1 & x_{t+2}^2 & \dots & x_{t+2}^k \\ \vdots & & & \\ x_{t+k-1}^1 & x_{t+k-1}^2 & \dots & x_{t+k-1}^k \end{pmatrix}.$$

Higher-order linear equations

Definition

Let $\{x_t^1, x_t^2, \ldots, x_t^k\}$ be a fundamental set of solutions of $k^{th}-$order linear homogeneous difference equation. Then the general solution of the $k^{th}-$order linear homogeneous difference equation is given by

$$x_t = \sum_{i=1}^{k} c_i x_t^i,$$

for arbitrary constants c_i, $i = 1, \ldots, k$

Higher-order linear equations with constant coefficients

An $m^{th}-$order linear homogeneous equation with constant coefficients is defined as

$$a_0 x_{t+m} + a_1 x_{t+m-1} + \cdots + a_m x_t = 0. \tag{1}$$

The fundamental set is composed of m linear independent solutions of the form $x_t = \lambda^t$, where λ are the roots (eigenvalues) of the characteristic equation

$$a_0 \lambda^m + a_1 \lambda^{m-1} + \cdots + a_m = 0.$$

- If eigenvalues are all real and distinct, the general solution takes the form

  $$x_t = c_1 \lambda_1^t + \cdots + c_m \lambda_m^t,$$

  where c_i, $i = 1, \ldots, m$ are arbitrary.
- If there is a real eigenvalue λ_1 of multiplicity k, then k linearly independent solutions can be formed by multiplying by powers of t:

  $$\lambda_1^t, t\lambda_1^t, t^2\lambda_1^t, \ldots, t^{k-1}\lambda_1^t.$$

- If there are complex eigenvalues $\lambda_{1,2} = r(\cos\phi \pm i \sin\phi)$ of multiplicty k, then there are $2k$ linearly independent solutions:

  $$r^t \cos(t\phi), r^t \sin(t\phi), tr^t \cos(t\phi), tr^t \sin(t\phi), \ldots, t^{k-1} r^t \cos(t\phi), t^{k-1} r^t \sin(t\phi).$$

Higher-order linear equations with constant coefficients

Let the general solution of (1) be

$$x_t = \sum_{i=1}^{m} c_i \lambda_i^t.$$

The asymptotic behavior of the general solution is determined by the behavior of the dominant solution (corresponding to the dominant eigenvalue).
Suppose that there exists a strictly dominant eigenvalue $|\lambda_1| > |\lambda_i|$, then

$$x_t = \lambda_1^t \left[c_1 + \sum_{i=2}^{m} c_i \left(\frac{\lambda_i}{\lambda_1} \right)^t \right].$$

Since $\left|\frac{\lambda_i}{\lambda_1}\right| < 1$, for all $i \neq 1$, it follows that $\left\{ \left(\frac{\lambda_i}{\lambda_1}\right)^t \right\}$ is a geometric sequence with general term $\left|\frac{\lambda_i}{\lambda_1}\right| < 1$, and thus

$$\left(\frac{\lambda_i}{\lambda_1} \right)^t \to 0 \quad \text{as } t \to +\infty.$$

Then

$$\lim_{t \to +\infty} x_t = \lim_{t \to +\infty} c_1 \lambda_1^t.$$

Definition

Suppose that the k eigenvalues of a characteristic equation are $\lambda_1, \cdots, \lambda_k$. An eigenvalue λ_i such as $|\lambda_i| \geq |\lambda_j|$ for all $j \neq i$ is called a dominant eigenvalue. If $|\lambda_i| > |\lambda_j|$ for all $j \neq i$, then λ_i is called a strictly dominant eigenvalue.

Higher-order linear equations with constant coefficients

If $\lambda_1 \in \mathbb{R}$. Then

- if $\lambda_1 > 1$, then $\lim_{t\to+\infty} c_1\lambda_1^t = \infty$ (monotonically diverges $\Rightarrow$ unstable equilibrium),
- if $\lambda_1 = 1$, then the solution is constant,
- if $0 < \lambda_1 < 1$: $\lim_{t\to+\infty} c_1\lambda_1^t = 0$ (monotonically decreases to 0 $\Rightarrow$ asymptotically stable equilibrium),
- if $-1 < \lambda_1 < 0$: $\lim_{t\to+\infty} c_1\lambda_1^t = 0$ (oscillating around zero and converging to 0 $\Rightarrow$ asymptotically stable equilibrium),
- if $\lambda_1 = -1$, the system oscillates between two values c_1 and $-c_1$,
- and if $\lambda_1 < -1$, the system is oscillating but increasing in magnitude (unstable equilibrium).

If $\lambda_1 \in \mathbb{C}$. Then

- if $|\lambda_1| > 1$, the system oscillates but increases in magnitude (unstable equilibrium),
- if $|\lambda_1| = 1$, the system oscillates but constant magnitude,
- if $|\lambda_1| < 1$, the system oscillates but converges to 0 (asymptotically stable equilibrium).

Outline

$$a_0 x_{t+m} + a_1 x_{t+m-1} + \cdots + a_m x_t = b(t).$$

Assume that $x_t = x(t)$. Let $Y(t) = (y_1(t), y_2(t), \ldots, y_m(t))$, which satisfies

$$\begin{aligned} y_1(t) &= x(t) \\ y_2(t) &= x(t+1) \\ &\vdots \\ y_m(t) &= x(t+m-1). \end{aligned}$$

The first element $y_1(t)$ is the solution $x(t)$. Hence a first-order system in Y is

$$\begin{aligned} y_1(t+1) &= y_2(t) \\ y_2(t+1) &= y_3(t) \\ &\vdots \\ y_{m-1}(t+1) &= y_m(t) \\ y_m(t+1) &= -a_1 y_m(t) - \cdots - a_{m-1} y_2(t) - a_m y_1(t) + b(t) \end{aligned}$$

In matrix form,

$$Y(t+1) = AY(t) + B,$$

where

$$A = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -a_m & -a_{m-1} & -a_{m-2} & \dots & -a_1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ b(t) \end{pmatrix}.$$

A solution to a first-order linear difference system

$$X(t+1) = AX(t) + B$$

is the superposition of two solutions:

- the general solution X_h to the homogeneous system $X_h(t+1) = AX_h(t)$
- and a particular solution X_p to the nonhomogeneous system $X_p(t+1) = AX_p(t) + B$.

The general solution to the nonhomogeneous system is

$$X(t) = X_h(t) + X_p(t).$$

The solution of

$$X(t+1) = AX(t)$$

is

$$X(t) = A^t X(0).$$

To solve

$$X(t+1) = AX(t)$$

where $A = (a_{ij})$ is an $m \times m$ constant matrix.

- Assume that a solution has the following form $X(t) = \lambda^t V$ where V is an nonzero $m-$column vector and λ is a constant.
- Substituting $\lambda^t V$ into the linear system gives

  $$\lambda^{t+1} V = A\lambda^t V,$$

  then

  $$(A - \lambda I)V = \mathbf{0} \qquad (2)$$

  where I is the $m \times m$ identity matrix and $\mathbf{0}$ is the zero vector.
- Hence, nonzero solutions V are obtained if and only if

  $$\det(A - \lambda I) = 0. \qquad (3)$$

  Equation (3) is the **characteristic equation of matrix** A.
- m solutions λ_i, $i = 1, \ldots, m$ of Eq. (3) are the **eigenvalues** of A. Nonzero solutions V_i are the **eigenvectors** corresponding to eigenvalues λ_i obtained by $(A - \lambda_i I)V_i = \mathbf{0}$.

$$X(t+1) = AX(t)$$

where $A = (a_{ij})$ is an $m \times m$ constant matrix.
The general solution is a linear combination of m linearly independent solutions $X_i = \lambda_i^t V_i$, $i = 1, \dots, m$:

$$X(t) = \sum_{i=1}^{m} c_i \lambda_i^t V_i$$

where c_i are arbitrary constants.

Understanding the asymptotic behavior of the solution

$$X(t) = \sum_{i=1}^{m} c_i \lambda_i^t V_i$$

does not require the knowledge of the eigenvectors.

The asymptotic behavior is determined by the eigenvalues and their magnitude, as the solution of $X(t+1) = AX(t)$ is $X(t) = A^t X(0)$.

Definition

Suppose the $k \times k$ matrix A has k eigenvalues $\lambda_1, \cdots, \lambda_k$. The spectral radius of matrix A is denoted as $\rho(A)$ and defined as

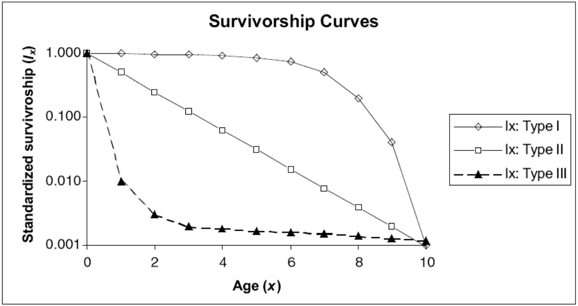
$$\rho(A) = \max_{i \in \{1,2,..,k\}} \{|\lambda_i|\}.$$

Theorem

Let A be a constant $k \times k$ matrix. Then the spectral radius of A satisfies $\rho(A) < 1$ if and only if $\lim_{t \to \infty} A^t = \mathbf{0}$.

Structured populations

In some species, the amount of reproduction varies greatly with the age of individuals.



Type I: organisms have high survivorship throughout life until old age sets in, and then survivorship declines dramatically to 0. Humans are type I organisms. Type III: organisms, in contrast, have very low survivorship early in life, and few individuals live to old age.

Age structure or developmental stage of population matter.

Example: US human population in 1964

To capture the effects of age structure on population growth: 10 age classes.

- $x_1(t)=$ # of individuals of age 0 to 4 at time t
- $x_2(t)=$ # of individuals of age 5 to 9 at time t
- $x_3(t)=$ # of individuals of age 10 to 14 at time t
- $x_4(t)=$ # of individuals of age 15 to 19 at time t
- $x_5(t)=$ # of individuals of age 20 to 24 at time t
- $x_6(t)=$ # of individuals of age 25 to 29 at time t
- $x_7(t)=$ # of individuals of age 30 to 34 at time t
- $x_8(t)=$ # of individuals of age 35 to 39 at time t
- $x_9(t)=$ # of individuals of age 40 to 44 at time t
- $x_{10}(t)=$ # of individuals of age 45 to 49 at time t

Structured population dynamics: discrete models, general case

- population categorized into a finite number of classes $i = 1, 2, \cdots, m$
- $x_i(t)$ number or density of individuals in the i^{th} class at time $t = 0, 1, 2, \cdots$
- If only birth and death processes (no migration):

$$x(t+1) = Px(t)$$

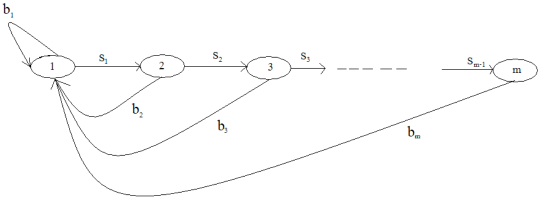
where $P = T + F$ is the projection matrix

Structured population dynamics: discrete models, general case

$$x(t+1) = Px(t)$$

- $T = [t_{ij}]$ transition matrix
  - t_{ij} fraction of $j-$class individual expected to survive and move to class i per unit of time
  - t_{ii} fraction of individuals in class i that survive and remain in class i after one unit of time
  - No individual can shrink or grow more than one class in one unit of time
  - $0 \leq t_{ij} \leq 1$ and $\sum_{i=1}^{m} t_{ij} \leq 1$ for all j
- $F = [f_{ij}]$ fertility matrix
  - f_{ij} the expected number of (surviving) $i-$class offspring per $j-$class individual per unit of time
  - $f_{ij} \geq 0$

A particular case: Leslie model (time interval coincides with the structure interval)



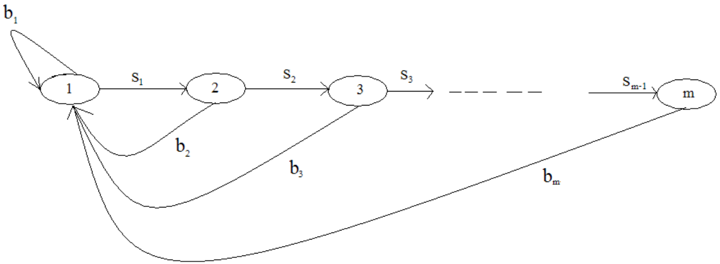
$$
\begin{aligned}
x_1(t+1) &= b_1 x_1(t) + b_2 x_2(t) + b_3 x_3(t) + \dots b_m x_m(t)\\
x_2(t+1) &= s_1 x_1(t)\\
&\vdots\\
x_m(t+1) &= s_{m-1} x_{m-1}(t)
\end{aligned}
$$

$$
X(t+1) = \begin{pmatrix} x_1(t+1) \\ x_2(t+1) \\ \vdots \\ x_m(t+1) \end{pmatrix} = \begin{pmatrix} b_1 & b_2 & \dots & b_{m-1} & b_m \\ s_1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & s_{m-1} & 0 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_m(t) \end{pmatrix} = LX(t)
$$

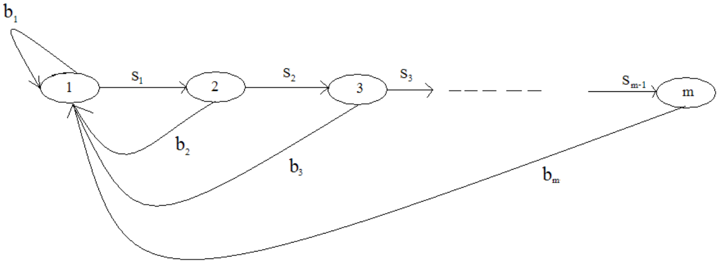
Leslie model

$$L = \begin{pmatrix} b_1 & b_2 & \dots & b_{m-1} & b_m \\ s_1 & 0 & \dots & 0 & 0 \\ 0 & s_2 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & s_{m-1} & 0 \end{pmatrix}$$

- L is called the Leslie matrix
- fertilities or fecundities on the first row
- survival probabilities on the subdiagonal
- all other entries in the Leslie matrix are zero.



Life cycle and projection matrix



Life cycle graph of the projection matrix on m age classes: each node represents each age group x_i, and arcs represent relation between two groups. An arrow connects the node j to i if the ij^{th} element in the projection matrix is nonzero.

Example: US population in 1964

Data:

$$b = (0, 0.001, 0.0878, 0.3487, 0.4761, 0.3377, 0.1833, 0.0761, 0.0174, 0.001)$$

$$s = (0.9966, 0.9983, 0.9979, 0.9968, 0.9961, 0.9947, 0.9923, 0.9987, 0.9831)$$

Perron-Frobenius Theorem

If M is a nonnegative primitive matrix, then:

- M has a positive eigenvalue λ_1 of maximum modulus.
- λ_1 is a simple root of the characteristic polynomial.
- For every other eigenvalue λ_i, $\lambda_1 > |\lambda_i|$ (it is strictly dominant).
- $$\min_i \sum_j m_{ij} \leq \lambda_1 \leq \max_i \sum_j m_{ij}$$
  $$\min_j \sum_i m_{ij} \leq \lambda_1 \leq \max_j \sum_i m_{ij}$$
- Row and column eigenvectors associated with λ_1 are strictly positive.
- The sequence M^t is asymptotically one-dimensional, its columns converge to the column eigenvector associated with λ_1; and its rows converges to the row eigenvector associated with λ_1.

Definition

A matrix A whose entries are nonnegative is called a **nonnegative matrix**, denoted $A \geq 0$.

Definition

A matrix A whose entries are positive is called a **positive matrix**, denoted $A > 0$.

Irreducible

Definition

If there exits a directed path from node i to j for every node i and j in the digraph, then the digraph is said to be **strongly connected**.

Theorem

The digraph of matrix A is strongly connected if and only if A is irreducible.

Primitivity

Definition (Primitivity)

If an irreducible, nonnegative matrix A has h eigenvalues $\lambda_1, \lambda_2, \dots \lambda_h$ of maximum modulus ($\rho(A) = |\lambda_i|, i = 1, 2, \dots, h$), then A is called **primitive** if $h = 1$ and **imprimitive** if $h > 1$. The value of h is called the **index of imprimitivity**.

The index of imprimitivity is the number of eigenvalues of matrix A with maximum modulus (with magnitude equal to $\rho(A)$).

Theorem

A nonnegative matrix A is primitive if and only if some power of A is positive (i.e. $A^p > 0$ for some integer $p \geq 1$).

Theorem (Berman and Plemmons, 1994)

An irreducible matrix is primitive if its trace is positive.

Fundamental Theorem of Demography

Suppose that the nonnegative matrix $P \geq 0$ is irreducible and primitive. Let λ_1 be the strictly dominant eigenvalue of P and $V_1 > 0$ be an associated eigenvector (called stable distribution). Let $x(t)$ be the solution of the linear matrix equation $x(t+1) = Px(t)$, $t \in [0, +\infty)$, with an initial state satisfying $0 \leq x(0) \neq 0$, and let $p(t) = |x(t)| = \sum_{i=1}^{n} x_i(t)$. Then

- $\displaystyle\lim_{t\to+\infty} \frac{x(t)}{p(t)} = \frac{V_1}{|V_1|}$
- $\displaystyle\lim_{t\to+\infty} p(t) = 0 \quad \text{if} \quad \lambda_1 < 1$, and $\displaystyle\lim_{t\to+\infty} p(t) = +\infty \quad \text{if} \quad \lambda_1 > 1$.

(The strictly dominant eigenvalue λ_1 is the (inherent) growth rate of the population)

$$x(t+1) = Px(t)$$

- λ_1 dominant eigenvalue of P
- **Stable stage distribution** V_1 (right eigenvector associated to λ_1)

  $$PV_1 = \lambda_1 V_1.$$

  For P, any initial population stage structure projected forward will approach the stable stage distribution V_1, where each stage class increases in size λ_1 times each time period.
- **Reproductive value of each stage** W_1 (left eigenvector associated to λ_1)

  $$W_1^T P = \lambda_1 W_1^T,$$

  These reproductive values estimate the expected reproductive contribution of each stage to population growth.
- Total population $p(t) = \sum_{i=1}^{n} x_i(t)$

  $$p(t) = \lambda_1^t p_0$$

  define $\lambda_1 = e^r$ then

  $$\frac{dp}{dt} = rp.$$

Outline

$1^{st}-$order scalar nonlinear equation

Cobwebbing is a graphical method to answer qualitative questions about the solution of

$$x_{t+1} = f(x_t).$$

In the $(x_t x_{t+1})-$plane, sketch $x_{t+1} = x_t$ and $x_{t+1} = f(x_t)$:

- any intersection of these graphs is an equilibrium solution of the difference equation.
- to investigate the behavior of the solutions
  - choose a starting value x_0, and begin at the point (x_0, x_0) in the $(x_t x_{t+1})-$plane.
  - draw a vertical line to the curve $x_{t+1} = f(x_t)$; this reaches the curve at $(x_0, f(x_0)) = (x_0, x_1)$.
  - draw a horizontal line to the diagonal $x_{t+1} = x_t$; this reaches the diagonal at the point (x_1, x_1).
  - Repeat the process to arrive at (x_2, x_2) and indefinitely until the behavior of the equation with this starting value becomes clear.
  - If necessary, repeat the process with other starting values.

Nonlinear systems of difference equations

Consider the system

$$\begin{aligned} x_{t+1} &= f(x_t, y_t) \\ y_{t+1} &= g(x_t, y_t), \end{aligned} \tag{4}$$

where f and g are nonlinear function. An equilibrium $(\bar{x}, \bar{y})$ to (4) satisfies the fixed point problem

$$\bar{x} = f(\bar{x}, \bar{y})$$
$$\bar{y} = g(\bar{x}, \bar{y}).$$

What is the stability of the equilibrium $(\bar{x}, \bar{y})$?

Local stability: linearization

Linearization of the nonlinear system (4) at the equilibrium $(\bar{x}, \bar{y})$ is

$$V_{t+1} = JV_t,$$

where $V_t = (u_t, v_t)^T$ (with small perturbations $u_t = x_t - \bar{x}$ and $v_t = y_t - \bar{y}$ about $(\bar{x}, \bar{y})$) and J is the Jacobian of $(f, g)^T$ evaluated at $(\bar{x}, \bar{y})$,

$$J = \begin{pmatrix} \frac{\partial f}{\partial x}\Big|_{\bar{x},\bar{y}} & \frac{\partial f}{\partial y}\Big|_{\bar{x},\bar{y}} \\ \frac{\partial g}{\partial x}\Big|_{\bar{x},\bar{y}} & \frac{\partial g}{\partial y}\Big|_{\bar{x},\bar{y}} \end{pmatrix}.$$

Eigenvalues of the Jacobian J determine the local stability of $(\bar{x}, \bar{y})$:

- if $|\lambda_i| < 1$ for $i = 1, 2$, i.e, if the spectral radius $\rho(J) < 1$, then $\lim_{t\to\infty} J^t = 0$. Therefore $\lim_{t\to\infty} V_t = 0$. This, in turn, implies that

$$\lim_{t\to\infty} x_t - \bar{x} = 0, \quad \lim_{t\to\infty} x_t = \bar{x}$$
$$\lim_{t\to\infty} y_t - \bar{y} = 0, \quad \lim_{t\to\infty} y_t = \bar{y}.$$

  $(\bar{x}, \bar{y})$ is locally asymptotically stable.
- If one of the eigenvalues has modulus larger than to 1, then V_t does not converge, implying that $(\bar{x}, \bar{y})$ is unstable.

Theorem: Condition for stability

Assume $\frac{df}{dx} = f'$ is continuous on an open interval $\mathcal{I}$ containing $\bar{x}$, and $\bar{x}$ is a fixed point of f. Then $\bar{x}$ is a locally asymptotically stable equilibrium of $x_{t+1} = f(x_t)$ if

$$\left|f'(\bar{x})\right| < 1$$

and unstable if

$$\left|f'(\bar{x})\right| > 1.$$

Another formulation of this result is as follows:

Theorem

Let f be a continuously differentiable function (C^1), and $\bar{x}$ be a fixed point of f.

1. If $|f'(\bar{x})| < 1$, then there is an open interval $\mathcal{I} \ni \bar{x}$ such that $\lim_{k\to\infty} f^k(x) = \bar{x}$ for all $x \in \mathcal{I}$.
2. If $|f'(\bar{x})| > 1$, then there is an open interval $\mathcal{I} \ni \bar{x}$ such that if $x \in \mathcal{I}$, $x \neq \bar{x}$, then there exists k such that $f^k(x) \notin \mathcal{I}$.

Theorem

Let $f(x, y)$ and $g(x, y)$ be two functions with continuous first-order partial derivatives in x and y on some set containing $(\bar{x}, \bar{y})$. Then the equilibrium $(\bar{x}, \bar{y})$ of the nonlinear system

$$\begin{aligned} x_{t+1} &= f(x_t, y_t) \\ y_{t+1} &= g(x_t, y_t) \end{aligned}$$

is locally asymptotically stable if the eigenvalues of the Jacobian matrix J evaluated at the equilibrium $(\bar{x}, \bar{y})$ satisfy $|\lambda_i| < 1$ if and only if

$$|\mathrm{tr}(J)| < 1 + \det(J) < 2.$$

The equilibrium is unstable if some $|\lambda_i| > 1$, that is, if any one of three inequalities is satisfied

- $\mathrm{tr}(J) > 1 + \det(J)$,
- or $\mathrm{tr}(J) < -1 - \det(J)$,
- or $\det(J) > 1$

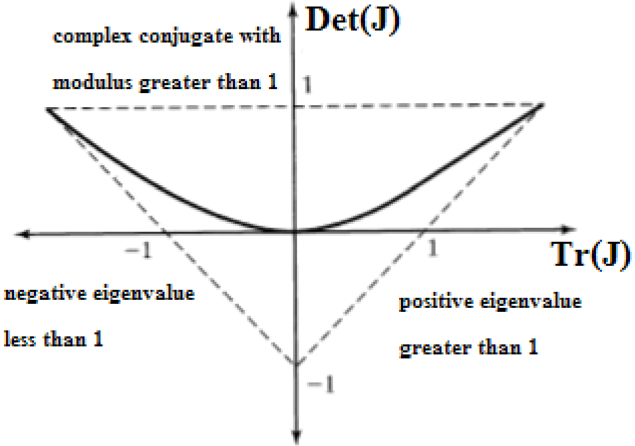


Figure: The triangular region inside the dashed lines is the region of local asymptotic stability ($|\lambda_i| < 1$) for the system of difference equations in the $\mathrm{tr}(J) - \det(J)-$plane. The solid curve represents $\mathrm{tr}(J)^2 = 4\det(J)$, below the curve the eigenvalues are real and above it the eigenvalues are complex conjugate. If the parameters lie outside of the triangular region then at least one eigenvalue satisfies $|\lambda_i| > 1$.

$n^{th}-$order criterium

Theorem

If the solutions λ_i, $i = 1, 2, \ldots, n$ of

$$p(\lambda) = \lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + a_3\lambda^{n-3} + \cdots + a_n = 0$$

satisfy $|\lambda_i| < 1$ then

- $p(1) > 0$
- $(-1)^n p(-1) > 0$
- $|a_n| < 1$

Jury conditions or Schur-Cohn Criteria, for $n = 3$

Consider the characteristic polynomial

$$p(\lambda) = \lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + a_3.$$

The solutions λ_i, $i = 1, 2, 3$, of $p(\lambda) = 0$ satisfy $|\lambda_i| < 1$ if and only if the following three conditions hold:

1. $p(1) = 1 + a_1 + a_2 + a_3 > 0$,
2. $(-1)^3 p(-1) = 1 - a_1 + a_2 - a_3 > 0$
3. $1 - (a_3)^2 > |a_2 - a_3 a_1|$