Toolbox to analyse Ordinary Differential Equation Models

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Outline

Linear / Nonlinear ODEs

$$\frac{dx}{dt} = f(t, x)$$

Definition

If the function f is a linear function of the dependent variable x then the ODE is said to be **linear**. Otherwise, the ODE is **nonlinear**.

Linear ODE: Homogeneous / Nonhomogeneous

$$\frac{dx}{dt} = p(t)x + g(t)$$

Definition

If the function $g(t)$ is zero for all t in the interval I, then the linear ODE is said to be **homogeneous**. Otherwise, the linear ODE is **nonhomogeneous**.

Autonomous equation

$$\frac{dx}{dt} = f(x)$$

Definition

If the independent variable t does not appear explicitly in the right-hand term $f(x)$, the ODE is **autonomous**. Otherwise, the ODE is **non-autonomous**.

nth Order Ordinary Differential Equation

Definition

Let $D \subset \mathbb{R}^{n+1}$ and $h \in C(D, \mathbb{R})$ (set of continuous functions from $D \to \mathbb{R}$). Use the notation $y^{(i)} = \frac{d^i y}{dt^i}$ and $y^{(0)} = y$.

$$y^{(n)} = h(t, y, y^{(1)}, \dots, y^{n-1}), \qquad (\mathrm{Y}_n)$$

Definition

Let $J = (a, b)$. A solution of the differential equation (Y_n) on J is $\varphi \in C^n(J, \mathbb{R})$ such that $(t, \varphi(t), \varphi^{(1)}(t), \dots, \varphi^{(n-1)}(t)) \in D$ and

$$\varphi^{(n)}(t) = h(t, \varphi(t), \varphi^{(1)}(t), \dots, \varphi^{(n-1)}(t)),$$

for all $t \in J$.

The theory of nth order ordinary differential equations actually reduces to the theory of systems of n first order ordinary differential equations.

To transform a n^{th} order equation into a system of n first order equations

$$y^{(n)} = F(t, y, y', y'', \dots, y^{(n-1)})$$

Define a change of variables

$$x_1 = y, \quad x_2 = y', \quad x_3 = y'', \quad \dots \quad x_n = y^{(n-1)}$$

then

$$\begin{cases} \frac{dx_1}{dt} = & x_2 \\ \frac{dx_2}{dt} = & x_3 \\ \frac{dx_3}{dt} = & x_4 \\ \vdots & \\ \frac{dx_{n-1}}{dt} = & x_n \\ \frac{dx_n}{dt} = & F(t, x_1, x_2, x_3, \dots, x_n) \end{cases}$$

IVP for *n*th Order Ordinary Differential Equation

Definition

Given $(\tau, \xi_1, \ldots, \xi_n) \in D$, the initial value problem for (Y_n) is given by

$$\begin{aligned} & y^{(n)} = h(t, y, y^{(1)}, \ldots, y^{(n-1)}) \\ & y^{(i-1)}(\tau) = \xi_i, \quad \text{for } i = 1, \ldots, n. \end{aligned} \tag{IY$_n$}$$

Definition

A function φ is a solution of (IY_n) if φ is a solution of (Y_n) on some interval J containing τ and also satisfies the initial conditions, $y^{(i-1)}(\tau) = \xi_i$ for $i = 1, \ldots, n$.

Different approaches to deal with initial value problems

1. Analytical methods - used to obtain the exact expression of solutions of a given equation
2. Qualitative methods - to investigate properties of solutions without necessarily finding those solutions (existence, uniqueness, stability, or chaotic or asymptotic behaviors)
3. Numerical methods - approximate, can be reasonably accurate. Yields approximations only locally on small intervals of the solution's domain

Outline

Well-posedness

- Existence of solutions
- Uniqueness of solutions
- Positivity of solutions

Existence and Uniqueness Theorem: general case

$$\begin{cases} \frac{dx_1}{dt} = & F_1(t, x_1, x_2, x_3, \ldots, x_n) \\ \frac{dx_2}{dt} = & F_2(t, x_1, x_2, x_3, \ldots, x_n) \\ \vdots & \\ \frac{dx_n}{dt} = & F_n(t, x_1, x_2, x_3, \ldots, x_n) \end{cases}$$

with $x_1(t_0) = x_1^0$, $x_2(t_0) = x_2^0$, $\ldots$, $x_n(t_0) = x_n^0$.

Theorem

Let each of the functions F_1, $\ldots$, F_n and the partial derivatives $\frac{\partial F_1}{\partial x_1}$, $\ldots$, $\frac{\partial F_1}{\partial x_n}$, $\ldots$, $\frac{\partial F_n}{\partial x_1}$, $\ldots$, $\frac{\partial F_n}{\partial x_n}$ be continuous in a region R of $tx_1x_2\cdots x_n-$space defined by $\alpha < t < \beta$, $\alpha_1 < x_1 < \beta_1$, $\ldots$, $\alpha_n < x_n < \beta_n$, and let the point $(t_0, x_1^0, \ldots, x_n^0)$ be in R. Then there is an interval $|t - t_0| < h$ in which there exists an unique solution $x_1 = \phi_1(t)$, $x_2 = \phi_2(t), \ldots x_n = \phi_n(t)$ of the system that also satisfies the initial conditions.

Existence and Uniqueness Theorem: linear ODEs

$$\frac{dx}{dt} = p(t)x + g(t)$$

with initial condition $x(t_0) = x_0$

Theorem (for linear ODEs)

If the functions p and g are continuous on an open interval I ($\alpha < t < \beta$) containing t_0, then there exists an unique solution $\phi(t)$ of the ODE that also satisfies the initial conditions. Moreover, the solution is defined on the whole interval I.

Outline

Methods to solve equations

- One equation of 1st order (scalar case)
  - Direct integration
  - Integrating factors for linear equations
  - Separable variables
  - Substitution methods (change of variables)
  - ..
- nth order linear equations or systems of n linear equations
  - with constant coefficients
  - ..

Integrating factors (To solve 1^{st} order linear equation with non-constant coefficients)

1. Put the DE in the standard form

$$\frac{dy}{dt} + p(t)y = g(t) \tag{1}$$

2. Determine the integrating factor $\mu(t)$
   - Multiply the DE (1) by an undetermied function $\mu(t)$

$$\mu(t)\frac{dy}{dt} + \mu(t)p(t)y = \mu(t)g(t) \tag{2}$$

   - State that the left side of (2) is equal to $\frac{d}{dt}(\mu(t)y)$

$$\frac{d}{dt}(\mu(t)y) = \mu(t)\frac{dy}{dt} + \underbrace{\frac{d\mu}{dt}}\, y = \mu(t)\frac{dy}{dt} + \underbrace{\mu(t)p(t)}\, y$$

   - Solve for $\mu(t)$

$$\frac{d\mu}{dt} = \mu(t)p(t) \quad \Rightarrow \quad \mu(t) = e^{\int p(t)dt}$$

3. Solve (2) for y with $\mu(t) = e^{\int p(t)dt}$

$$\left(\frac{d}{dt}e^{\int p(t)dt}y = \right) e^{\int p(t)dt}\frac{dy}{dt} + p(t)e^{\int p(t)dt}y = e^{\int p(t)dt}g(t) \quad \Rightarrow \quad \frac{d}{dt}\mu(t)y = \mu(t)g(t)$$

   Integrate with respect to t

$$\mu(t)y = \int \mu(t)g(t)dt + c$$

Hence the general solution of (1) is

$$y(t) = \frac{1}{\mu(t)}\left[\int_{t_0}^{t} \mu(s)g(s)ds + c\right] \quad with \quad \mu(t) = e^{\int p(t)dt}$$

Separable equations

Definition

A first order DE

$$\frac{dy}{dx} = f(x, y)$$

is said to be separable or to have separable variables if it can be expressed as follows

$$\frac{dy}{dx} = g(x)h(y).$$

(the vector field f can be expressed as a product of a function of the independent variable times a function of the dependent variable)

Method to solve separable equations $\frac{dy}{dx} = g(x)h(y)$

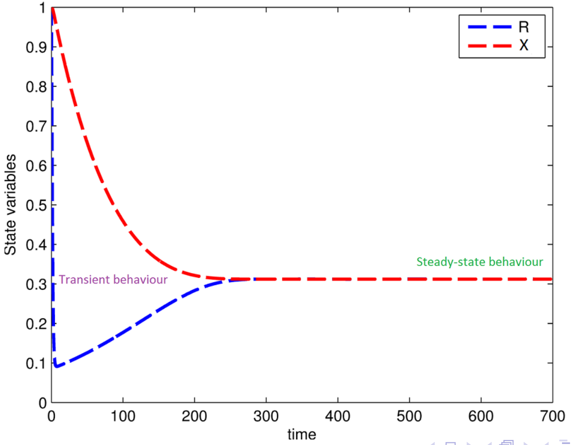
1. Express the separable equation as follows
$$\frac{1}{h(y(x))}\frac{dy}{dx} = g(x)$$
2. As y, $\frac{dy}{dx}$, and $g(x)$ are functions of x, integrate with respect to x
$$\int \frac{1}{h(y(x))}\frac{dy}{dx}dx = \int g(x)dx$$
3. Use the Change of variable Theorem [if $u = v(x)$, $\int f(v(x))v'(x)dx = \int f(u)du$] for the left side with $u = y(x)$
$$\int \frac{1}{h(u)}du = \int g(x)dx$$
$$\int \frac{1}{h(y)}dy = \int g(x)dx$$
4. Integrate
$$H(y) = G(x) + c \tag{3}$$

c is the combination of the left and right integration constants, H and G are antiderivatives of $\frac{1}{h(y)}$ and $g(x)$ respectively.

Outline

Dynamics of the model

- Transient behaviour: that leads from the initial state to the long-time behaviour
- Steady-state behaviour: in the long run (asymptotic), persistent operating state, steady oscillations..



Qualitative analysis: asymptotic behavior

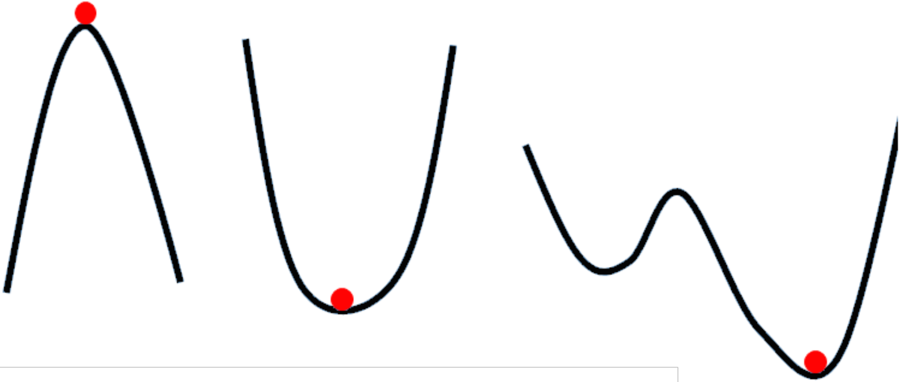
For autonomous problems:

$$\frac{dX}{dt} = f(X)$$

- Nondimensionalize the system (reduce the number of parameter)
- Find equilibria
- Characterize the nature of equilibria (stability analysis)
  - Local stability analysis (phase line analysis, phase plane analysis, linearization of the system near the equilibrium of interest..)
  - Global stability analysis (Poincaré-Bendixson, LaSalle's invariance principle, Lyapunov functions..)

Asymptotic behaviour of solutions - Equilibria and their nature

$$\frac{dN}{dt} = f(N, parameter), \quad N(t) = \cdots$$



Equilibria

Consider a nonlinear autonomous system

$$\begin{cases} \frac{dx_1}{dt} = & f_1(x_1, x_2, x_3, \ldots, x_n) \\ \frac{dx_2}{dt} = & f_2(x_1, x_2, x_3, \ldots, x_n) \\ \vdots & \\ \frac{dx_n}{dt} = & f_n(x_1, x_2, x_3, \ldots, x_n) \end{cases}$$

To find equilibria $(\bar{x}_1, \bar{x}_2, \bar{x}_3, \ldots, \bar{x}_n)^T$, solve

$$\begin{cases} 0 = & f_1(\bar{x}_1, \bar{x}_2, \bar{x}_3, \ldots, \bar{x}_n) \\ 0 = & f_2(\bar{x}_1, \bar{x}_2, \bar{x}_3, \ldots, \bar{x}_n) \\ \vdots & \\ 0 = & f_n(\bar{x}_1, \bar{x}_2, \bar{x}_3, \ldots, \bar{x}_n) \end{cases}$$

Stability (1/2)

Let define a system of n autonomous differential equations, $\frac{dY}{dt} = F(Y)$, where $Y = (y_1, \ldots, y_n)^T$ and $F(Y) = (f_1(y_1, \ldots, y_n), \ldots, f_n(y_1, \ldots, y_n))^T$, and F does not depend explicit on t.

(Locally stable)

An equilibrium solution Y^* of the system is said to be locally stable if for each $\epsilon > 0$ there exits a $\delta > 0$ such that every solution $Y(t)$ of the system with the initial condition $Y(t_0) = Y_0$,

$$\|Y_0 - Y^*\| < \delta,$$

satisfies the condition that

$$\|Y(t) - Y^*\| < \epsilon$$

for all $t \geq t_0$.
If the equilibrium solution is not locally stable it is said to be unstable.

Let $Y_1 = (y_1^1, y_2^1, \ldots, y_n^1)$ and $Y_2 = (y_1^2, y_2^2, \ldots, y_n^2)$ be 2 points in $\mathbb{R}^n$:

$$\|Y_1 - Y_2\| = \sqrt{\sum_{i=1}^{n} (y_i^1 - y_i^2)^2}.$$

Stability (2/2)

(Locally asymptotically stable)

An equilibrium solution Y^* of the system is said to be locally asymptotically stable if it is locally stable and if there exist $\gamma > 0$ such that $\|Y_0 - Y^*\| < \gamma$ implies

$$\lim_{t\to\infty} \|Y(t) - Y^*\| = 0.$$

Outline

$n^{th}-$dimensional nonhomogeneous linear system

Consider the initial value problem (IVP)

$$\frac{d}{dt}\mathbf{X}(t) = \mathbf{A}(t)\mathbf{X}(t) + \mathbf{B}(t), \quad \mathbf{X}(t_0) = \mathbf{X}_0.$$

The unique solution to the IVP can be expressed as

$$\mathbf{X}(t) = \mathbf{\Phi}(t)\mathbf{\Phi}^{-1}(t_0)\mathbf{X}_0 + \mathbf{\Phi}(t)\int_{t_0}^{t}\mathbf{\Phi}^{-1}(s)\mathbf{B}(s)ds$$

where $\mathbf{\Phi}(t)$ is a fundamental matrix of the corresponding homogeneous system

Definition

A **fundamental matrix** of solutions of the homogeneous system $\frac{d}{dt}\mathbf{X}(t) = \mathbf{A}(t)\mathbf{X}(t)$ is $\mathbf{\Phi}(\mathbf{t}) = (\mathbf{X}_1(t), \ldots, \mathbf{X}_n(t))$, where the columns of $\mathbf{\Phi}(t)$ are the n linearly independent solution vectors $\mathbf{X}_i(t)$.

Homogeneous linear system with constant coefficients

$$\frac{d}{dt}\mathbf{X}(t) = \mathbf{A}\mathbf{X}(t)$$

where $\mathbf{A} = (a_{ij})$ is a $n \times n$ constant matrix with real elements.

- If $\det(\mathbf{A}) \neq 0$, the unique equilibrium solution is $\mathbf{X}(t) = \mathbf{0}$, $\forall t \in \mathbb{R}$.
- The general solution is

$$\mathbf{X}(t) = e^{\mathbf{A}t}\mathbf{C}, \quad \forall t \in \mathbb{R}$$

  where $e^{\mathbf{A}t}$ (matrix exponential) is an $n \times n$ matrix, and $\mathbf{C}$ is a $n \times 1$ arbitrary constant vector.
- $\mathbf{\Phi}(t) = e^{\mathbf{A}t}$ is the fundamental matrix such as $\mathbf{\Phi}(0) = I_n$.
- $e^{\mathbf{A}t} = \mathbf{I_n} + \mathbf{A}t + \frac{t^2}{2!}\mathbf{A}^2 + \frac{t^3}{3!}\mathbf{A}^3 + \cdots = \sum_{i=0}^{\infty} \frac{t^i}{i!}\mathbf{A}^i, \quad \forall t \in \mathbb{R}.$

Homogeneous linear system with constant coefficients

$$\frac{d}{dt}\mathbf{X}(t) = \mathbf{A}\mathbf{X}(t)$$

where $\mathbf{A} = (a_{ij})$ is a $n \times n$ constant matrix with real elements.

Instead of computing $e^{\mathbf{A}t}$, we need to find n linearly independent solutions $\mathbf{X}_i(t)$ (to form a fundamental matrix)

- Let $\mathbf{X}_i(t) = e^{\lambda_i t}\mathbf{u_i}$ ($\lambda_i=$ unknown scalar, $\mathbf{u_i}$ = unknown $n \times 1-$vector).
- So $\mathbf{A}\mathbf{u_i} = \lambda_i \mathbf{u_i}$ where λ_i is an eigenvalue of $\mathbf{A}$ and $\mathbf{u_i}$ is an eigenvector associated to λ_i.
- To find λ_i ($i \in 1, \ldots, n$), solve the characteristic polynomial

$$\det(\mathbf{A} - \lambda \mathbf{I}_n) = 0.$$

- To find $\mathbf{u_i}$ ($i \in 1, \ldots, n$), solve $(\mathbf{A} - \lambda_i \mathbf{I_n})\,\mathbf{u_i} = \mathbf{0}$.

Homogeneous linear system with constant coefficients : n distinct eigenvalues

Theorem

Let $\lambda_1, \ldots, \lambda_n$ be n distinct eigenvalues of the coefficient matrix $\mathbf{A}$ of the homogeneous system

$$\frac{d}{dt}\mathbf{X} = \mathbf{A}\mathbf{X},$$

and let $\mathbf{u}_1, \ldots, \mathbf{u}_n$ be the corresponding eigenvectors. Then the general solution of the homogeneous system on the interval $(-\infty, \infty)$ is given by

$$\mathbf{X}(t) = c_1\mathbf{u}_1 e^{\lambda_1 t} + \cdots + c_n\mathbf{u}_n e^{\lambda_n t}$$

with $c_1, \ldots, c_n$ arbitrary constants.

Complex conjugate eigenvalues

Theorem

Let $\mathbf{A}$ be the coefficient matrix having real entries of the homogeneous system $\frac{d}{dt}\mathbf{X} = \mathbf{A}\mathbf{X}$, and let $\mathbf{u}_1$ be an eigenvector corresponding to the complex eigenvalue $\lambda_1 = \alpha + i\beta$, with α and β real. Then,

$$\mathbf{X}_1(t) = \mathbf{u}_1 e^{\lambda_1 t}, \qquad \mathbf{X}_2(t) = \bar{\mathbf{u}}_1 e^{\bar{\lambda_1} t}$$

are solutions of the homogeneous system.

Theorem

Let $\lambda_1 = \alpha + i\beta$ be a complex eigenvalue of the coefficient matrix having real entries of the homogeneous system $\frac{d}{dt}\mathbf{X} = \mathbf{A}\mathbf{X}$, and let $\mathbf{u}_1 = \mathbf{a} + i\mathbf{b}$ be an eigenvector corresponding to the complex eigenvalue λ_1. Then

$$\mathbf{X}_1(t) = \left(\mathbf{a}\cos(\beta t) - \mathbf{b}\sin(\beta t)\right) e^{\alpha t}, \quad \mathbf{X}_2(t) = \left(\mathbf{b}\cos(\beta t) + \mathbf{a}\sin(\beta t)\right) e^{\alpha t}$$

are linearly independent solutions of the homogeneous system on $\mathbb{R}$.

Asymptotical behavior of solutions

$$\frac{d\mathbf{X}}{dt} = \mathbf{AX}$$

Theorem

If all the roots of the characteristic polynomial $P(\lambda) = \det(\mathbf{A} - \lambda \mathbf{I}_n) = 0$ (or the eigenvalues of $\mathbf{A}$) are negative or have negative real part, then given any solution $\mathbf{X}(t)$ of $\frac{d\mathbf{X}}{dt} = \mathbf{AX}$, there exist positive constant M and positive constant b such that

$$||\mathbf{X}(t)|| \leq Me^{-bt}, \quad \forall t > 0$$

and

$$\lim_{t \to \infty} ||\mathbf{X}(t)|| = 0.$$

$\frac{d\mathbf{X}}{dt} = \mathbf{AX}$ with $\mathbf{A}$ a 2×2 constant matrix. The characteristic polynomial of $\mathbf{A}$ is $\lambda^2 - \mathrm{tr}\mathbf{A}\lambda + \det\mathbf{A} = 0$.

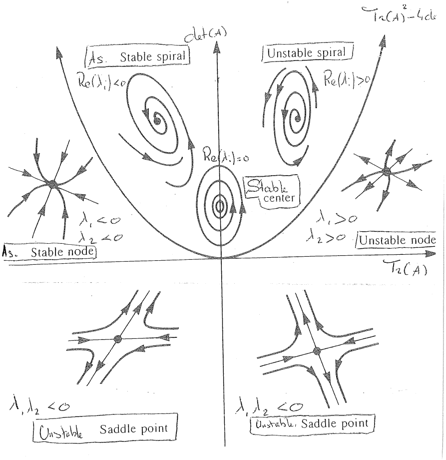
Theorem

- if $\det\mathbf{A} > 0$ and $\mathrm{tr}\mathbf{A}^2 - 4\det\mathbf{A} \geq 0$ then the origin is a **node** (real eigenvalues having the same signs)
  - an **asymptotic stable node** if $\mathrm{tr}\mathbf{A} < 0$ (real $\lambda_{1,2} < 0$),
  - an **unstable node** if $\mathrm{tr}\mathbf{A} > 0$ (real $\lambda_{1,2} > 0$).
- if $\det\mathbf{A} < 0$ then the origin is a **saddle point** (real eigenvalues have opposite signs, $\lambda_1\lambda_2 < 0$). It is an **unstable point**.
- if $\det\mathbf{A} > 0$ and $\mathrm{tr}\mathbf{A}^2 - 4\det\mathbf{A} < 0$ and $\mathrm{tr}\mathbf{A} \neq 0$, the origin is a **spiral point** (complex conjugate with nonzero real part)
  - an **asymptotic stable point** if $\mathrm{tr}\mathbf{A} < 0$ (negative real part)
  - an **unstable point** if $\mathrm{tr}\mathbf{A} > 0$ (positive real part).
- if $\det\mathbf{A} > 0$ and $\mathrm{tr}\mathbf{A} = 0$ then the origin is a **center** (purely imaginary eigenvalues $\lambda_{1,2} = \pm i\beta$). It is a **stable point**.

Nature of eigenvalues for a 2×2 matrix

$\dot{X} = AX$ $\quad A$ = 2x2 real matrix

λ_1, λ_2 eigenvalues of A



Stability

Theorem

Suppose $\frac{d\mathbf{X}}{dt} = \mathbf{A}\mathbf{X}$ where $\mathbf{A}$ is a 2×2 constant matrix with $\det\mathbf{A} \neq 0$.
The origin is asymptotically stable iff

$$\mathrm{tr}\mathbf{A} < 0 \quad \textit{and} \quad \det\mathbf{A} > 0.$$

The orign is stable iff

$$\mathrm{tr}\mathbf{A} \leq 0 \quad \textit{and} \quad \det\mathbf{A} > 0.$$

The origin is unstable iff

$$\mathrm{tr}\mathbf{A} > 0 \quad \textit{or} \quad \det\mathbf{A} < 0.$$

Real distinct eigenvalues of the same sign

$\frac{d\mathbf{X}}{dt} = \mathbf{AX}$ with $\mathbf{A}$ a 2×2 constant matrix.

When the eigenvalues λ_1 and λ_2 are both positive or both negative, the general solution
$\mathbf{X}(t) = c_1\xi_1 e^{\lambda_1 t} + c_2\xi_2 e^{\lambda_2 t}, \quad t \in \mathbb{R}$,
with ξ_1 (ξ_2) eigenvector corresponding to λ_1 (λ_2)

- $\lambda_1 < \lambda_2 < 0$: as $t \to \infty$, trajectories approach to the origin with the direction of the eigenvector corresponding to the $\max_{i=1,2} Re(\lambda_i) = \lambda_2$ (the origin is an **asymptotic stable node**)
- $\lambda_1 > \lambda_2 > 0$: as $t \to \infty$, trajectories flow away from the origin with the direction of the eigenvector corresponding to the $\max_{i=1,2} Re(\lambda_i) = \lambda_1$ (the origin is an **unstable node**)

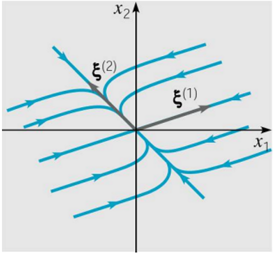


Figure: $\lambda_1 < \lambda_2 < 0$

Real distinct eigenvalues of opposite sign

$\frac{d\mathbf{X}}{dt} = \mathbf{AX}$ with $\mathbf{A}$ a 2×2 constant matrix.

When the eigenvalues λ_1 and λ_2 have opposite signs, the general solution
$\mathbf{X}(t) = c_1\xi_1 e^{\lambda_1 t} + c_2\xi_2 e^{\lambda_2 t}, \quad t \in \mathbb{R}$,
with ξ_1 (ξ_2) eigenvector corresponding to λ_1 (λ_2)

- $\lambda_2 < 0 < \lambda_1$: the positive exponential term is dominant for large t, so all solutions approach infinity asymptotic to the line determined by the eigenvector ξ_1 corresponding to $\max_{i=1,2} Re(\lambda_i) = \lambda_1 > 0$.
- the origin is a **saddle point**, it is **unstable**.

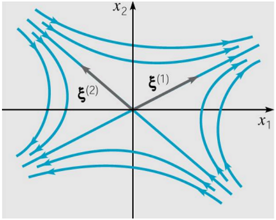


Figure: $\lambda_2 < 0 < \lambda_1$

Real repeated eigenvalues 1/2

$\frac{d\mathbf{X}}{dt} = \mathbf{A}\mathbf{X}$ with $\mathbf{A}$ a 2×2 constant matrix.

When the eigenvalues λ_1 and λ_2 are equal and there exist 2 linearly independent eigenvectors, the general solution
$\mathbf{X}(t) = c_1\xi_1 e^{\lambda_1 t} + c_2\xi_2 e^{\lambda_1 t}, \quad t \in \mathbb{R},$
with ξ_1 (ξ_2) eigenvector corresponding to λ_1 (λ_2)

- Trajectories lies on straight line through the origin.
- $\lambda_1 = \lambda_2 > 0$: trajectories flow away from the origin. The origin is **unstable proper node**.
- $\lambda_1 = \lambda_2 < 0$: trajectories flow in the origin. The origin is **asymptotic stable proper node**.

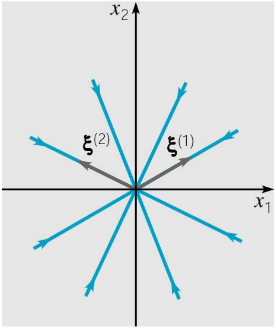


Figure: $\lambda_2 = \lambda_1 < 0$

Real repeated eigenvalues 2/2

$\frac{d\mathbf{X}}{dt} = \mathbf{A}\mathbf{X}$ with $\mathbf{A}$ a 2×2 constant matrix.

When $\lambda_1 = \lambda_2$ and only one eigenvector ξ, general solution
$\mathbf{X}(t) = c_1 \xi e^{\lambda_1 t} + c_2 \left(\xi t e^{\lambda_1 t} + \eta e^{\lambda_1 t} \right),\ t \in \mathbb{R}$,
with η generalized eigenvector ($(\mathbf{A} - \lambda_1 \mathbf{I})\eta = \xi$).

- $t \to +\infty$, the dominant term is $c_2 \xi t e^{\lambda_1 t}$
- Orientation of trajectories depends on the relative positions of ξ and η
  $\mathbf{X} = (c_2 \xi t + c_2 \eta + c_1 \xi)\, e^{\lambda_1 t}$
  - $c_2 \xi t + c_2 \eta + c_1 \xi$ determines the direction of trajectories
  - $e^{\lambda_1 t}$ determines the magnitude of trajectories.
- $\lambda_1 = \lambda_2 < 0$: Trajectories approach origin tangent to line through the eigenvector ξ
- The origin is an **improper node**

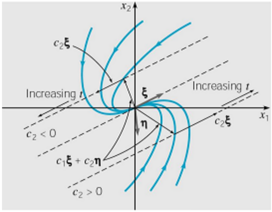


Figure: $\lambda_2 = \lambda_1 < 0$

Complex conjugate eigenvalues

$\frac{d\mathbf{X}}{dt} = \mathbf{AX}$ with $\mathbf{A}$ a 2×2 constant matrix.

When the eigenvalues λ_1 and λ_2 are complex conjugate, the general solution

$$\mathbf{X}(t) = c_1 e^{Re(\lambda_1)t} \left(Re(\xi_1)\cos(Im(\lambda_1)t) - Im(\xi_1)\sin(Im(\lambda_1)t)\right) +$$
$$c_2 e^{Re(\lambda_1)t} \left(Im(\xi_1)\cos(Im(\lambda_1)t) + Re(\xi_1)\sin(Im(\lambda_1)t)\right), \quad t \in \mathbb{R}$$

with ξ_1 eigenvector corresponding to λ_1.

- the trajectories are **spirals**, which approach or recede from the origin depending on the sign of $Re(\lambda_1)$.
- Stability of spiral points depends on the sign of $Re(\lambda_1)$.

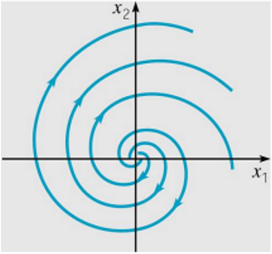


Figure: $Re(\lambda_1) > 0$

Purely imaginary eigenvalues

$\frac{d\mathbf{X}}{dt} = \mathbf{AX}$ with $\mathbf{A}$ a 2×2 constant matrix.

When the eigenvalues $\lambda_{1,2} = \pm i\beta$ are purely imaginary, the general solution

$\mathbf{X}(t) = c_1 \left(Re(\xi_1) \cos(\beta t) - Im(\xi_1) \sin(\beta t) \right) +$

$c_2 \left(Im(\xi_1) \cos(\beta t) + Re(\xi_1) \sin(\beta t) \right), \quad t \in \mathbb{R}$

with ξ_1 eigenvector corresponding to λ_1.

- Trajectories are circles or ellipses (**closed curves**) centered at the origin.
- Solutions are **periodic**.
- The origin is called a **center**; it is **stable**.

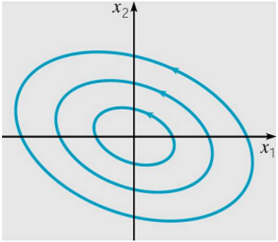


Figure: $Re(\lambda_1) = 0$

Outline

Phase Plane analysis

To study the qualitative behavior of a system without solving it.

$$\begin{aligned} \frac{dx}{dt} &= f(x,y), \\ \frac{dy}{dt} &= g(x,y). \end{aligned}$$

At any point (x, y) of the $xy-$plane (called the **Phase-Plane**),

$$\frac{dy}{dx} = \frac{g(x,y)}{f(x,y)}$$

is the slope of the trajectory $(x(t), y(t))$ in the $xy-$plane, and the tangent vector that gives the direction of the trajectory is $(f(x,y), g(x,y))$. The collection of vectors evaluated at any point of the $xy-$plane defines the **direction field**.

Determination of trajectories in the $xy-$plane

$$\left.\begin{array}{rl}\frac{dx}{dt} = & f(x,y) \\ \frac{dy}{dt} = & g(x,y)\end{array}\right\} \Rightarrow \frac{dy}{dx} = \frac{g(x,y)}{f(x,y)}$$

If $\frac{dy}{dx} = \frac{g(x,y)}{f(x,y)}$ can be solved, and if the implicit solution can be written as

$$H(x,y) = c,$$

then $H(x,y) = c$ is an equation for the trajectories for the system. Trajectories lie on the level curves of $H(x,y)$.

Nullclines (1/2)

$$\begin{aligned} \frac{dx}{dt} &= f(x,y) \\ \frac{dy}{dt} &= g(x,y) \end{aligned}$$

$x-$nullcline

The $x-$nullcline for the system is the set of all points in the $xy-$plane satisfying $f(x,y) = 0$.

$y-$nullcline

The $y-$nullcline for the system is the set of all points in the $xy-$plane satisfying $g(x,y) = 0$.

Equilibrium point

At any intersection of $x-$nullcline and $y-$nullcline, there is an equilibrium point.

Nullclines (2/2)

$$\begin{aligned} \frac{dx}{dt} &= f(x,y) \\ \frac{dy}{dt} &= g(x,y) \end{aligned}$$

- On the $x-$nullcline, all vectors are vertical.
- On the $y-$nullcline, all vectors are horizontal.
- We need to check if the direction of flow is up or down on the $x-$nullcline.
- We need to ckeck if the direction of flow is left or right on the $y-$nullcline.

Local stability: linearization

Consider a nonlinear autonomous system

$$\begin{cases} \frac{dx_1}{dt} = & f_1(x_1, x_2, x_3, \ldots, x_n) \\ \frac{dx_2}{dt} = & f_2(x_1, x_2, x_3, \ldots, x_n) \\ \vdots & \\ \frac{dx_n}{dt} = & f_n(x_1, x_2, x_3, \ldots, x_n) \end{cases}$$

with an equilibrium $(\bar{x}_1, \bar{x}_2, \bar{x}_3, \ldots, \bar{x}_n)^T$. Consider

$$\begin{cases} u_1(t) = & x_1(t) - \bar{x}_1 \\ u_2(t) = & x_2(t) - \bar{x}_2 \\ \vdots & \\ u_n(t) = & x_n(t) - \bar{x}_n \end{cases}$$

With $\frac{d\mathbf{u}}{dt} = \frac{d\mathbf{x}}{dt}$ and expanding $\mathbf{f}$ about $\bar{\mathbf{x}}$ using Taylor's formula $\Rightarrow$ Linearization of system

Taylor's expansion of $f(x,y)$ about $(\bar{x},\bar{y})$

assuming that $f(x,y)$ has continuous second-order partial derivatives in an open set containing $(\bar{x},\bar{y})$

$$\begin{aligned} f(x,y) =& f(\bar{x},\bar{y}) + \frac{\partial f}{\partial x}(\bar{x},\bar{y})(x-\bar{x}) + \frac{\partial f}{\partial y}(\bar{x},\bar{y})(y-\bar{y}) \\ &+ \frac{\partial^2 f}{\partial x^2}(\bar{x},\bar{y})\frac{(x-\bar{x})^2}{2} + \frac{\partial^2 f}{\partial x \partial y}(\bar{x},\bar{y})(x-\bar{x})(y-\bar{y}) \\ &+ \frac{\partial^2 f}{\partial y^2}(\bar{x},\bar{y})\frac{(y-\bar{y})^2}{2} + \dots \end{aligned}$$

Linearization

Consider a nonlinear autonomous system

$$\begin{cases} \frac{dx_1}{dt} = & f_1(x_1, x_2, x_3, \dots, x_n) \\ \frac{dx_2}{dt} = & f_2(x_1, x_2, x_3, \dots, x_n) \\ \vdots & \\ \frac{dx_n}{dt} = & f_n(x_1, x_2, x_3, \dots, x_n) \end{cases}$$

with an equilibrium $(\bar{x}_1, \bar{x}_2, \bar{x}_3, \dots, \bar{x}_n)^T$.
The linearized system about $(\bar{x}_1, \bar{x}_2, \bar{x}_3, \dots, \bar{x}_n)^T$ is

$$\frac{d}{dt}\begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix} = \mathbf{J}_{(\bar{x}_1, \dots, \bar{x}_n)} \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix}.$$

where $\mathbf{J}_{(\bar{x}_1, \dots, \bar{x}_n)}$ is the Jacobian matrix evaluated at the equilibrium.

Jacobian

Consider a nonlinear autonomous system

$$\begin{cases} \frac{dx_1}{dt} = & f_1(x_1, x_2, x_3, \dots, x_n) \\ \frac{dx_2}{dt} = & f_2(x_1, x_2, x_3, \dots, x_n) \\ \vdots & \\ \frac{dx_n}{dt} = & f_n(x_1, x_2, x_3, \dots, x_n) \end{cases}$$

For this system, the Jacobian matrix $\mathbf{J}_{(x_1,\dots,x_n)}$ evaluated at $(x_1, \dots, x_n)$ is

$$\mathbf{J}_{(x_1,\dots,x_n)} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(x_1, \dots, x_n) & \frac{\partial f_1}{\partial x_2}(x_1, \dots, x_n) & \dots & \frac{\partial f_1}{\partial x_n}(x_1, \dots, x_n) \\ \frac{\partial f_2}{\partial x_1}(x_1, \dots, x_n) & \frac{\partial f_2}{\partial x_2}(x_1, \dots, x_n) & \dots & \frac{\partial f_2}{\partial x_n}(x_1, \dots, x_n) \\ \vdots & \vdots & \dots & \vdots \\ \frac{\partial f_n}{\partial x_1}(x_1, \dots, x_n) & \frac{\partial f_n}{\partial x_2}(x_1, \dots, x_n) & \dots & \frac{\partial f_n}{\partial x_n}(x_1, \dots, x_n) \end{pmatrix}$$

Nonlinear systems

$$\frac{d\mathbf{X}}{dt} = \mathbf{f}(\mathbf{X})$$

Hartman-Grobman Theorem

Assume that $\overline{\mathbf{X}}$ is a hyperbolic (all eigenvalues of the Jacobian matrix evaluated at $\overline{\mathbf{X}}$ have nonzero real part) equilibrium. Then, in a small neighborhood of $\overline{\mathbf{X}}$, the nonlinear system behaves in a similar manner as the linearized system.

Planar systems

$$\frac{dx}{dt} = f(x, y)$$
$$\frac{dy}{dt} = g(x, y)$$

Theorem

Assume the first order partial derivatives of f and g are continuous in some open set containing the equilibrium $(\bar{x}, \bar{y})$ of the system. Then the equilibrium is locally asymptotically stable if

$$\mathrm{tr}(J_{(\bar{x},\bar{y})}) < 0 \quad \text{and} \quad \det(J_{(\bar{x},\bar{y})}) > 0,$$

where $J_{(\bar{x},\bar{y})}$ is the Jacobian matrix evaluated at the equilibrium. In addition, the equilibrium is unstable if either $\mathrm{tr}(J_{(\bar{x},\bar{y})}) > 0$ or $\det(J_{(\bar{x},\bar{y})}) > 0$.

Nonlinear systems

$$\frac{d\mathbf{X}}{dt} = \mathbf{f}(\mathbf{X})$$

Theorem

If all eigenvalues of the Jacobian matrix evaluated at the equilibrium have negative real part, then the equilibrium is locally asymptotically stable.

Tools to determine the sign of eigenvalues

Descartes' rule of signs

Let $p(x) = \sum_{i=0}^{m} a_i x^i$ be a polynomial with real coefficients such that $a_m \neq 0$. Define v to be the number of *variations in sign* of the sequence of coefficients $a_m, \ldots, a_0$. By 'variations in sign' we mean the number of values of n such that the sign of a_n differs from the sign of a_{n-1}, as n ranges from m down to 1. Then

- the number of positive real roots of $p(x)$ is $v - 2N$ for some integer N satisfying $0 \leq N \leq \dfrac{v}{2}$,
- the number of negative roots of $p(x)$ may be obtained by the same method by applying the rule of signs to $p(-x)$.

Tools to determine the sign of eigenvalues

Routh-Hurwitz Criteria

Given the polynomial,

$$P(\lambda) = \lambda^n + a_1\lambda^{n-1} + \cdots + a_{n-1}\lambda + a_n$$

where the coefficients a_i are real constants, $i = 1, \ldots, n$ define the n Hurwitz matrices using the coefficients a_i of the characteristic polynomial:

$$H_1 = (a_1), \quad H_2 = \begin{pmatrix} a_1 & 1 \\ a_3 & a_2 \end{pmatrix}, \quad H_3 = \begin{pmatrix} a_1 & 1 & 0 \\ a_3 & a_2 & a_1 \\ a_5 & a_4 & a_3 \end{pmatrix}, \cdots$$

and

$$H_n = \begin{pmatrix} a_1 & 1 & 0 & 0 & \ldots & 0 \\ a_3 & a_2 & a_1 & 1 & \ldots & 0 \\ a_5 & a_4 & a_3 & a_2 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & a_n \end{pmatrix}$$

where $a_j = 0$ if $j > n$. All of the roots of the polynomial $P(\lambda)$ are negative or have negative real part if and only if the determinants of all Hurwitz matrices are positive:

$$\mathrm{det} H_i > 0, \quad j = 1, 2, \ldots, n.$$

Tools to determine the sign of eigenvalues

Corollary

Routh-Hurwitz criteria for $n = 2, 3, 4, 5$

- $n = 2 : a_1 > 0$ and $a_2 > 0$.
- $n = 3 : a_1 > 0,\ a_3 > 0$ and $a_1 a_2 > a_3$.
- $n = 4 : a_1 > 0,\ a_3 > 0,\ a_4 > 0$ and $a_1 a_2 a_3 > a_3^2 + a_1^2 a_4$.
- $n = 5 : a_i > 0,\ i = 1, 2, 3, 4, 5,\ a_1 a_2 a_3 > a_3^2 + a_1^2 a_4$ and $(a_1 a_4 - a_5)(a_1 a_2 a_3 - a_3^2 - a_1^2 a_4) > a_5(a_1 a_2 - a_3)^2 + a_1 a_5^2$

Tools to determine the sign of eigenvalues

Gerhgorin's Theorem

Let A be an $n \times n$ matrix. Let D_i be the disk in the complex plane with the center at a_{ii} and radius $r_i = \sum_{j=1, j\neq i}^{n} |a_{ij}|$. Then all eigenvalues of the matrix A lie in the union of the disks D_i, $i = 1, 2, \ldots, n$, $\cup_{i=1}^{n} D_i$. In particular, if λ_i is an eigenvalue of A, then for some $i = 1, 2, \ldots, n$

$$|\lambda_i - a_{ii}| \leq r_i.$$

Corollary

Let A be an $n \times n$ matrix with real entries. If the diagonal elements of A satisfy

$$a_{ii} < -r_i \quad where \quad r_i = \sum_{j=1, j\neq i}^{n} |a_{ij}|$$

for $i = 1, 2, \ldots, n$ then the eigenvalues of A are negative or have negative real part.

Global stability: Planar systems

$$\frac{dx}{dt} = f(x, y)$$
$$\frac{dy}{dt} = g(x, y)$$

with the initial conditions $X_0 = (x(t_0), y(t_0)) = (x_0, y_0)$.

$\Rightarrow$ Poincaré-Bendixson Theorem (for global stability analysis)

Planar systems

$$\frac{dx}{dt} = f(x, y)$$
$$\frac{dy}{dt} = g(x, y)$$

with initial conditions $X_0 = (x(t_0), y(t_0))^T = (x_0, y_0)^T$.

- $\Gamma(X_0, t)$: solution trajectory (as a function of time) starting at X_0
- $\Gamma^+(X_0, t)$: part of solution trajectory where $t \geq t_0$ (positive orbit)
- $\Gamma^-(X_0, t)$: part of solution trajectory where $t \leq t_0$ (negative orbit)
- $\alpha-$limit set, $\alpha(X_0)$: set of points in the plane that are approached by the negative orbit $\Gamma^-(X_0, t)$, as $t \to -\infty$
- $\omega-$limit set, $\omega(X_0)$: set of points in the plane that are approached by the positive orbit $\Gamma^+(X_0, t)$, as $t \to +\infty$

Definition

A periodic solution $\mathbf{X}(t)$ of $\frac{d\mathbf{X}}{dt} = \mathbf{f}(\mathbf{X})$ is a non-constant solution satisfying $\mathbf{X}(t+T) = \mathbf{X}(t)$ for all t on the interval of existence ($T > 0$ is called the period).

(No periodic solutions in autonomous scalar differential equations)

Definition

A limit cycle is the orbit of an isolated periodic solution.

Existence of periodic solutions

Poincaré-Bendixson theorem

Let $\Gamma^+(X_0, t)$ be a positive orbit of

$$\begin{aligned}\frac{dx}{dt} &= f(x,y)\\ \frac{dy}{dt} &= g(x,y)\end{aligned}$$

that remains in a closed and bounded region of the plane. Suppose that the $\omega-$limit set does not contain any equilibria. Then either

- $\Gamma^+(X_0, t)$ is a periodic orbit ($\Gamma^+(X_0, t) = \omega(X_0)$),
- or $\omega-$limit set, $\omega(X_0)$, is a periodic orbit.

Theorem

Every periodic orbit (closed orbit) must enclose an equilibrium (has an equilibrium in its interior).

Poincaré-Bendixson trichotomy

Let $\Gamma^+(X_0, t)$ be a positive orbit of

$$\frac{dx}{dt} = f(x,y)$$
$$\frac{dy}{dt} = g(x,y)$$

that remains in a closed and bounded region B of the plane. Suppose B contains only a finite number of equilibria. Then the $\omega-$limit set takes ones of the following 3 forms:

- $\omega(X_0)$ is an equilibrium,
- $\omega(X_0)$ is a periodic orbit,
- $\omega(X_0)$ (cycle graph) contains a finite number of equilibria and a set of trajectories Γ_i whose $\alpha-$ and $\omega-$limit sets consist of one of these equilibria for each trajectory Γ_i.

Dulac's criterion

Suppose D is a simply connected open subset of the plane and $\beta(x, y)$ is a real-valued continuously differentiable function in D. If

$$\frac{\partial(\beta f)}{\partial x} + \frac{\partial(\beta g)}{\partial y}$$

is not identically zero and does not change sign in D, then there is no periodic solutions of the autonomous system

$$\begin{aligned}\frac{dx}{dt} &= f(x, y)\\ \frac{dy}{dt} &= g(x, y)\end{aligned}$$

in D.

Definition

A region D of the plane is said to be simply connected if every closed loop within D can be shrunk to a point without leaving D.

Bendixson's criterion

Suppose D is a simply connected open subset of the plane. If

$$\frac{\partial f}{\partial x}+\frac{\partial g}{\partial y}$$

is not identically zero and does not change sign in D, then there is no periodic solutions of the autonomous system

$$\begin{aligned}\frac{dx}{dt} &= f(x,y)\\ \frac{dy}{dt} &= g(x,y)\end{aligned}$$

in D.

Outline

Transient behaviour

Resilience represents the minimum return rate to a stable equilibrium (or the inverse of the average time to return to a stable equilibrium)

Reactivity quantifies the transient behaviour of a system in response to a perturbation from a stable equilibrium (the maximum amplification rate, over all initial perturbations, immediately following the perturbation)

Neubert and Caswell, Ecology 78 (1997). Hastings *et al. Science 361 (2018). Lutscher and Wang J. Theor. Biol. 493 (2020)*

Resilience

Consider

$$\frac{dX}{dt} = AX$$

where $\det(A) \neq 0$ and the eigenvalues λ of A satisfied $\mathrm{Re}(\lambda) < 0$ (there is a unique equilibrium which is asymptotically stable).

$$\text{resilience} := -\max\left\{\mathrm{Re}(\lambda) : \det\left(\lambda I_{n\times n} - A\right) = 0\right\} > 0,$$

$1/\text{resilience}$ is the average time to return to the stable equilibrium

Reactivity

Consider

$$\frac{dX}{dt} = AX$$

where $\det(A) \neq 0$ and the eigenvalues λ of A satisfied $\mathrm{Re}(\lambda) < 0$ (there is a unique equilibrium which is asymptotically stable).

$$\text{reactivity} := \max\left\{\lambda : \det\left(\lambda I_{n\times n} - H(A)\right) = 0\right\},$$

with $H(A) = (A + A^T)/2$ where A^T represents the transpose matrix of A. Notice that $H(A)$ is a symmetric matrix and all its eigenvalues are real.

A system with a positive reactivity is said to be reactive and is sensitive to any changes in initial conditions. A small change in the initial conditions might cause drastic changes in the early dynamics before the system returns to the equilibrium.