MATH 1300
Definitions and Results

The following are the definitions, theorems and other results you should know for the final examination of Math 1300. Additionally, the proofs of theorems indicated by **<u>Theorem</u>** should be known. Note that this does not mean that you are not responsible for the remaining material in the course, simply that course questions can only be relative to results that are stated here.

Notation

In this document, the following (standard) notation are sometimes used:

- $\forall$ means "for all".
- $\exists$ means "there exists".
- $\mathbb{N}$ is the set of natural integers.
- $\mathbb{R}$ is the set of real numbers.
- $x \in \mathbb{R}$ means "x belongs to $\mathbb{R}$".
- $X \subset Y$ means "X is a subset of Y".
- $\mathcal{M}_{mn}$ is the set of $m \times n$-matrices, $\mathcal{M}_{nn}$ is the set of $n \times n$ (square) matrices. $A \in \mathcal{M}_{nn}$ means "the square $n \times n$-matrix A", $A \in \mathcal{M}_{mn}$ means "the $m \times n$-matrix A".
- $u \perp v$ indicates that the vectors u and v are orthogonal.
- iff means "if and only if"

1 Systems of linear equations and matrices

Definition 1. *We define a* ***linear equation*** *in the* n *variables* $x_1, x_2, \ldots, x_n$ *to be one that can be expressed in the form*

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b,$$

where $a_1, a_2, \ldots, a_n$ *and* b *are real constants. The variables in a linear equation are sometimes called* ***unknowns****.*

Definition 2. *A* ***solution*** *of a linear equation* $a_1x_1+a_2x_2+\cdots+a_nx_n = b$ *is a sequence of* n *numbers* $s_1, \ldots, s_n$ *such that the equation is satisfied when we substitute* $x_1 = s_1, x_2 = s_2, \ldots, x_n = s_n$*. The set of all solutions of the equation is called its* ***solution set*** *or sometimes the* ***general solution*** *of the equation.*

Definition 3. *A finite set of linear equations in the variables* $x_1, \ldots, x_n$ *is called a* ***system of linear equations*** *or a* ***linear system****. A sequence of numbers* $s_1, \ldots, s_n$ *is called a* ***solution*** *of the system if* $x_1 = s_1, \ldots, x_n = s_n$ *is a solution of every equation in the system.*

Definition 4. *A system of equations that has no solutions is said to be* ***inconsistent****; if there is at least one solution of the system, it is called* ***consistent****.*

Theorem 5. *Every system of linear equations has no solutions, or has exactly one solution, or has infinitely many solutions.*

Definition 6. *Consider the linear system*

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m. \end{aligned}$$

Then the rectangular array of numbers

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]$$

is the ***augmented matrix*** *associated to the system.*

Definition 7. *Let* A *be a matrix. Then* ***elementary row operations*** *on* A *are the following operations:*

1. *Multiply a row through by a nonzero constant.*
2. *Interchange two rows.*
3. *Add a multiple of one row to another row.*

Definition 8. *A matrix* M *that satisfies properties 1-3 below is in* ***row-echelon form****. If, additionally, it satisfies property 4, it is in* ***reduced row-echelon form****.*

1. *If a row does not consist entirely of zeros, then the first nonzero number in the row is a 1. We call this a* ***leading 1****.*
2. *If there are any rows that consist entirely of zeros, then they are grouped together at the bottom of the matrix.*

3. *In any two successive rows that do not consist entirely of zeros, the leading 1 in the lower row occurs farther to the right than the leading 1 in the higher row.*

4. *Each column that contains a leading 1 has zeros everywhere else in that column.*

Definition 9. *Solving a linear system of equations by computing a row-echelon form of the associated augmented matrix and then performing a **back-substitution** is called a **Gaussian elimination**. If one continues on to the reduced row-echelon form, then one is performing a **Gauss-Jordan elimination**.*

Definition 10. *The linear system*

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= 0 \end{aligned}$$

*is called a **homogeneous** linear system. It always has at least the **trivial** solution $x_1 = \cdots = x_n = 0$.*

Theorem 11. *A homogeneous system of linear equations with more unknowns than equations has infinitely many solutions.*

Definition 12. *A **matrix** is a rectangular array of numbers. The numbers in the array are called the **entries** in the matrix.*

Definition 13. *Two matrices are defined to be **equal** if they have the same size and their corresponding entries are equal.*

In other words, if $A = [a_{ij}]$ and $B = [b_{ij}]$ have the same size, then $A = B$ iff

$$\forall i, j, \quad a_{ij} = b_{ij}.$$

Definition 14. *If A and B are matrices of the same size, then the sum $A+B$ is the matrix obtained by adding the entries of B to the corresponding entries of A, and the difference $A-B$ is the matrix obtained by subtracting the entries of B from the corresponding entries of A. Matrices of different sizes cannot be added or subtracted.*

In other words, if $A = [a_{ij}]$ and $B = [b_{ij}]$ have the same size, then

$$(A+B)_{ij} = (A)_{ij} + (B)_{ij} = a_{ij} + b_{ij}, \quad (A-B)_{ij} = (A)_{ij} - (B)_{ij} = a_{ij} - b_{ij}.$$

Definition 15. *If A is any matrix and c is any scalar, then the product cA is the matrix obtained by multiplying each entry of the matrix A by c. The matrix cA is said to be a scalar multiple of A.*

In other words, if $A = [a_{ij}]$ and $c \in \mathbb{R}$, then

$$\forall i, j, \quad (cA)_{ij} = c(A)_{ij} = ca_{ij}.$$

Definition 16. *If A is an $m \times r$ matrix and B is an $r \times n$ matrix, then the product AB is the $m \times n$ matrix whose entries are determined as follows. To find the entry in row i and column j of AB, single out row i from the matrix A and column j from the matrix B. Multiply the corresponding entries from the row and column together, and then add up the resulting products.*

Definition 17. *If A is any $m \times n$ matrix, then the **transpose** of A, denoted by A^T, is defined to be the $n \times n$ matrix that results from interchanging the rows and columns of A; that is, the first column of A^T is the first row of A, the second column of A^T is the second row of A, and so forth.*

In other words,

$$\forall i, j, \quad (A^T)_{ij} = (A)_{ji}.$$

Theorem 18. *If the sizes of the matrices are such that the stated operations can be performed, then*

1. $\left((A^T)\right)^T = A$.
2. $(A+B)^T = A^T + B^T$ *and* $(A-B)^T = A^T - B^T$.
3. $(kA)^T = kA^T$, *where k is any scalar.*
4. $(AB)^T = B^T A^T$.

Definition 19. *If A is a square matrix, then the **trace** of A, denoted by $tr(A)$, is defined to be the sum of the entries on the main diagonal of A. The trace of A is undefined if A is not a square matrix.*

In other words, if $A \in \mathcal{M}_{nn}$, then

$$tr(A) = \sum_{i=1}^{n} a_{ii}.$$

Theorem 20. *Assuming that the sizes of the matrices are such that the indicated operations can be performed, the following rules of matrix arithmetic are valid.*

1. $A + B = B + A$ *[Addition commutes]*
2. $A + (B + C) = (A + B) + C$ *[Addition is associative]*
3. $A(BC) = (AB)C$ *[Matrix multiplication is associative]*
4. $A(B + C) = AB + AC$ *[Matrix multiplication is left distributive over matrix addition]*
5. $(A + B)C = AC + BC$ *[Matrix multiplication is right distributive over matrix addition]*
6. $A(B - C) = AB - AC$
7. $(A - B)C = AC - BC$
8. $a(B + C) = aB + aC$
9. $a(B - C) = aB - aC$
10. $(a + b)C = aC + bC$
11. $(a - b)C = aC - bC$
12. $a(bC) = (ab)C$
13. $a(BC) = (aB)C = B(aC)$.

Definition 21. *If R is the reduced row-echelon form of $A \in \mathcal{M}_{nn}$, then either R has a row of zeros or R is the identity matrix I_n.*

Definition 22. *If A is a square matrix, and if a matrix B of the same size can be found such that $AB = BA = I$, then A is **invertible** and B is an **inverse** of A. If no such matrix B can be found, then A is **singular**.*

<u>Theorem</u> 23. *If B and C are both inverses of the matrix A, then $B = C$.*

<u>Theorem</u> 24. *Let*

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Then A is invertible iff $\det(A) = ad - bc \neq 0$, *and in that case,*

$$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

<u>Theorem</u> 25. *If A and B are invertible matrices of the same size, then AB is invertible and*

$$(AB)^{-1} = B^{-1}A^{-1}.$$

Theorem 26. *Let $A_1, \ldots, A_p$ be p invertible matrices in $\mathcal{M}_{nn}$. Then $A_1 \cdots A_p$ is invertible, and*

$$(A_1 \cdots A_p)^{-1} = A_p^{-1} \cdots A_1^{-1}.$$

Definition 27. *If $A \in \mathcal{M}_{nn}$ is a square matrix, then we define the nonnegative integer powers of A to be*

$$A^0 = I_n \qquad A^n = \underbrace{AA\cdots A}_{n \text{ times}} \quad (n > 0)$$

Moreover, if A is invertible, then we define the negative integer powers to be

$$A^{-n} = \underbrace{A^{-1}A^{-1}\cdots A^{-1}}_{n \text{ times}} \quad (n > 0)$$

Theorem 28. *If $A \in \mathcal{M}_{nn}$ and $r, s \in \mathbb{N}$, then*

$$A^rA^s = A^{r+s}, \quad (A^r)^s = A^{rs}.$$

Theorem 29 (Laws of Exponents)**.** *If A is an invertible matrix, then:*

1. *A^{-1} is invertible and $(A^{-1})^{-1} = A$.*
2. *A^n is invertible and $(A^n)^{-1} = (A^{-1})^n$ for $n \in \mathbb{N}$.*
3. *For any nonzero scalar k, the matrix kA is invertible and $(kA)^{-1} = \frac{1}{k}A^{-1}$.*

<u>Theorem</u> 30. *If A is an invertible matrix, then A^T is also invertible and*

$$(A^T)^{-1} = (A^{-1})^T.$$

Definition 31. *An $n \times n$ matrix is called an* ***elementary matrix*** *if it can be obtained from the $n \times n$ identity matrix I_n by performing a single elementary row operation.*

Theorem 32. *If the elementary matrix E results from performing a certain row operation on I_m and if $A \in \mathcal{M}_{mn}$, then the product EA is the matrix that results when this same row operation is performed on A.*

Theorem 33. *Every elementary matrix is invertible, and the inverse is also an elementary matrix.*

Theorem 34. *If $A \in \mathcal{M}_{nn}$ is invertible, then for each matrix $b \in \mathcal{M}_{n1}$, the system of equations $Ax = b$ has exactly one solution, namely, $x = A^{-1}b$.*

<u>Theorem</u> 35. *Let A be a square matrix.*

1. *If B is a square matrix satisfying $BA = I$, then $B = A^{-1}$.*

2. *If B is a square matrix satisfying $AB = I$, then $B = A^{-1}$.*

Theorem 36. *Let A and B be square matrices of the same size. If AB is invertible, then A and B must also be invertible.*

Definition 37. *Let $A \in \mathcal{M}_{nn}$.*

1. *If A has all its entries off the main diagonal equal to zero, then A is called a* ***diagonal*** *matrix.*

2. *If A has all its entries above the main diagonal equal to zero, then A is called* ***lower triangular****.*

3. *If A has all its entries below the main diagonal equal to zero, then A is called* ***upper triangular****.*

4. *A matrix that is either upper triangular or lower triangular is called* ***triangular****.*

Theorem 38.

1. *The transpose of a lower triangular matrix is upper triangular, and the transpose of an upper triangular matrix is lower triangular.*

2. *The product of lower triangular matrices is lower triangular, and the product of upper triangular matrices is upper triangular.*

3. *A triangular matrix is invertible if and only if its diagonal entries are all nonzero.*

4. *The inverse of an invertible lower triangular matrix is lower triangular, and the inverse of an invertible upper triangular matrix is upper triangular.*

Definition 39. *A square matrix A is called* ***symmetric*** *if $A = A^T$.*

Theorem 40. *If A and B are symmetric matrices with the same size, and if k is any scalar, then:*

1. *A^T is symmetric.*

2. *$A + B$ and $A - B$ are symmetric.*

3. *kA is symmetric.*

Theorem 41. *If A is an invertible symmetric matrix, then A^{-1} is symmetric.*

Theorem 42. *If A is an invertible matrix, then AA^T and A^TA are also invertible.*

2 Determinants

Definition 43. *If $A \in \mathcal{M}_{nn}$ is a square matrix, then the* ***minor*** *of entry a_{ij} is denoted by M_{ij} and is defined to be the determinant of the submatrix that remains after the ith row and jth column are deleted from A. The number $(-1)^{i+j}M_{ij}$ is denoted by C_{ij} and is called the* ***cofactor*** *of entry a_{ij}.*

Theorem 44. *The determinant of an $n \times n$ matrix A can be computed by multiplying the entries in any row (or column) by their cofactors and adding the resulting products; that is, for each $1 \leq i \leq n$ and $1 \leq j \leq n$,*

$$\det(A) = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj},$$

which is called a ***cofactor expansion*** *along the jth column, and*

$$\det(A) = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in},$$

a cofactor expansion along the ith row.

Definition 45. *If $A \in \mathcal{M}_{nn}$ is any matrix and C_{ij} is the cofactor of a_{ij}, then the matrix*

$$\begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ \vdots & \vdots & & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{bmatrix}$$

is called the ***matrix of cofactors*** *from A. The transpose of this matrix is called the* ***adjoint*** *of A and is denoted by adj(A).*

Theorem 46. *If A is an invertible matrix, then*

$$A^{-1} = \frac{1}{\det(A)} adj(A).$$

<u>Theorem</u> 47. *If $A \in \mathcal{M}_{nn}$ is a triangular matrix (upper triangular, lower triangular, or diagonal), then $\det(A)$ is the product of the entries on the main diagonal of the matrix; that is, $\det(A) = a_{11}a_{22}\cdots a_{nn} = \prod_{i=1}^{n} a_{ii}$.*

Theorem 48 (Cramer's rule)**.** *If $Ax = b$ is a system of n linear equations in n unknowns such that $\det(A) \neq 0$, then the system has a unique solution. This solution is*

$$x_1 = \frac{\det(A_1)}{\det(A)}, x_2 = \frac{\det(A_2)}{\det(A)}, \ldots, x_n = \frac{\det(A_n)}{\det(A)},$$

where A_n is the matrix obtained by replacing the entries in the jth column of A by the entries in the matrix

$$b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}.$$

<u>Theorem</u> 49. *Let A be a square matrix. If A has a row of zeros or a column of zeros, then $\det(A) = 0$.*

Theorem 50. *Let A be a square matrix. Then $\det(A) = \det(A^T)$.*

Theorem 51. *Let $A \in \mathcal{M}_{nn}$ be a matrix.*

1. *If B is the matrix that results when a single row or single column of A is multiplied by a scalar k, then $\det(B) = k\det(A)$.*

2. *If B is the matrix that results when two rows or two columns of A are interchanged, then* $\det(B) = -\det(A)$.

3. *If B is the matrix that results when a multiple of one row of A is added to another row or when a multiple of one column is added to another column, then* $\det(B) = \det(A)$.

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Theorem 52. *Let $E \in \mathcal{M}_{nn}$ be an elementary matrix.*

1. *If E results from multiplying a row of I_n by k, then* $\det(E) = k$.
2. *If E results from interchanging two rows of I_n, then* $\det(E) = -1$.
3. *If E results from adding a multiple of one row of I_n to another, then* $\det(E) = 1$.

Theorem 53. *If A is a square matrix with two proportional rows or two proportional columns, then* $\det(A) = 0$.

<u>Theorem</u> 54. *Let $A \in \mathcal{M}_{nn}$, $k \in \mathbb{R}$. Then*

$$\det(kA) = k^n \det(A).$$

Theorem 55. *Let A, B and C be $n \times n$-matrices that differ only in a single row, say the rth, and assume that the rth row of C can be obtained by adding corresponding entries in the rth rows of A and B. Then*

$$\det(C) = \det(A) + \det(B).$$

The same result holds for columns.

Theorem 56. *If A is an $n \times n$ matrix and E is an $n \times n$ elementary matrix, then*

$$\det(EA) = \det(E)\det(A).$$

<u>Theorem</u> 57. *A square matrix A is invertible if and only if* $\det(A) \neq 0$.

Theorem 58. *If A and B are square matrices of the same size, then*

$$\det(AB) = \det(A)\det(B).$$

<u>Theorem</u> 59. *If A is invertible, then*

$$\det(A^{-1}) = \frac{1}{\det(A)}.$$

Theorem 60. *Let $A \in \mathcal{M}_n$. Then the following statements are equivalent.*

1. *A is invertible.*
2. *$Ax = 0$ has only the trivial solution.*
3. *The reduced row-echelon form of A is I_n.*
4. *A can be expressed as a product of elementary matrices.*
   [$A = E_1 \cdots E_n$, with $E_1, \dots, E_n$ elementary matrices.]
5. *$Ax = b$ is consistent for every matrix .*
   [$\forall b$, $Ax = b$ is consistent.]
6. *$Ax = b$ has exactly one solution for every matrix .*
   [$\forall b$, $Ax = b$ has exactly one solution.]
7. $\det(A) \neq 0$.

3 Vectors in 2-space and 3-space

Definition 61. *If v and w are any two vectors, then the sum $v + w$ is the vector determined as follows: Position the vector w so that its initial point coincides with the terminal point of v. The vector $v + w$ is represented by the arrow from the initial point of v to the terminal point of w.*

Definition 62. *The vector of length zero is called the* ***zero vector*** *and is denoted by 0. We define $0 + v = v + 0 = v$ for every vector v. If v is any nonzero vector, then $-v$, the* ***negative*** *of v, is defined to be the vector that has the same magnitude as v but is oppositely directed, i.e., satisfying*

$$v + (-v) = 0.$$

Definition 63. *If v and w are any two vectors, then the difference of w from v is defined by*

$$v - w = v + (-w).$$

Definition 64. *If v is a nonzero vector and k is a nonzero real number (scalar), then the product kv is defined to be the vector whose length is k times the length of v and whose direction is the same as that of v if $k > 0$ and opposite to that of v if $k < 0$. We define $kv = 0$ if $k = 0$ or $v = 0$.*

Theorem 65 (Properties of vector arithmetic)**.** *Let u, v, w be vectors in 2- or 3-space and $k, \ell \in \mathbb{R}$, then the following hold true.*

1. $u + v = v + u$.
2. $(u + v) + w = u + (v + w)$.
3. $u + 0 = 0 + u = u$.
4. $u + (-u) = 0$.
5. $k(\ell u) = (k\ell)u$.
6. $k(u + v) = ku + kv$.
7. $(k + \ell)u = ku + \ell u$.
8. $1u = u$.

Definition 66. *The (Euclidean)* ***norm*** *of $u = (u_1, u_2)$ (also called its length) is*

$$\|u\| = \sqrt{u_1^2 + u_2^2}$$

in 2-space, that of $u = (u_1, u_2, u_3)$ is

$$\|u\| = \sqrt{u_1^2 + u_2^2 + u_3^2}$$

in 3-space. A vector u is a ***unit vector*** *if $\|u\| = 1$.*

Definition 67. *Let P_1 and P_2 be two points. Then the distance between P_1 and P_2, $d(P_1, P_2)$, is given by $d(P_1, P_2) = \|\overrightarrow{P_1P_2}\|$.*

Definition 68. *If u and v are vectors in 2-space or 3-space and $\theta \in [0, \pi]$ is the angle between u and v, then the* ***dot product*** *or* ***Euclidean inner product*** *$u \bullet v$ is defined by*

$$u \bullet v = \begin{cases} \|u\| \, \|v\| \, \cos\theta, & \text{if } u \neq 0 \text{ and } v \neq 0 \\ 0, & \text{if } u = 0 \text{ or } v = 0. \end{cases}$$

Theorem 69. *Let* $u = (u_1, u_2)$, $v = (v_1, v_2)$. *Then* $u \bullet v = u_1v_1 + u_2v_2$. *Let* $u = (u_1, u_2, u_3)$, $v = (v_1, v_2, v_3)$. *Then* $u \bullet v = u_1v_1 + u_2v_2 + u_3v_3$.

Theorem 70. *Let* u *be a vector in 2- or 3-space. Then*

$$u \bullet u = \|u\|^2,$$

or, in other words,

$$\|u\| = (u \bullet u)^{1/2}.$$

Theorem 71. *Let* u, v, w *be vectors in 2- or 3-space,* $k \in \mathbb{R}$, *then*

1. $u \bullet v = v \bullet u$.
2. $u \bullet (v + w) = u \bullet v + u \bullet w$.
3. $k(u \bullet v) = (ku) \bullet v = u \bullet (kv)$.
4. $v \bullet v > 0$ *if* $v \neq 0$, *and* $v \bullet v = 0$ *if* $v = 0$.

Theorem 72. *Let* u *and* a *be vectors in 2- or 3-space, with* $a \neq 0$. *Then the* ***orthogonal projection of*** u ***onto*** a *(or* ***vector component of*** u ***along*** a*) is given by*

$$proj_a u = \frac{u \bullet a}{\|a\|^2} a$$

and the ***vector component of*** u ***orthogonal to*** a *is given by*

$$u - proj_a u = u - \frac{u \bullet a}{\|a\|^2} a.$$

Theorem 73. *Let* u *and* a *be vectors in 2- or 3-space, with* $a \neq 0$. *Then*

$$\|proj_a u\| = \frac{|u \bullet a|}{\|a\|} = \|u\| |\cos\theta|,$$

if θ *is the angle between* u *and* a.

Definition 74. *Let* $u = (u_1, u_2, u_3)$ *and* $v = (v_1, v_2, v_3)$ *be vectors in 3-space. The* ***cross product*** *of* u *and* v, *denoted* $u \times v$, *is the vector defined by*

$$u \times v = (u_2v_3 - u_3v_2, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1),$$

or, in determinant form,

$$u \times v = \left(\begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}, - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}, \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \right).$$

Theorem 75. *The cross product of the 3-space vectors* $u = (u_1, u_2, u_3)$ *and* $v = (v_1, v_2, v_3)$ *can be found by computing symbolically the determinant*

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

by expansion along the first row, where $\vec{i} = (1, 0, 0)$, $\vec{j} = (0, 1, 0)$ *and* $\vec{k} = (0, 0, 1)$ *are the* ***standard unit vectors*** *of 3-space.*

Theorem 76. *Let u, v, w be vectors in 3-space, $k \in \mathbb{R}$. Then*

1. $u \bullet (u \times v) = 0$. *[$u \times v \perp u$]*
2. $v \bullet (u \times v) = 0$. *[$u \times v \perp v$]*
3. $\|u \times v\|^2 = \|u\|^2\|v\|^2 - (u \bullet v)^2$. *[Lagrange's identity]*
4. $u \times (v \times w) = (u \bullet w)v - (u \bullet v)w$.
5. $(u \times v) \times w = (u \bullet w)v - (v \bullet w)u$.
6. $u \times v = -(v \times u)$.
7. $u \times (v + w) = (u \times v) + (u \times w)$.
8. $(u + v) \times w = (u \times w) + (v \times w)$.
9. $k(u \times v) = (ku) \times v = u \times (kv)$.
10. $u \times 0 = 0 \times u = 0$.
11. $u \times u = 0$.

Theorem 77. *Let u, v be vectors in 3-space, $\theta \in [0, \pi]$ the angle between them. Then*

$$\|u \times v\| = \|u\| \ \|v\| \ \sin\theta.$$

$\|u \times v\|$ is equal to the area of the parallelogram determined by u and v.

Definition 78. *Let u, v, w be vectors in 3-space. Then*

$$u \bullet (v \times w)$$

is called the ***scalar triple product*** *of u, v and w.*

Theorem 79. *Let $u = (u_1, u_2, u_3)$, $v = (v_1, v_2, v_3)$ and $w = (w_1, w_2, w_3)$ be vectors in 3-space. Then*

$$u \bullet (v \times w) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}.$$

<u>**Theorem**</u> **80.** *Let $u = (u_1, u_2, u_3)$, $v = (v_1, v_2, v_3)$ and $w = (w_1, w_2, w_3)$ be vectors in 3-space. Then*

$$u \bullet (v \times w) = w \bullet (u \times v) = v \bullet (w \times u).$$

Theorem 81. *Let $u = (u_1, u_2)$ and $v = (v_1, v_2)$ be vectors in 2-space. Then*

$$\left| \det \begin{bmatrix} u_1 & u_2 \\ v_1 & v_2 \end{bmatrix} \right|$$

equals the area of the parallelogram in 2-space determined by u and v. Let $u = (u_1, u_2, u_3)$, $v = (v_1, v_2, v_3)$ and $w = (w_1, w_2, w_3)$ be vectors in 3-space. Then

$$|u \bullet (v \times w)|$$

equals the volume of the parallelepiped in 3-space determined by u, v and w.

Theorem 82. *Let $u = (u_1, u_2, u_3)$, $v = (v_1, v_2, v_3)$ and $w = (w_1, w_2, w_3)$ be vectors in 3-space that have the same initial point. Then u, v and w lie in the same plane if and only if $u \bullet (v \times w) = 0$.*

Theorem 83. *Let $P_0(x_0, y_0, z_0)$ be a point in 3-space and $n = (a, b, c)$ be a vector in 3-space. Then the **point-normal** form of the equation of the plane through P_0 and orthogonal to n is given by*

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0.$$

Theorem 84. *Let $a, b, c, d \in \mathbb{R}$ with a, b, c not all zero. Then the graph of the equation*

$$ax + by + cz + d = 0$$

is a plane with $n = (a, b, c)$ as a normal.

Theorem 85. *Let $r = (x, y, z)$ be the vector from the origin to $P(x, y, z)$, $r_0 = (x_0, y_0, z_0)$ be the vector from the origin to $P_0(x_0, y_0, z_0)$. Then the **vector form** of the equation of the plane through P_0 and perpendicular to $n = (a, b, c)$ is given by*

$$n \bullet (r - r_0) = 0.$$

Theorem 86. *Let $P_0(x_0, y_0, z_0)$ be a point in 3-space, $u = (a, b, c)$ be a vector in 3-space. Then the **parametric** equation of the line through P_0 and in the direction of u takes the form*

$$(x, y, z) = P_0 + tu, \quad t \in \mathbb{R},$$

or, in other words,

$$x = x_0 + ta, \quad y = y_0 + tb, \quad z = z_0 + tc, \quad t \in \mathbb{R}.$$

Theorem 87. *Let L be the line in 2-space with equation $ax + by + c = 0$ and $P_0(x_0, y_0)$ be a point in 2-space. Then the distance D between the point P_0 and the line L is given by*

$$D = \frac{|ax_0 + by_0 + c|}{\sqrt{a^2 + b^2}}.$$

Let L be the plane in 3-space with equation $ax + by + cz + d = 0$ and $P(x_0, y_0, z_0)$ be a point in 3-space. Then the distance D between the point P_0 and the plane L is given by

$$D = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}.$$

4 Vectors in n-space

Definition 88. *If n is a positive integer, then an **ordered n-tuple** is a sequence of n real numbers $(a_1, \ldots, a_n)$. The set of all ordered n-tuples is called n-space and is denoted by $\mathbb{R}^n$.*

Definition 89. *Two vectors $u = (u_1, \ldots, u_n)$ and $v = (v_1, \ldots, v_n)$ in $\mathbb{R}^n$ are called **equal** if*

$$u_1 = v_1, \ldots, u_n = v_n.$$

*The **sum** $u + v$ is defined by*

$$u + v = (u_1 + v_1, \ldots, u_n + v_n)$$

and if k is any scalar, the scalar multiple ku is defined by

$$ku = (ku_1, \ldots, ku_n).$$

*The operations of addition and scalar multiplication in this definition are called the **standard operations** on $\mathbb{R}^n$. The zero vector in $\mathbb{R}^n$ is denoted by 0 (sometimes $0_{\mathbb{R}^n}$) and is defined to be the vector*

$$0 = (0, \ldots, 0).$$

*If u is any vector in $\mathbb{R}^n$, then the **negative** (or **additive inverse**) of u is denoted by $-u$ and is defined by*

$$-u = (-u_1, \ldots, -u_n).$$

The difference of vectors in $\mathbb{R}^n$ is defined by

$$u - v = u + (-v)$$

or, in terms of components,

$$u - v = (u_1 - v_1, \ldots, u_n - v_n).$$

Theorem 90. *Let $u, v \in \mathbb{R}^n$ be two vectors, $k, \ell \in \mathbb{R}$ two scalars. Then:*

1. $u + v = v + u$.
2. $u + (v + w) = (u + v) + w$.
3. $u + 0 = u$.
4. $u + (-u) = 0$.
5. $k(\ell u) = (k\ell)u$.
6. $k(u + v) = ku + kv$.
7. $(k + \ell)u = ku + ellu$.
8. $1u = u$.

Definition 91. *Let $u = (u_1, \ldots, u_n)$ and $v = (v_1, \ldots, v_n)$ be two vectors in $\mathbb{R}^n$. Then the **Euclidean inner product** $u \bullet v$ of u and v is defined by*

$$u \bullet v = u_1 v_1 + \cdots + u_n v_n = \sum_{i=1}^{n} u_i v_i.$$

Theorem 92. *Let $u, v, w \in \mathbb{R}^n$, $k \in \mathbb{R}$. Then:*

1. $u \bullet v = v \bullet u$.

2. $(u + v) \bullet w = u \bullet w + v \bullet w$.

3. $(ku) \bullet v = k(u \bullet v)$.

4. $u \bullet u \geq 0$. *Further,* $u \bullet u = 0$ *if and only if* $u = 0$.

Definition 93. *Let* $u = (u_1, \ldots, u_n) \in \mathbb{R}^n$. *Then the* ***Euclidean norm*** *of* u *is given by*

$$\|u\| = (u \bullet u)^{1/2} = \sqrt{u_1^2 + \cdots + u_n^2}.$$

Additionally, let $v = (v_1, \ldots, v_n) \in \mathbb{R}^n$. *Then the* ***Euclidean distance*** *between* u *and* v *is given by*

$$d(u, v) = \|u - v\| = \sqrt{(u_1 - v_1)^2 + \cdots + (u_n - v_n)^2}.$$

<u>Theorem</u> 94 (Cauchy-Schwarz inequality)**.** *Let* $u, v \in \mathbb{R}^n$. *Then*

$$|u \bullet v| \leq \|u\| \, \|v\|.$$

<u>Theorem</u> 95. *Let* $u, v \in \mathbb{R}^n$, $k \in \mathbb{R}$. *Then:*

1. $\|u\| \geq 0$.

2. $\|u\| = 0$ *iff* $u = 0$.

3. $\|ku\| = |k| \, \|u\|$.

4. $\|u + v\| \leq \|u\| + \|v\|$. *[Triangle inequality]*

<u>Theorem</u> 96. *Let* $u, v, w \in \mathbb{R}^n$, $k \in \mathbb{R}$. *Then:*

1. $d(u, v) \geq 0$.

2. $d(u, v) = 0$ *iff* $u = v$.

3. $d(u, v) = d(v, u)$.

4. $d(u, v) \leq d(u, w) + d(w, v)$. *[Triangle inequality]*

Theorem 97. *Let* $u, v \in \mathbb{R}^n$ *with the Euclidean inner product, then*

$$u \bullet v = \frac{1}{4}\|u + v\|^2 - \frac{1}{4}\|u - v\|^2.$$

Definition 98. *Let* $u, v \in \mathbb{R}^n$. u *and* v *are called* ***orthogonal*** *if* $u \bullet v = 0$.

Theorem 99 (Pythagorean theorem in $\mathbb{R}^n$)**.** *Let* $u, v \in \mathbb{R}^n$ *be orthogonal in* $\mathbb{R}^n$ *equiped with the Euclidean inner product. Then*

$$\|u + v\|^2 = \|u\|^2 + \|v\|^2.$$

5 General vector spaces

Definition 100. *Let V be an arbitrary nonempty set of objects on which two operations are defined: addition, and multiplication by scalars (numbers). By addition we mean a rule for associating with each pair of objects u and v in V an object $u+v$, called the sum of u and v; by scalar multiplication we mean a rule for associating with each scalar k and each object u in V an object ku, called the scalar multiple of u by k. If the following axioms are satisfied by all objects u, v, w in V and all scalars k and m, then we call V a* ***vector space*** *and we call the objects in V* ***vectors***.

1. *If u and v are objects in V, then $u+v$ is in V.*
2. $u+v = v+u$.
3. $u+(v+w) = (u+v)+w$.
4. *There is an object 0 in V, called a zero vector for V, such that $u+0 = u$ for all u in V.*
5. *For each u in V, there is an object $-u$ in V, called a negative of u, such that $u+(-u) = (-u)+u = 0$.*
6. *If k is any scalar and u is any object in V, then ku is in V.*
7. $k(u+v) = ku+kv$.
8. $(k+m)u = ku+mu$.
9. $k(mu) = (km)u$.
10. $1u = u$.

Theorem 101. *The following are vectors spaces when equiped with the usual addition and scalar multiplication.*

1. $\mathbb{R}^n$.
2. *The set $\mathcal{M}_{mn}$ of $m \times n$-matrices.*
3. *The set P_n of polynomials of degree $\leq n$.*
4. *The set $\{0_V\}$.*
5. *The set of real-valued functions defined on $\mathbb{R}$.*

Theorem 102. *Let V be a vector space, $u \in V$ and $k \in \mathbb{R}$. Then:*

1. $0u = 0_V$.
2. $k0_V = 0_V$.
3. $(-1)u = -u$.
4. *If $ku = 0_V$, then $k = 0$ or $u = 0_V$.*

Definition 103. *A subset W of a vector space V is called a* ***subspace*** *of V if W is itself a vector space under the addition and scalar multiplication defined on V.*

Theorem 104. *If W is a set of one or more vectors from a vector space V, then W is a subspace of V if and only if the following conditions hold.*

1. *If u and v are vectors in W, then $u + v$ is in W.*
   [W ***closed under addition****]*

2. *If k is any scalar and u is any vector in W, then ku is in W.*
   [W ***closed under scalar multiplication****]*

Theorem 105. *1. The following are subspaces of $\mathbb{R}^2$:*

   (a) $\{0_{\mathbb{R}^2}\}$.
   (b) Lines through the origin.
   (c) $\mathbb{R}^2$.

2. *The following are subspaces of $\mathbb{R}^3$:*

   (a) $\{0_{\mathbb{R}^3}\}$.
   (b) Lines through the origin.
   (c) Planes through the origin.
   (d) $\mathbb{R}^3$.

3. *The following are subspaces of $\mathbb{R}^n$:*

   (a) $\{0_{\mathbb{R}^n}\}$.
   (b) Any hyperplane through the origin, i.e., of the form $a_1x_1 + \cdots + a_px_p = 0$, with $p \leq n$.
   (c) $\mathbb{R}^n$.

4. *The following are subspaces of $\mathcal{M}_n$, the vector space of $n \times n$-matrices:*

   (a) $\{0_{\mathcal{M}_n}\}$.
   (b) The set of upper triangular $n \times n$-matrices.
   (c) The set of lower triangular $n \times n$-matrices.
   (d) The set of diagonal $n \times n$-matrices.
   (e) The set of symmetric $n \times n$-matrices.
   (f) $\mathcal{M}_n$.

Theorem 106. *If $Ax = 0$ is a homogeneous linear system of m equations in n unknowns, then the set of solution vectors is a subspace of $\mathbb{R}^n$.*

Definition 107. *A vector w is called a* ***linear combination*** *of the vectors $v_1, \ldots, v_p$ if it can be expressed in the form*

$$w = k_1v_1 + \cdots + k_pv_p,$$

where $k_1, \ldots, k_p \in \mathbb{R}$.

Theorem 108. *Let $v_1, \ldots, v_p$ be vectors in a vector space V. Then*

1. *The set W of all linear combinations of $v_1, \ldots, v_p$ is a subspace of V.*

2. *W is the smallest subspace of V that contains $v_1, \ldots, v_p$ in the sense that every other subspace of V that contains $v_1, \ldots, v_p$ must contain W.*

Definition 109. *Let $S = \{v_1, \ldots, v_p\}$ be a set of vectors in a vector space V. The* ***span*** *of S (or* ***space spanned*** *by S), denoted span(S), is the subspace of V consisting of all linear combinations of elements of S; in other words,*

$$span(S) = \{w \in V; w = k_1 v_1 + \cdots + k_p v_p, \quad \forall k_1, \ldots, k_p \in \mathbb{R}\}.$$

Theorem 110. *Let $S = \{v_1, \ldots, v_p\}$ and $S' = \{w_1, \ldots, w_q\}$ be two sets of vectors in a vector space V. Then*

$$span(S) = span(S')$$

if and only if each vector in S is linear combination of those in S' and each vector in S' is linear combination of those in S.

Definition 111. *Let $S = \{v_1, \ldots, v_p\}$ be a nonempty set of vectors in the vector space V. The equation*

$$k_1 v_1 + \cdots + k_p v_p = 0$$

has always at least the trivial solution $k_1 = \cdots = k_p = 0$. If this is the only solution, then S is called ***linearly independent*** *(or a linearly independent set). If there are other solutions, then S is* ***linearly dependent****.*

Theorem 112. *A set S with two or more vectors is*

1. *linearly dependent if and only if at least one of the vectors in S is linear combination of other vectors in S,*

2. *linearly independent if and only if no vector in S is linear combination of other vectors in S.*

Theorem 113.
1. *A finite set of vectors that contains the zero vector is linearly dependent.*

2. *A set with exactly two vectors is linearly independent if and only if neither vector is a scalar multiple of the other.*

Theorem 114. *Let $S = \{v_1, \ldots, v_p\}$ be a set of vectors in $\mathbb{R}^n$. If $r > n$, then S is linearly dependent.*

Definition 115. *If V is any vector space and $S = \{v_1, \ldots, v_p\}$ is a set of vectors in V, then S is called a* ***basis*** *for V if the following two conditions hold:*

1. *S is linearly independent.*

2. *S spans V.*

Theorem 116. *If $S = \{v_1, \ldots, v_p\}$ is a basis for a vector space V, then every vector v in V can be expressed in the form $v = c_1 v_1 + \cdots + c_p v_p$ in exactly one way.*

Definition 117. *A nonzero vector space V is called* ***finite-dimensional*** *if it contains a finite set of vectors $S = \{v_1, \ldots, v_n\}$ that forms a basis. If no such set exists, V is called* ***infinite-dimensional****. In addition, we shall regard the zero vector space to be finite dimensional.*

Theorem 118. *Let V be a finite-dimensional vector space, and let $\{v_1, \ldots, v_n\}$ be any basis of V.*

1. *If a set has more than n vectors, then it is linearly dependent.*

2. *If a set has fewer than n vectors, then it does not span V.*

Theorem 119. *All bases for a finite-dimensional vector space have the same number of vectors.*

Definition 120. *The **dimension** of a finite-dimensional vector space V, denoted by $\dim(V)$, is defined to be the number of vectors in a basis for V. In addition, we define the zero vector space to have dimension zero.*

Theorem 121 (Plus/Minus Theorem)**.** *Let S be a nonempty set of vectors in a vector space V.*

1. *If S is a linearly independent set, and if v is a vector in V that is outside of $span(S)$, then the set that results by inserting v into S is still linearly independent.*

2. *If v is a vector in S that is expressible as a linear combination of other vectors in S, and if $S - \{v\}$ denotes the set obtained by removing v from S, then S and $S - \{v\}$ span the same space; that is,*
$$span(S) = span(S - \{v\}).$$

Theorem 122. *If V is an n-dimensional vector space, and if S is a set in V with exactly n vectors, then S is a basis for V if either S spans V or S is linearly independent.*

Theorem 123. *Let S be a finite set of vectors in a finite-dimensional vector space V.*

1. *If S spans V but is not a basis for V, then S can be reduced to a basis for V by removing appropriate vectors from S.*

2. *If S is a linearly independent set that is not already a basis for V, then S can be enlarged to a basis for V by inserting appropriate vectors into S.*

Theorem 124. *If W is a subspace of a finite-dimensional vector space V, then $\dim(W) \leq \dim(V)$; moreover, if $\dim(W) = \dim(V)$, then $W = V$.*

Definition 125. *For a matrix $A \in \mathcal{M}_{mn}$,*
$$A = \begin{bmatrix} a_{11} & & a_{1n} \\ & & \\ a_{m1} & & a_{mn} \end{bmatrix},$$
the m vectors
$$\begin{aligned} r_1 &= [a_{11} \quad \cdots \quad a_{1n}] \\ &\vdots \\ r_m &= [a_{m1} \quad \cdots \quad a_{mn}] \end{aligned}$$
*in $\mathbb{R}^n$ formed from the rows of A are called the **row vectors** of A, and the n vectors*
$$c_1 = \begin{bmatrix} a_{11} \\ \vdots a_{m1} \end{bmatrix}, \ldots c_n = \begin{bmatrix} a_{1n} \\ \vdots a_{mn} \end{bmatrix},$$
*in $\mathbb{R}^m$ formed from the columns of A are called the **column vectors** of A.*

Definition 126. *If $A \in \mathcal{M}_{mn}$ is a matrix, then the subspace of $\mathbb{R}^n$ spanned by the row vectors of A is called the **row space** of A, and the subspace of $\mathbb{R}^m$ spanned by the column vectors of A is called the **column space** of A. The solution space of the homogeneous system of equations $Ax = 0$, which is a subspace of $\mathbb{R}^n$, is called the **nullspace** of A.*

Theorem 127. *A system of linear equations $Ax = b$ is consistent if and only if b is in the column space of A.*

Theorem 128. *If x_0 denotes any single solution of a consistent linear system $Ax = b$, and if $v_1, \ldots, v_p$ form a basis for the nullspace of A –that is, the solution space of the homogeneous system $Ax = 0$– then every solution of $Ax = b$ can be expressed in the form*

$$x = x_0 + c_1 v_1 + \cdots + c_p v_p$$

and, conversely, for all choices of scalars $c_1, \ldots, c_p$, the vector x in this formula is a solution of $Ax = b$.

Theorem 129. *Elementary row operations do not change the nullspace of a matrix.*

Theorem 130. *Elementary row operations do not change the row space of a matrix.*

Theorem 131. *If A and B are row equivalent matrices, then*

1. *A given set of column vectors of A is linearly independent if and only if the corresponding column vectors of B are linearly independent.*
2. *A given set of column vectors of A forms a basis for the column space of A if and only if the corresponding column vectors of B form a basis for the column space of B.*

Theorem 132. *If a matrix R is in row-echelon form, then the row vectors with the leading 1's (the nonzero row vectors) form a basis for the row space of R, and the column vectors with the leading 1's of the row vectors form a basis for the column space of R.*