Single population growth models

Objective

We are given a table with the population census at different time intervals between a date a and a date b, and want to get an expression for the population. This allows us to:

- compute a value for the population at any time between the date a and the date b (*interpolation*),
- predict a value for the population at a date before a or after b (*extrapolation*).

PROCEEDINGS

OF THE

NATIONAL ACADEMY OF SCIENCES

Volume 6 JUNE 15, 1920 Number 6

ON THE RATE OF GROWTH OF THE POPULATION OF THE UNITED STATES SINCE 1790 AND ITS MATHEMATICAL REPRESENTATION[1]

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Read before the Academy, April 26, 1920

Showing the Dates of the Taking of the Census and the Recorded Populations from 1790 to 1910

Date of Census		Recorded Population (Revised Figures from Statistical Abst., 1918)
Year	Month and Day	
1790	First Monday in August	3,929,214
1800	First Monday in August	5,308,483
1810	First Monday in August	7,239,881
1820	First Monday in August	9,638,453
1830	June 1	12,866,020
1840	June 1	17,069,453
1850	June 1	23,191,876
1860	June 1	31,443,321
1870	June 1	38,558,371
1880	June 1	50,155,783
1890	June 1	62,947,714
1900	June 1	75,994,575
1910	April 15	91,972,266

The US population from 1790 to 1910

Year	Population (millions)
1790	3.929
1800	5.308
1810	7.240
1820	9.638
1830	12.866
1840	17.069
1850	23.192

Year	Population (millions)
1860	31.443
1870	38.558
1880	50.156
1890	62.948
1900	75.995
1910	91.972

PLOT THE DATA !!! (here, to 1910)

Using MatLab (or Octave), create two vectors using commands such as

```
t=1790:10:1910;
```

Format is

$$\text{Vector}=\text{Initial value:Step:Final value}$$

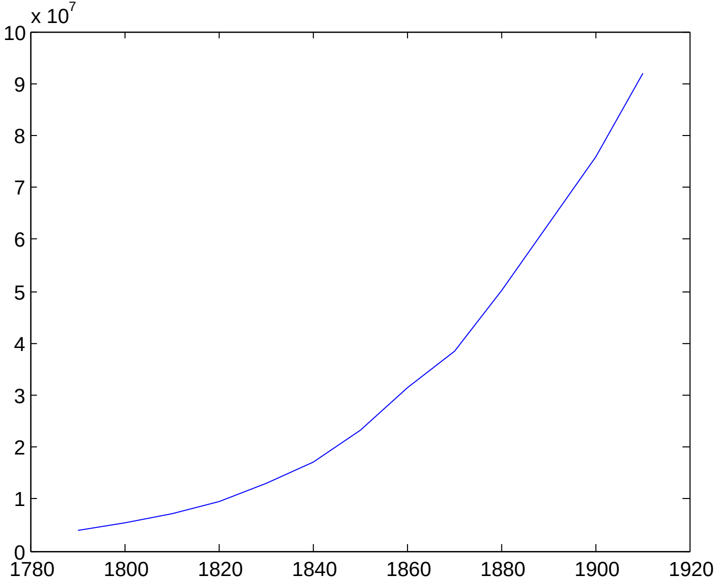
(semicolumn hides result of the command.)

```
P=[3929214,5308483,7239881,9638453,12866020,...
17069453,23191876,31443321,38558371,50155783,...
62947714,75994575,91972266];
```

Here, elements were just listed (`...` indicates that the line continues below).

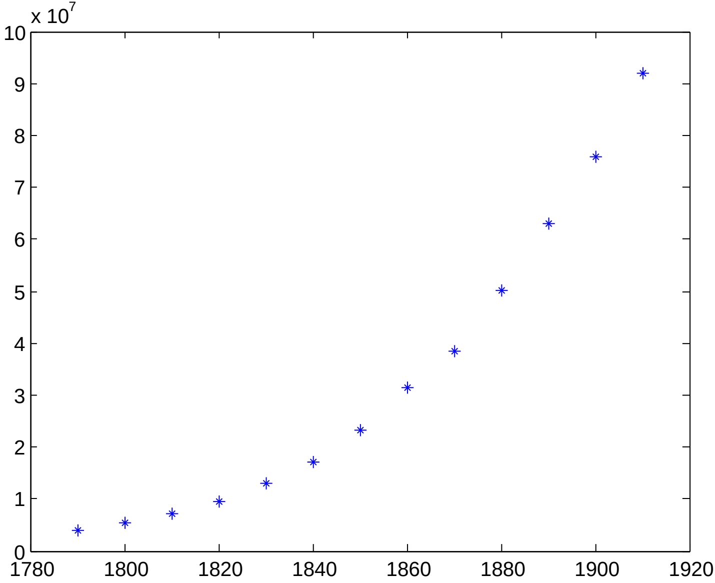
Then plot using

```
plot(t,P);
```



To get points instead of a line

```
plot(t,P,'*');
```



First idea

The curve looks like a piece of a parabola. So let us fit a curve of the form

$$P(t) = a + bt + ct^2.$$

To do this, we want to minimize

$$S = \sum_{k=1}^{13} \left(P(t_k) - P_k\right)^2,$$

where t_k are the known dates, P_k are the known populations, and $P(t_k) = a + bt_k + ct_k^2$.

We proceed as in the notes (but note that the role of a, b, c is reversed):

$$S = S(a, b, c) = \sum_{k=1}^{13} \left(a + bt_k + ct_k^2 - P_k\right)^2$$

is maximal if (necessary condition) $\partial S/\partial a = \partial S/\partial b = \partial S/\partial c = 0$, with

$$\frac{\partial S}{\partial a} = 2\sum_{k=1}^{13}(a + bt_k + ct_k^2 - P_k)$$

$$\frac{\partial S}{\partial b} = 2\sum_{k=1}^{13}(a + bt_k + ct_k^2 - P_k)t_k$$

$$\frac{\partial S}{\partial c} = 2\sum_{k=1}^{13}(a + bt_k + ct_k^2 - P_k)t_k^2$$

So we want

$$2\sum_{k=1}^{13}(a + bt_k + ct_k^2 - P_k) = 0$$

$$2\sum_{k=1}^{13}(a + bt_k + ct_k^2 - P_k)t_k = 0$$

$$2\sum_{k=1}^{13}(a + bt_k + ct_k^2 - P_k)t_k^2 = 0,$$

that is

$$\sum_{k=1}^{13}(a + bt_k + ct_k^2 - P_k) = 0$$

$$\sum_{k=1}^{13}(a + bt_k + ct_k^2 - P_k)t_k = 0$$

$$\sum_{k=1}^{13}(a + bt_k + ct_k^2 - P_k)t_k^2 = 0.$$

Rearranging the system

$$\sum_{k=1}^{13}(a + bt_k + ct_k^2 - P_k) = 0$$
$$\sum_{k=1}^{13}(a + bt_k + ct_k^2 - P_k)t_k = 0$$
$$\sum_{k=1}^{13}(a + bt_k + ct_k^2 - P_k)t_k^2 = 0,$$

we get

$$\sum_{k=1}^{13}(a + bt_k + ct_k^2) = \sum_{k=1}^{13} P_k$$
$$\sum_{k=1}^{13}(at_k + bt_k^2 + ct_k^3) = \sum_{k=1}^{13} P_k t_k$$
$$\sum_{k=1}^{13}(at_k^2 + bt_k^3 + ct_k^4) = \sum_{k=1}^{13} P_k t_k^2.$$

$$\sum_{k=1}^{13}(a + bt_k + ct_k^2) = \sum_{k=1}^{13} P_k$$
$$\sum_{k=1}^{13}(at_k + bt_k^2 + ct_k^3) = \sum_{k=1}^{13} P_k t_k$$
$$\sum_{k=1}^{13}(at_k^2 + bt_k^3 + ct_k^4) = \sum_{k=1}^{13} P_k t_k^2,$$

after a bit of tidying up, takes the form

$$\left(\sum_{k=1}^{13} 1\right) a + \left(\sum_{k=1}^{13} t_k\right) b + \left(\sum_{k=1}^{13} t_k^2\right) c = \sum_{k=1}^{13} P_k$$
$$\left(\sum_{k=1}^{13} t_k\right) a + \left(\sum_{k=1}^{13} t_k^2\right) b + \left(\sum_{k=1}^{13} t_k^3\right) c = \sum_{k=1}^{13} P_k t_k$$
$$\left(\sum_{k=1}^{13} t_k^2\right) a + \left(\sum_{k=1}^{13} t_k^3\right) b + \left(\sum_{k=1}^{13} t_k^4\right) c = \sum_{k=1}^{13} P_k t_k^2.$$

So the aim is to solve the linear system

$$\begin{pmatrix} 13 & \sum_{k=1}^{13} t_k & \sum_{k=1}^{13} t_k^2 \\ \sum_{k=1}^{13} t_k & \sum_{k=1}^{13} t_k^2 & \sum_{k=1}^{13} t_k^3 \\ \sum_{k=1}^{13} t_k^2 & \sum_{k=1}^{13} t_k^3 & \sum_{k=1}^{13} t_k^4 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} \sum_{k=1}^{13} P_k \\ \sum_{k=1}^{13} P_k t_k \\ \sum_{k=1}^{13} P_k t_k^2 \end{pmatrix}$$

With MatLab (or Octave), getting the values is easy.

- To apply an operation to every element in a vector or matrix, prefix the operation with a dot, hence

  ```
  t.^2;
  ```

  gives, for example, the vector with every element t_k squared.
- Also, the function `sum` gives the sum of the entries of a vector or matrix.
- When entering a matrix or vector, separate entries on the same row by `,` and create a new row by using `;`.

Thus, to set up the problem in the form of solving $Ax = b$, we need to do the following:

```
format long g;
A=[13,sum(t),sum(t.^2);sum(t),sum(t.^2),sum(t.^3);...
sum(t.^2),sum(t.^3),sum(t.^4)];
b=[sum(P);sum(P.*t);sum(P.*(t.^2))];
```

The `format long g` command is used to force the display of digits (normally, what is shown is in "scientific" notation, not very informative here).

Then, solve the system using

```
A\b
```

We get the following output:

```
>> A\b
Warning: Matrix is close to singular or badly scaled.
         Results may be inaccurate. RCOND = 1.118391e-020.


ans =


        22233186177.8195
        -24720291.325476
        6872.99686313725
```

(note that here, Octave gives a solution that is not as good as this one, provided by MatLab).

Thus

$$P(t) = 22233186177.8195 - 24720291.325476t + 6872.99686313725t^2$$

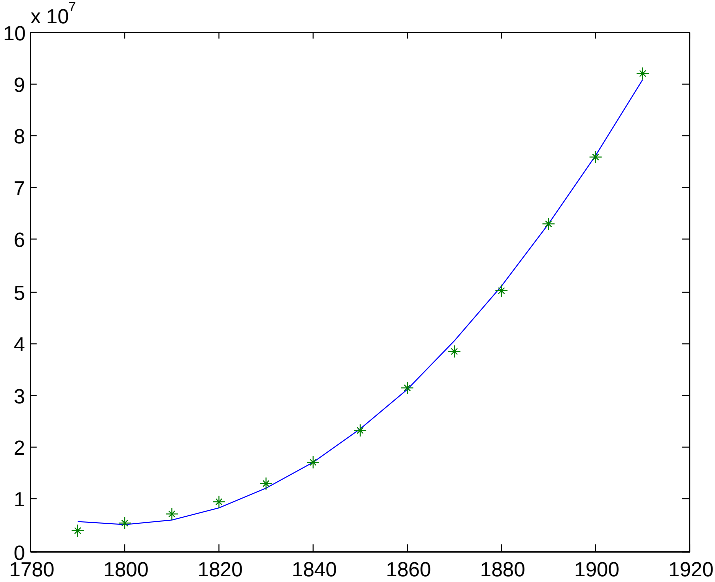
To see what this looks like,

```
plot(t,22233186177.8195-24720291.325476.*t...
+6872.99686313725.*t.^2);
```

(note the dots before multiplication and power, since we apply this function to every entry of t). In fact, to compare with original data:

```
plot(t,22233186177.8195-24720291.325476.*t...
+6872.99686313725.*t.^2,t,P,'*');
```

Our first guess, in pictures



Now we want to generate the table of values, to compare with the true values and thus compute the error. To do this, we can proceed directly:

```
computedP=22233186177.8195-24720291.325476.*t...
+6872.99686313725.*t.^2;
```

We get

```
computedP =

 Columns 1 through 4:

     5633954.39552689      5171628.52739334      6083902.03188705      8370774.90901184

 Columns 5 through 8:

     12032247.1587601      17068318.7811356      23478989.7761383      31264260.1437798

 Columns 9 through 12:

      40424129.884037      50958598.9969215      62867667.4824371      76151335.3405762

 Column 13:

     90809602.5713463
```

We can also create an *inline* function

```
f=inline('22233186177.8195-24720291.325476.*t+6872.99686313725.*t.^2')
f =

     Inline function:
     f(t) = 22233186177.8195-24720291.325476.*t+6872.99686313725.*t.^2
```

This function can then easily be used for a single value

```
octave:24> f(1880)
ans =      50958598.9969215
```

as well as for vectors..

(Recall that t has the dates; t in the definition of the function is a dummy variable, we could have used another letter-.)

```
octave:25> f(t)
```

```
ans =

 Columns 1 through 4:

     5633954.39552689      5171628.52739334      6083902.03188705      8370774.90901184

 Columns 5 through 8:

     12032247.1587601      17068318.7811356      23478989.7761383      31264260.1437798

 Columns 9 through 12:

      40424129.884037      50958598.9969215      62867667.4824371      76151335.3405762
12186176863781.4
 Column 13:

     90809602.5713463
```

Form the vector of errors, and compute sum of errors squared:

```
octave:26> E=f(t)-P;
octave:27> sum(E.^2)
ans =      12186176863781.4
```

Quite a large error (12,186,176,863,781.4), which is normal since we have used actual numbers, not thousands or millions of individuals, and we are taking the square of the error.

To present things legibly, one way is to put everything in a matrix..

```
M=[P;f(t);E;E./P];
```

This matrix will have each type of information as a row, so to display it in the form of a table, show its transpose, which is achieved using the function `transpose` or the operator $'$.

```
M'
ans =
 3929214       5633954.39552689     1704740.39552689    0.433862954658
 5308483       5171628.52739334    -136854.472606659   -0.0257803354756
 7239881       6083902.03188705    -1155978.96811295    -0.15966822771l
 9638453       8370774.90901184    -1267678.09098816    -0.131522983095
12866020       12032247.1587601    -833772.841239929   -0.0648042550252
17069453       17068318.7811356    -1134.21886444092  -6.644728828e-05
23191876       23478989.7761383     287113.776138306    0.0123799289086
31443321       31264260.1437798    -179060.856220245  -0.00569471832254
38558371        40424129.884037     1865758.88403702    0.0483879073635
50155783       50958598.9969215     802815.996921539    0.0160064492846
62947714       62867667.4824371    -80046.5175628662  -0.00127163502018
75994575       76151335.3405762     156760.340576172   0.00206278330494
91972266       90809602.5713463    -1162663.42865372    -0.012641456813
```

Now for the big question...

How does our formula do for present times?

```
f(2006)
ans =      301468584.066013
```

Actually, quite well: 301,468,584, compared to the 298,444,215 July 2006 estimate, overestimates the population by 3,024,369, a relative error of approximately 1%.

The US population from 1790 to 2000 (revised numbers)

Year	Population (millions)
1790	3.929
1800	5.308
1810	7.240
1820	9.638
1830	12.866
1840	17.069
1850	23.192
1860	31.443
1870	38.558
1880	50.156
1890	62.948

Year	Population (millions)
1900	76.212
1910	92.228
1920	106.021
1930	123.202
1940	132.164
1950	151.325
1960	179.323
1970	203.302
1980	226.542
1990	248.709
2000	281.421

Other similar approaches

Pritchett, 1891:

$$P = a + bt + ct^2 + dt^3.$$

(we have done this one, and found it to be quite good too).
Pearl, 1907:

$$P(t) = a + bt + ct^2 + d \ln t.$$

Finds

$$P(t) = 9,064,900 - 6,281,430t + 842,377t^2 + 19,829,500 \ln t.$$

SHOWING (*a*) THE ACTUAL POPULATION[1] ON CENSUS DATES, (*b*) ESTIMATED POPULATION FROM PRITCHETT'S THIRD-ORDER PARABOLA, (*c*) ESTIMATED POPULATION FROM LOGARITHMIC PARABOLA, AND (*d*) (*e*) ROOT-MEAN SQUARE ERRORS OF BOTH METHODS

CENSUS YEAR	(*a*) OBSERVED POPULATION	(*b*) PRITCHETT ESTIMATE	(*c*) LOGARITHMIC PARABOLA ESTIMATE	(*d*) ERROR OF (*b*)	(*e*) ERROR OF (*c*)
1790	3,929,000	4,012,000	3,693,000	+ 83,000	− 236,000
1800	5,308,000	5,267,000	5,865,000	− 41,000	+ 557,000
1810	7,240,000	7,059,000	7,293,000	− 181,000	+ 53,000
1820	9,638,000	9,571,000	9,404,000	− 67,000	− 234,000
1830	12,866,000	12,985,000	12,577,000	+ 119,000	− 289,000
1840	17,069,000	17,484,000	17,132,000	+ 415,000	+ 63,000
1850	23,192,000	23,250,000	23,129,000	+ 58,000	− 63,000
1860	31,443,000	30,465,000	30,633,000	− 978,000	− 810,000
1870	38,558,000	39,313,000	39,687,000	+ 755,000	+1,129,000
1880	50,156,000	49,975,000	50,318,000	− 181,000	+ 162,000
1890	62,948,000	62,634,000	62,547,000	− 314,000	− 401,000
1900	75,995,000	77,472,000	76,389,000	+1,477,000	+ 394,000
1910	91,972,000	94,673,000	91,647,000	+2,701,000	− 325,000
				935,000[2]	472,000[2]
1920		114,416,000	108,214,000		

[1] To the nearest thousand.

[2] Root-mean square error.

The logistic curve

Pearl and Reed try

$$P(t) = \frac{be^{at}}{1 + ce^{at}}$$

or

$$P(t) = \frac{b}{e^{-at} + c}.$$

The logistic equation

The logistic curve is the solution to the ordinary differential equation

$$N' = rN\left(1 - \frac{N}{K}\right),$$

which is called the *logistic equation*. r is the *intrinsic growth rate*, K is the *carrying capacity*.

This equation was introduced by Pierre-François Verhulst (1804-1849), in 1844.

Deriving the logistic equation

The idea is to represent a population with the following components:

- birth, at the *per capita* rate b,
- death, at the *per capita* rate d,
- competition of individuals with other individuals reduces their ability to survive, resulting in death.

This gives

$$N' = bN - dN - \text{competition}.$$

Accounting for competition

Competition describes the mortality that occurs when two individuals meet.

- In chemistry, if there is a concentration X of one product and Y of another product, then XY, called *mass action*, describes the number of interactions of molecules of the two products.
- Here, we assume that X and Y are of the same type (individuals). So there are N^2 contacts.
- These N^2 contacts lead to death of one of the individuals at the rate c.

Therefore, the *logistic* equation is

$$N' = bN - dN - cN^2.$$

Reinterpreting the logistic equation

The equation

$$N' = bN - dN - cN^2$$

is rewritten as

$$N' = (b - d)N - cN^2.$$

- $b - d$ represents the rate at which the population increases (or decreases) in the absence of competition. It is called the *intrinsic growth rate* of the population.
- c is the rate of *intraspecific* competition. The prefix *intra* refers to the fact that the competition is occurring between members of the same species, that is, within the species. [We will see later examples of *interspecific* competition, that is, between different species.]

Another (..) interpretation of the logistic equation

We have

$$N' = (b - d)N - cN^2.$$

Factor out an N:

$$N' = \big((b - d) - cN\big)N.$$

This gives us another interpretation of the logistic equation. Writing

$$\frac{N'}{N} = (b - d) - cN,$$

we have N'/N, the *per capita growth rate* of N, given by a constant, $b - d$, minus a *density dependent inhibition* factor, cN.

Equivalent equations

$$\begin{aligned}
N' &= (b-d)N - cN^2 \\
&= \big((b-d) - cN\big)N \\
&= \left(r - \frac{r}{r}cN\right)N, \quad \text{with } r = b-d \\
&= rN\left(1 - \frac{c}{r}N\right) \\
&= rN\left(1 - \frac{N}{K}\right),
\end{aligned}$$

with

$$\frac{c}{r} = \frac{1}{K},$$

that is, $K = r/c$.

3 ways to tackle this equation

1. The equation is separable. [explicit method]
2. The equation is a Bernoulli equation. [explicit method]
3. Use qualitative analysis.

Studying the logistic equation qualitatively

We study

$$N' = rN\left(1 - \frac{N}{K}\right). \tag{ODE1}$$

For this, write

$$f(N) = rN\left(1 - \frac{N}{K}\right).$$

Consider the initial value problem (IVP)

$$N' = f(N), \quad N(0) = N_0 > 0. \tag{IVP1}$$

- f is C^1 (differentiable with continuous derivative) so solutions to (IVP1) exist and are unique.

Equilibria of (ODE1) are points such that $f(N) = 0$ (so that $N' = f(N) = 0$, meaning N does not vary). So we solve $f(N) = 0$ for N. We find two points:

- $N = 0$
- $N = K$.

By uniqueness of solutions to (IVP1), solutions cannot cross the lines $N(t) = 0$ and $N(t) = K$.

There are several cases.

- $N = 0$ for some t, then $N(t) = 0$ for all $t \geq 0$, by uniqueness of solutions.
- $N \in (0, K)$, then $rN > 0$ and $N/K < 1$ so $1 - N/K > 0$, which implies that $f(N) > 0$. As a consequence, $N(t)$ increases if $N \in (0, K)$.
- $N = K$, then $rN > 0$ but $N/K = 1$ so $1 - N/K = 0$, which implies that $f(N) = 0$. As a consequence, $N(t) = K$ for all $t \geq 0$, by uniqueness of solutions.
- $N > K$, the $rN > 0$ and $N/K > 1$, implying that $1 - N/K < 0$ and in turn, $f(N) < 0$. As a consequence, $N(t)$ decreases if $N \in (K, +\infty)$.

Therefore,

Theorem

Suppose that $N_0 > 0$*. Then the solution* $N(t)$ *of* (IVP1) *is such that*

$$\lim_{t\to\infty} N(t) = K,$$

so that K *is the number of individuals that the environment can support, the carrying capacity of the environment.*
If $N_0 = 0$*, then* $N(t) = 0$ *for all* $t \geq 0$*.*

The delayed logistic equation

Consider the equation as

$$\frac{N'}{N} = (b - d) - cN,$$

that is, the per capita rate of growth of the population depends on the net growth rate $b - d$, and some density dependent inhibition cN (resulting of competition).
Suppose that instead of instantaneous inhibition, there is some delay τ between the time the inhibiting event takes place and the moment where it affects the growth rate. (For example, two individuals fight for food, and one later dies of the injuries sustained when fighting).

The delay logistic equation

In the of a time τ between inhibiting event and inhibition, the equation would be written as

$$\frac{N'}{N} = (b - d) - cN(t - \tau).$$

Using the change of variables introduced earlier, this is written

$$N'(t) = rN(t)\left(1 - \frac{N(t-\tau)}{K}\right). \tag{DDE1}$$

Such an equation is called a *delay* differential equation. It is much more complicated to study than (ODE1). In fact, some things remain unknown about (DDE1).

Delayed initial value problem

The IVP takes the form

$$N'(t) = rN(t)\left(1 - \frac{N(t-\tau)}{K}\right),$$
$$N(t) = \phi(t) \text{ for } t \in [-\tau, 0], \tag{IVP2}$$

where $\phi(t)$ is some continuous function. Hence, initial conditions (called initial data in this case) must be specific on an interval, instead of being specified at a point, to guarantee existence and uniqueness of solutions.
We will not learn how to study this type of equation (this is graduate level mathematics). I will give a few results.

To find equilibria, remark that delay should not play a role, since N should be constant. Thus, equilibria are found by considering the equation with no delay, which is (ODE1).

Theorem

Suppose that $r\tau < 22/7$. Then all solutions of (IVP2) *with positive initial data $\phi(t)$ tend to K. If $r\tau > \pi/2$, then K is an unstable equilibrium and all solutions of* (IVP2) *with positive initial data $\phi(t)$ on $[-\tau, 0]$ are oscillatory.*

Note that there is a gray zone between $22/7$ and $\pi/2$.. The first part of the theorem was proved in 1945 by Wright. Although there is very strong numerical evidence that this is in fact true up to $\pi/2$, nobody has yet managed to prove it.

Discrete-time systems

So far, we have seen continuous-time models, where $t \in \mathbb{R}_+$. Another way to model natural phenomena is by using a discrete-time formalism, that is, to consider equations of the form

$$x_{t+1} = f(x_t),$$

where $t \in \mathbb{N}$ or $\mathbb{Z}$, that is, t takes values in a discrete valued (countable) set.

Time could for example be days, years, etc.

The logistic map

The logistic *map* is, for $t \geq 0$,

$$N_{t+1} = rN_t \left(1 - \frac{N_t}{K}\right). \tag{DT1}$$

To transform this into an initial value problem, we need to provide an initial condition $N_0 \geq 0$ for $t = 0$.

Some mathematical analysis

Suppose we have a system in the form

$$x_{t+1} = f(x_t),$$

with initial condition given for $t = 0$ by x_0. Then,

$$\begin{aligned} x_1 &= f(x_0) \\ x_2 &= f(x_1) = f(f(x_0)) \stackrel{\Delta}{=} f^2(x_0) \\ &\vdots \\ x_k &= f^k(x_0). \end{aligned}$$

The $f^k = \underbrace{f \circ f \circ \cdots \circ f}_{k \text{ times}}$ are called the *iterates* of f.

Fixed points

Definition (Fixed point)

Let f be a function. A point p such that $f(p) = p$ is called a *fixed point* of f.

Theorem

Consider the closed interval $I = [a, b]$. If $f : I \to I$ is continuous, then f has a fixed point in I.

Theorem

Let I be a closed interval and $f : I \to \mathbb{R}$ be a continuous function. If $f(I) \supset I$, then f has a fixed point in I.

Periodic points

Definition (Periodic point)

Let f be a function. If there exists a point p and an integer n such that

$$f^n(p) = p, \quad \text{but} \quad f^k(p) \neq p \text{ for } k < n,$$

then p is a periodic point of f with (least) period n (or a n-periodic point of f).

Thus, p is a n-periodic point of f iff p is a 1-periodic point of f^n.

Stability of fixed points, of periodic points

Theorem

Let f be a continuously differentiable function (that is, differentiable with continuous derivative, or C^1), and p be a fixed point of f.

1. *If $|f'(p)| < 1$, then there is an open interval $\mathcal{I} \ni p$ such that $\lim_{k\to\infty} f^k(x) = p$ for all $x \in \mathcal{I}$.*
2. *If $|f'(p)| > 1$, then there is an open interval $\mathcal{I} \ni p$ such that if $x \in \mathcal{I}$, $x \neq p$, then there exists k such that $f^k(x) \notin \mathcal{I}$.*

Definition

Suppose that p is a n-periodic point of f, with $f \in C^1$.

- If $|(f^n)'(p)| < 1$, then p is an *attracting* periodic point of f.
- If $|(f^n)'(p)| > 1$, then p is an *repelling* periodic point of f.

Parametrized families of functions

Consider the equation (DT1), which for convenience we rewrite as

$$x_{t+1} = rx_t(1 - x_t), \tag{DT2}$$

where r is a parameter in $\mathbb{R}_+$, and x will typically be taken in $[0, 1]$. Let

$$f_r(x) = rx(1 - x).$$

The function f_r is called a *parametrized family* of functions.

Bifurcations

Definition (Bifurcation)

Let f_μ be a parametrized family of functions. Then there is a *bifurcation* at $\mu = \mu_0$ (or μ_0 is a bifurcation point) if there exists $\varepsilon > 0$ such that, if $\mu_0 - \varepsilon < a < \mu_0$ and $\mu_0 < b < \mu_0 + \varepsilon$, then the dynamics of $f_a(x)$ are "different" from the dynamics of $f_b(x)$.

An example of "different" would be that f_a has a fixed point (that is, a 1-periodic point) and f_b has a 2-periodic point.

Back to the logistic map

Consider the simplified version (DT2),

$$x_{t+1} = rx_t(1 - x_t) \triangleq f_r(x_t).$$

Are solutions well defined? Suppose $x_0 \in [0, 1]$, do we stay in $[0, 1]$? f_r is continuous on $[0, 1]$, so it has a extrema on $[0, 1]$. We have

$$f_r'(x) = r - 2rx = r(1 - 2x),$$

which implies that f_r increases for $x < 1/2$ and decreases for $x > 1/2$, reaching a maximum at $x = 1/2$.

$f_r(0) = f_r(1) = 0$ are the minimum values, and $f(1/2) = r/4$ is the maximum. Thus, if we want $x_{t+1} \in [0, 1]$ for $x_t \in [0, 1]$, we need to consider $r \leq 4$.

- Note that if $x_0 = 0$, then $x_t = 0$ for all $t \geq 1$.
- Similarly, if $x_0 = 1$, then $x_1 = 0$, and thus $x_t = 0$ for all $t \geq 1$.
- This is true for all t: if there exists t_k such that $x_{t_k} = 1$, then $x_t = 0$ for all $t \geq t_k$.
- This last case might occur if $r = 4$, as we have seen.
- Also, if $r = 0$ then $x_t = 0$ for all t.

For these reasons, we generally consider

$$x \in (0, 1)$$

and

$$r \in (0, 4).$$

Fixed points: existence

Fixed points of (DT2) satisfy $x = rx(1-x)$, giving:

- $x = 0$;
- $1 = r(1-x)$, that is, $p \triangleq \dfrac{r-1}{r}$.

Note that $\lim_{r \to 0^+} p = 1 - \lim_{r \to 0^+} 1/r = -\infty$, $\frac{\partial}{\partial r} p = 1/r^2 > 0$ (so p is an increasing function of r), $p = 0 \Leftrightarrow r = 1$ and $\lim_{r \to \infty} p = 1$. So we come to this first conclusion:

- 0 always is a fixed point of f_r.
- If $0 < r < 1$, then p tales negative values so is not relevant.
- If $1 < r < 4$, then p exists.

Stability of the fixed points

Stability of the fixed points is determined by the (absolute) value f_r' at these fixed points. We have

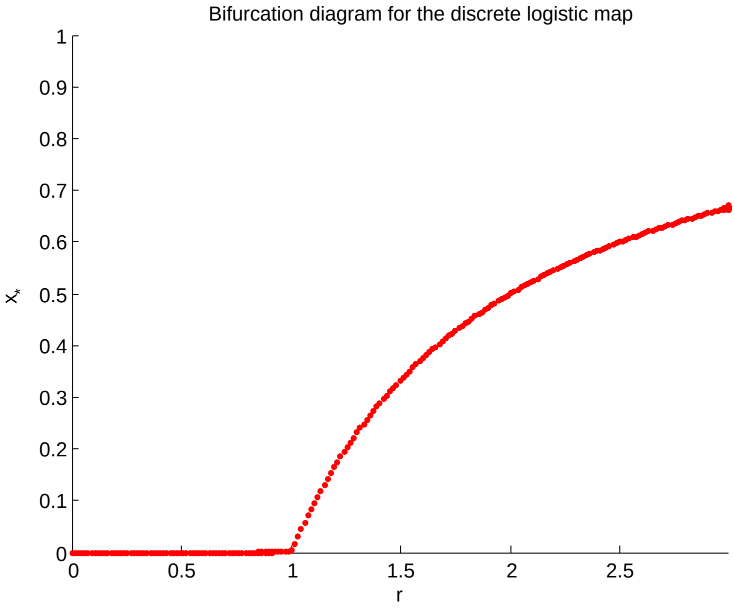
$$|f_r'(0)| = r,$$

and

$$\begin{aligned} |f_r'(p)| &= \left| r - 2r\frac{r-1}{r} \right| \\ &= |r - 2(r-1)| \\ &= |2 - r| \end{aligned}$$

Therefore, we have

- if $0 < r < 1$, then the fixed point $x = p$ does not exist and $x = 0$ is attracting,
- if $1 < r < 3$, then $x = 0$ is repelling, and $x = p$ is attracting,
- if $r > 3$, then $x = 0$ and $x = p$ are repelling.


Bifurcation diagram for the discrete logistic map
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Another bifurcation

Thus the points $r = 1$ and $r = 3$ are bifurcation points. To see what happens when $r > 3$, we need to look for period 2 points.

$$\begin{aligned} f_r^2(x) &= f_r(f_r(x)) \\ &= rf_r(x)(1 - f_r(x)) \\ &= r^2x(1-x)(1-rx(1-x)). \end{aligned} \tag{1}$$

0 and p are points of period 2, since a fixed point x^* of f satisfies $f(x^*) = x^*$, and so, $f^2(x^*) = f(f(x^*)) = f(x^*) = x^*$.
This helps localizing the other periodic points. Writing the fixed point equation as

$$Q(x) \stackrel{\Delta}{=} f_r^2(x) - x = 0,$$

we see that, since 0 and p are fixed points of f_μ^2, they are roots of $Q(x)$. Therefore, Q can be factorized as

$$Q(x) = x(x-p)(-r^3x^2 + Bx + C),$$

Substitute the value $(r-1)/r$ for p in Q, develop Q and (1) and equate coefficients of like powers gives

$$Q(x) = x\left(x - \frac{r-1}{r}\right)\left(-r^3x^2 + r^2(r+1)x - r(r+1)\right). \quad (2)$$

We already know that $x = 0$ and $x = p$ are roots of (2). So we search for roots of

$$R(x) := -r^3x^2 + r^2(r+1)x - r(r+1).$$

Discriminant is

$$\begin{aligned} \Delta &= r^4(r+1)^2 - 4r^4(r+1) \\ &= r^4(r+1)(r+1-4) \\ &= r^4(r+1)(r-3). \end{aligned}$$

Therefore, R has distinct real roots if $r > 3$. Remark that for $r = 3$, the (double) root is $p = 2/3$. For $r > 3$ but very close to 3, it follows from the continuity of R that the roots are close to $2/3$.

Descartes' rule of signs

Theorem (Descartes' rule of signs)

Let $p(x) = \sum_{i=0}^{m} a_i x^i$ be a polynomial with real coefficients such that $a_m \neq 0$. Define v to be the number of variations in sign of the sequence of coefficients $a_m, \ldots, a_0$. By 'variations in sign' we mean the number of values of n such that the sign of a_n differs from the sign of a_{n-1}, as n ranges from m down to 1. Then

- *the number of positive real roots of $p(x)$ is $v - 2N$ for some integer N satisfying $0 \leq N \leq \dfrac{v}{2}$,*
- *the number of negative roots of $p(x)$ may be obtained by the same method by applying the rule of signs to $p(-x)$.*

Example of use of Descartes' rule

Example

Let

$$p(x) = x^3 + 3x^2 - x - 3.$$

Coefficients have signs $++--$, i.e., 1 sign change. Thus $v = 1$. Since $0 \leq N \leq 1/2$, we must have $N = 0$. Thus $v - 2N = 1$ and there is exactly one positive real root of $p(x)$.

To find the negative roots, we examine $p(-x) = -x^3 + 3x^2 + x - 3$. Coefficients have signs $-++-$, i.e., 2 sign changes. Thus $v = 2$ and $0 \leq N \leq 2/2 = 1$. Thus, there are two possible solutions, $N = 0$ and $N = 1$, and two possible values of $v - 2N$. Therefore, there are either two or no negative real roots. Furthermore, note that $p(-1) = (-1)^3 + 3 \cdot (-1)^2 - (-1) - 3 = 0$, hence there is at least one negative root. Therefore there must be exactly two.

Back to the logistic map and the polynomial R..

We use Descartes' rule of signs.

- R has signed coefficients $- + -$, so 2 sign changes imlying 0 or 2 positive real roots.
- $R(-x)$ has signed coefficients $- - -$, so no negative real roots.
- Since $\Delta > 0$, the roots are real, and thus it follows that both roots are positive.

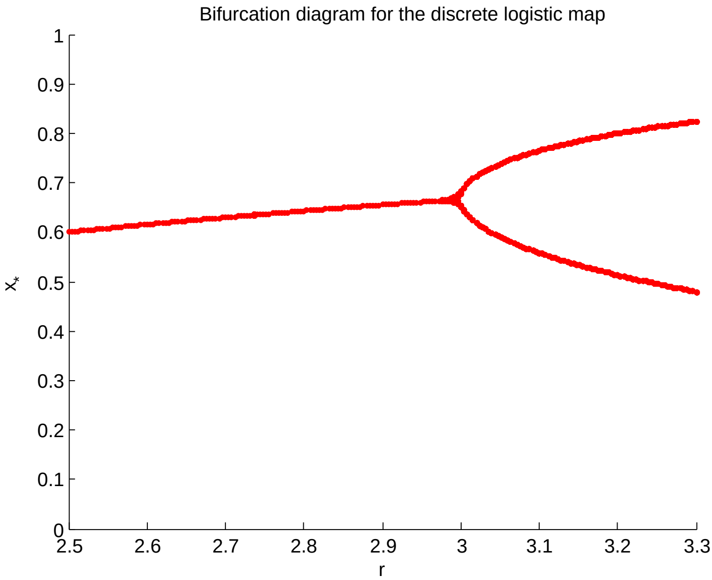
To show that the roots are also smaller than 1, consider the change of variables $z = x - 1$. The polynomial R is transformed into

$$\begin{aligned} R_2(z) &= -r^3(z+1)^2 + r^2(r+1)(z+1) - r(r+1) \\ &= -r^3 z^2 + r^2(1-r)z - r. \end{aligned}$$

For $r > 1$, the signed coefficients are $- - -$, so R_2 has no root $z > 0$, implying in turn that R has no root $x > 1$.

Summing up

- If $0 < r < 1$, then $x = 0$ is attracting, p does not exist and there are no period 2 points.
- At $r = 1$, there is a bifurcation (called a *transcritical* bifurcation).
- If $1 < r < 3$, then $x = 0$ is repelling, p is attracting, and there are no period 2 points.
- At $r = 3$, there is another bifurcation (called a *period-doubling* bifurcation).
- For $r > 3$, both $x = 0$ and $x = p$ are repelling, and there is a period 2 point.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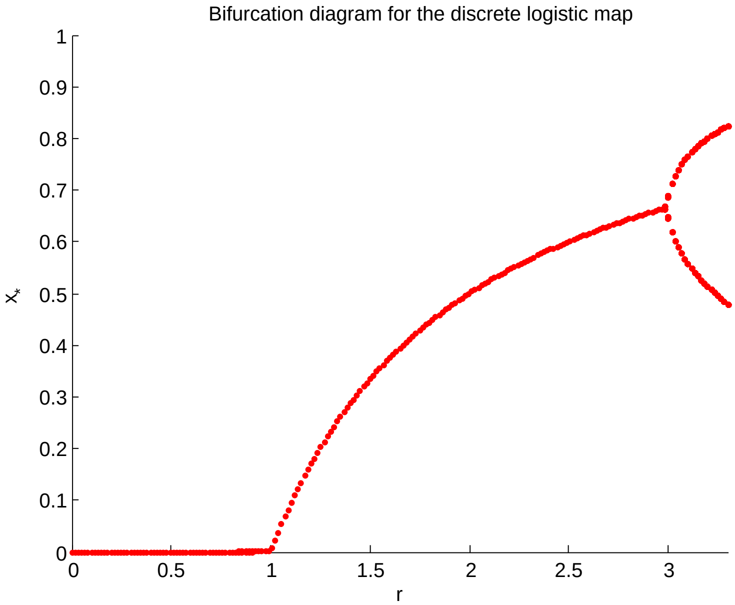
Bifurcation diagram for the discrete logistic map
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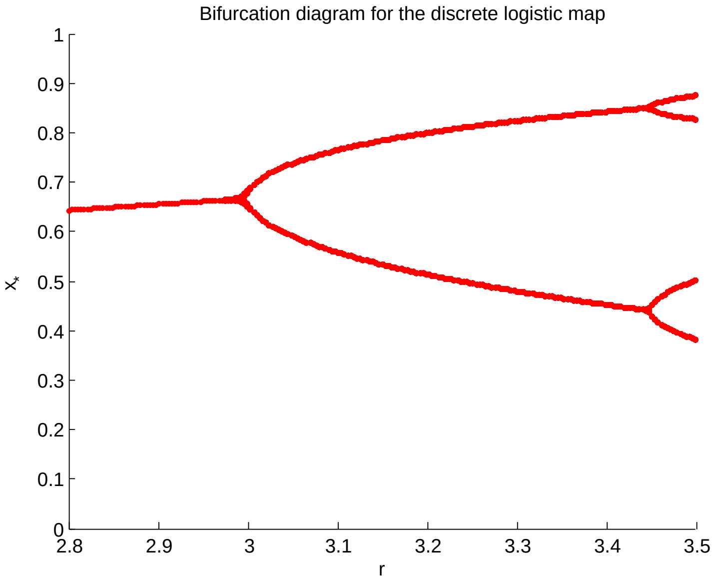
3

This process continues



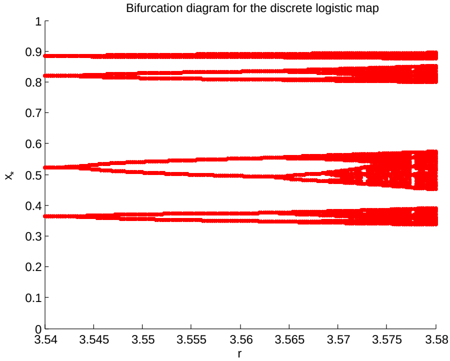
The period-doubling cascade to chaos

The logistic map undergoes a sequence of period doubling bifurcations, called the *period-doubling cascade*, as r increases from 3 to 4.

- Every successive bifurcation leads to a doubling of the period.
- The bifurcation points form a sequence, $\{r_n\}$, that has the property that

  $$\lim n \to \infty \frac{r_n - r_{n-1}}{r_{n+1} - r_n}$$

  exists and is a constant, called the Feigenbaum constant, equal to $4.669202\ldots$
- This constant has been shown to exist in many of the maps that undergo the same type of cascade of period doubling bifurcations.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0.2
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x*
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3.545
3.55
3.555
3.56
3.565
3.57
3.575
3.58
r

Chaos

After a certain value of r, there are periodic points with all periods. In particular, there are periodic points of period 3.

By a theorem (called the Sarkovskii theorem), the presence of period 3 points implies the presence of points of all periods.

At this point, the system is said to be in a *chaotic regime*, or *chaotic*.

A word of caution

We have used three different modelling paradigms to describe the growth of a population in a *logistic* framework:

- The ODE version has monotone solutions converging to the carrying capacity K.
- The DDE version has oscillatory solutions, either converging to K or, if the delay is too large, periodic about K.
- The discrete time version has all sorts of behaviors, and can be chaotic.

It is important to be aware that the **choice of modelling method** is almost **as important** in the outcome of the model as the precise formulation/hypotheses of the **model**.