Shallow water

Partial differential equations

Model formulation

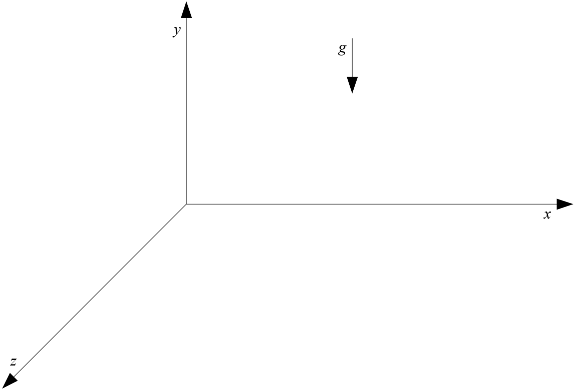
Case of smooth solutions

Linearization

Traveling wave solutions

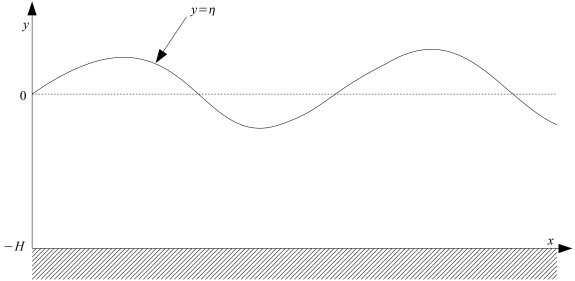
Spatial domain

We consider the motion of a body of water that is infinite in the z direction, with or without boundary in the x direction, and the vertical direction of gravity taken as the y direction.



From now on, suppose z direction uniform (the same for all z), so ignore z except for the sake of argument.

- Water depth at rest, H, small compared to distance L_0 over which significant changes can occur in the x direction.
- Undisturbed water surface, $y = 0$.
- Moving upper free surface $y = \eta$, measured from $y = 0$.
- Sea floor $y = -H$.

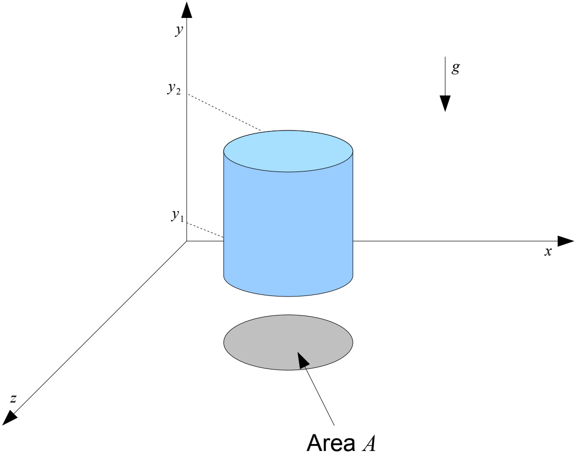


- u velocity in the x direction. Assume independent of depth y.
- ρ mass density of water.
- $p(x, y, t)$ pressure in fluid at point (x, y) at time t. In water, magnitude at any (x, y) is same in all directions.

Fluid motion independent of z, so

- $u = u(x, t)$
- $\eta = \eta(x, t)$.

Take a cylindrical water column, with base area A, between y_1 and $y_2 > y_1$.



Force equilibrium in the y direction in this cylinder requires balance of weight of water column and pressure differential between bottom face $y = y_1$ and top face $y = y_2$.

Weight of water column:

$$\iint_A \int_{y_1}^{y_2} (-\rho g)\ dydxdz$$

Pressure differential:

$$\iint_A (p(x, y_2, t) - p(x, y_1, t))\ dxdz$$

So we must have

$$\iint_A \int_{y_1}^{y_2} (-\rho g)\ dydxdz = \iint_A (p(x, y_2, t) - p(x, y_1, t))\ dxdz$$

$$\iint\limits_{A} \int\limits_{y_1}^{y_2} (-\rho g) \; dydxdz = \iint\limits_{A} \left(p(x, y_2, t) - p(x, y_1, t)\right) \; dxdz$$

is equivalent to

$$\iint\limits_{A} \int\limits_{y_1}^{y_2} \left(\frac{\partial p}{\partial y} + \rho g\right) \; dydxdz = 0$$

This must be true for any water column, i.e., any A, y_1, y_2.
Therefore,

$$\frac{\partial p}{\partial y} + \rho g = 0$$

(otherwise, we would be able to find a water column where the integrand is positive, leading to a positive value of the integral on that column).

Water is incompressible

If you force a body of water to deform, the volume of that body of water remains constant, i.e., water is an *incompressible fluid*.

$\Rightarrow \rho$, the density, is a constant, and from

$$\frac{\partial p}{\partial y} + \rho g = 0$$

we get

$$p = -\rho g y + C,$$

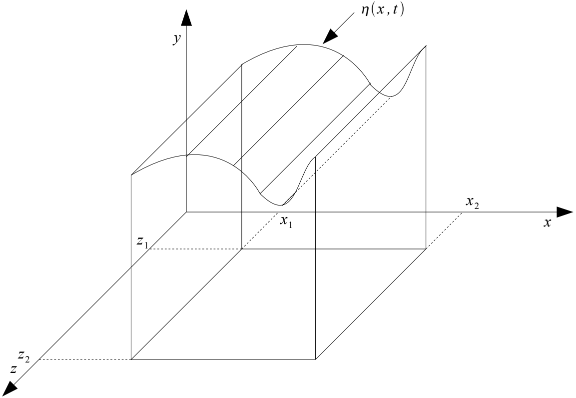
so if p is measured relative to the pressure above the free upper surface $y = \eta$,

$$p = \rho g(\eta - y)$$

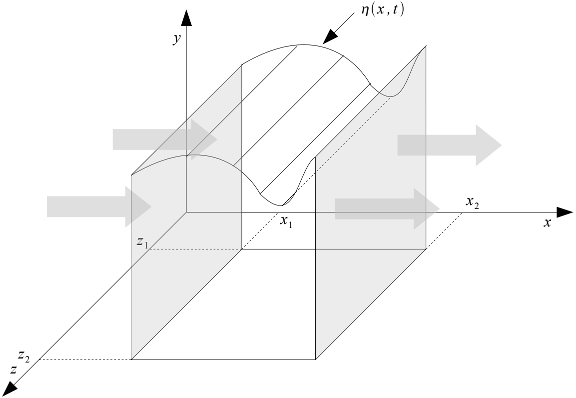
Water accumulation

Consider a fixed volume V,

$$V = \{z_1 \leq z \leq z_2, x_1 \leq x \leq x_2, -H \leq y \leq \eta\}$$



Water enters V through x_1 face and leaves V through x_2 face.



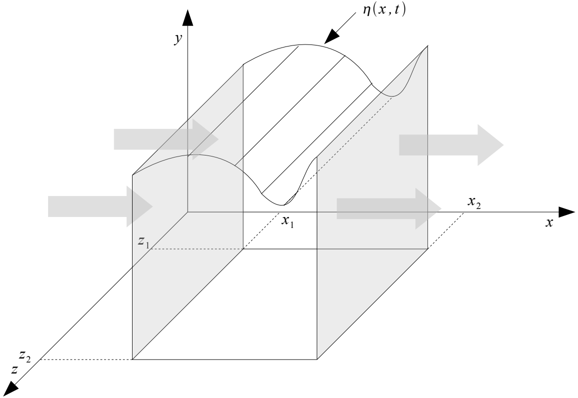
Rate of water accumulation in V is

$$\frac{d}{dt}\int_{z_1}^{z_2}\int_{x_1}^{x_2}\int_{-H}^{\eta}\rho \; dydxdz = \Delta z \frac{d}{dt}\int_{x_1}^{x_2}\rho h \; dx,$$

with $\Delta z = z_2 - z_1$, and $h(x,t) = \eta + H$ the height of water at time t at spatial location x.

Water flux

Net flux of water entering V through its faces $x = x_1$ and $x = x_2$ is



$$\left[\int_{z_1}^{z_2} \int_{-H}^{\eta} u \; dydz\right]_{x=x_1} - \left[\int_{z_1}^{z_2} \int_{-H}^{\eta} u \; dydz\right]_{x=x_2} = -\Delta z \left[\rho u h\right]_{x_1}^{x_2}$$

There is no flux through $y = -H$ and $y = \eta$, and no net flux through $z = z_1$ and $z = z_2$.

Conservation of mass

Of course, the mass must conserve in V, so the two expressions must be equal, i.e.,

$$\frac{d}{dt}\int_{x_1}^{x_2} \rho h \; dx + [\rho u h]_{x_1}^{x_2} = 0$$

Newton's second law for deformable media (Euler): rate of increase of horizontal momentum (in the x direction) in V must equal the sum of the net influx of momentum into the volume and the net horizontal force acting on the column.

(Momentum: product of mass and velocity of an object).

Rate of increase of momentum

$$\frac{d}{dt}\int_{z_1}^{z_2}\int_{x_1}^{x_2}\int_{-H}^{\eta}\rho u \; dydxdz = \Delta z \frac{d}{dt}\int_{x_1}^{x_2}\rho u h dx$$

Momentum flux

Net influx of momentum through faces $x = x_1$ and $x = x_2$ is

$$\left[\int_{z_1}^{z_2}\int_{-H}^{\eta}(\rho u)u \; dydz\right]_{x=x_1} - \left[\int_{z_1}^{z_2}\int_{-H}^{\eta}(\rho u)u \; dydz\right]_{x=x_2}$$
$$= -\Delta z \left[\rho u^2 h\right]_{x_1}^{x_2}$$

There is no flux through $y = -H$ and $y = \eta$, and no net flux through $z = z_1$ and $z = z_2$.

Forces acting on V

Ignore friction at $y = -H$. Then only contributions to horizontal forces come from pressure at $x = x_1$ and $x = x_2$, so net horizontal forces acting on V is

$$\left[\int_{z_1}^{z_2}\int_{-H}^{\eta} p \; dydz\right]_{x_1}^{x_2} = -\left[\Delta z \int_{-H}^{\eta} \rho g(\eta - y) \; dy\right]_{x_1}^{x_2}$$

$$= \left[-\Delta z \rho g(\eta y - \frac{1}{2}y^2)\Big|_{-H}^{\eta}\right]_{x_1}^{x_2}$$

$$= \left[-\frac{1}{2}\Delta z \rho g h^2\right]_{x_1}^{x_2}$$

Conclusion from Newton's second law

$$\frac{d}{dt}\int_{x_1}^{x_2} \rho u h \; dx + \left[\rho u^2 h + \frac{1}{2}\rho g h^2\right]_{x_1}^{x_2} = 0$$

The general model

Pressure magnitude:

$$p = \rho g(\eta - y) \tag{1}$$

Horizontal velocity:

$$\frac{d}{dt}\int_{x_1}^{x_2} \rho h \ dx + [\rho u h]_{x_1}^{x_2} = 0 \tag{2}$$

Free surface height:

$$\frac{d}{dt}\int_{x_1}^{x_2} \rho u h \ dx + \left[\rho u^2 h + \frac{1}{2}\rho g h^2\right]_{x_1}^{x_2} = 0 \tag{3}$$

Suppose u and h are smooth (with continuous first order partial derivatives), then (2) and (3) take a much simpler form,

$$\int_{x_1}^{x_2} \left(\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(uh) \right) \, dx = 0$$

and

$$\int_{x_1}^{x_2} \left(\frac{\partial}{\partial t}(uh) + \frac{\partial}{\partial x}(u^2 h + \frac{1}{2} g h^2) \right) \, dx = 0$$

Since the intervals of integration $[x_1, x_2]$ are arbitrary, and that the integrands are continuous, we have

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(uh) = 0$$

and

$$\frac{\partial}{\partial t}(uh) + \frac{\partial}{\partial x}(u^2 h + \frac{1}{2} g h^2) = 0$$

We write

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(uh) = 0$$

and

$$\frac{\partial}{\partial t}(uh) + \frac{\partial}{\partial x}(u^2 h + \frac{1}{2} g h^2) = 0$$

as

$$h_t + (uh)_x = 0 \tag{4}$$

and

$$(uh)_t + (u^2 h + \frac{1}{2} g h^2)_x = 0 \tag{5}$$

From (4),

$$h_t = -(uh)_x = -(u_x h + uh_x)$$

Equation (5) can be rewritten as

$$\begin{aligned}
(5) &\Leftrightarrow u_t h + uh_t + (u^2 h + \frac{1}{2} g h^2)_x = 0 \\
&\Leftrightarrow u_t h - u(u_x h + uh_x) + 2uu_x h + u^2 h_x + ghh_x = 0 \\
&\Leftrightarrow u_t h - uu_x h - \cancel{u^2 h_x} + 2uu_x h + \cancel{u^2 h_x} + ghh_x = 0 \\
&\Leftrightarrow u_t h + uu_x h + ghh_x = 0
\end{aligned}$$

Therefore, provided $h \neq 0$, we get

$$h_t + (uh)_x = 0 \tag{6a}$$

$$u_t + uu_x + gh_x = 0 \tag{6b}$$

which describes the evolution of u and h.

The model for smooth solutions

$$h_t + (uh)_x = 0 \tag{6a}$$

$$u_t + uu_x + gh_x = 0 \tag{6b}$$

If $-\infty < x < \infty$, then all we need is an initial condition, i.e., functions describing the initial state of u and h:

$$u(x,0) = u_0(x), \quad h(x,0) = h_0(x), \qquad -\infty < x < \infty.$$

If x has a boundary, then we need boundary conditions.

Suppose the bottom is flat (H is constant), and that the deviation from the undisturbed depth H is small compared to H itself, then

$$h = (H + \zeta) = H(1 + \frac{\zeta}{H}) \simeq H, \qquad h_t = \zeta_t, \qquad h_x = \zeta_x.$$

If $|u|$ is also small, then uu_x can be neglected. Then we can linearize

$$h_t + (uh)_x = 0 \tag{6a}$$
$$u_t + uu_x + gh_x = 0, \tag{6b}$$

getting

$$\zeta_t + Hu_x = 0 \tag{7a}$$
$$u_t + g\zeta_x = 0 \tag{7b}$$

Differentiate (7b) with respect to x:

$$u_{tx} + g\zeta_{xx} = 0$$

and therefore,

$$u_{tx} = -g\zeta_{xx} \tag{8}$$

Differentiate (7a) with respect to t:

$$\zeta_{tt} + Hu_{xt} = 0 \tag{9}$$

If u has continuous second-order partial derivatives, then from Clairaut's theorem, $u_{tx} = u_{xt}$. Therefore, substituting (8) into (9),

$$\zeta_{tt} - HG\zeta_{xx} = 0$$

that is

$$\zeta_{tt} = c^2\zeta_{xx}, \qquad c^2 = Hg$$

The one-dimensional wave equation (1)

The partial differential equation

$$\zeta_{tt} = c^2 \zeta_{xx} \tag{10}$$

with $c^2 = Hg$, is the one-dimensional wave equation. Initial conditions are given by

$$\begin{aligned} \zeta(x,0) &= h_0(x) - H \equiv \zeta_0(x) \\ \zeta_t(x,0) &= -Hu_x(x,0) = -H[u_0(x)]_x \equiv \nu_0(x) \end{aligned}$$

The one-dimensional wave equation (2)

Things can also be expressed in terms of u. Using the same type of simplification used before for ζ, we get

$$u_{tt} = c^2 u_{xx} \tag{11}$$

with $c^2 = Hg$. Initial conditions are given by

$$\begin{aligned} u(x,0) &= u_0(x) \\ u_t(x,0) &= -g\zeta_x(x,0) = -g[h_0(x)]_x \equiv v_0(x) \end{aligned}$$

Traveling wave solutions

This was obtained by d'Alembert. Consider

$$u_{tt} = c^2 u_{xx} \tag{11}$$

Note that this can be written as

$$\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right) u = 0$$

This implies that for any F, G, the sum

$$u(x,t) = F(x-ct) + G(x+ct)$$

satisfies (11).

Derivation of the solution

Introduce the new variables

$$a = x - ct \qquad \text{and} \qquad b = x + ct$$

We have

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial a} + \frac{\partial u}{\partial b} \qquad \frac{\partial u}{\partial t} = -c\frac{\partial u}{\partial a} + c\frac{\partial u}{\partial b}$$

$$\frac{\partial^2}{\partial x^2}u = \left(\frac{\partial}{\partial a} + \frac{\partial}{\partial b}\right)^2 u = \frac{\partial^2 u}{\partial a^2} + 2\frac{\partial^2 u}{\partial a \partial b} + \frac{\partial^2 u}{\partial b^2}$$

$$\frac{\partial^2}{\partial t^2}u = \left(-c\frac{\partial}{\partial a} + c\frac{\partial}{\partial b}\right)^2 u = c^2\left(\frac{\partial^2 u}{\partial a^2} - 2\frac{\partial^2 u}{\partial a \partial b} + \frac{\partial^2 u}{\partial b^2}\right)$$

So the equation

$$u_{tt} = c^2 u_{xx} \tag{11}$$

is written

$$4\frac{\partial^2 u}{\partial a \partial b} = 0$$

Integrate with respect to b:

$$\frac{\partial u}{\partial a} = \xi(a)$$

and thus

$$\begin{aligned} u(x,t) = u(a,b) &= \int \xi(a) da + G(b) \\ &= F(a) + G(b) \\ &= F(x - ct) + G(x + ct) \end{aligned}$$

Set

$$u(x,0) = f(x) \qquad u_t(x,0) = g(x)$$

Then d'Alembert's formula gives

$$u(x,t) = \frac{f(x-ct) + f(x+ct)}{2} + \frac{1}{2c} \int_{x-ct}^{x+ct} g(s)ds$$

Case of a Dirac delta initial condition

Suppose $u_0(x) = 0$ and $v_0(x) = \delta(x)$, for $-\infty < x < \infty$, with δ the Dirac delta,

$$\delta(x) = \begin{cases} \infty & \text{if } x = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Therefore,

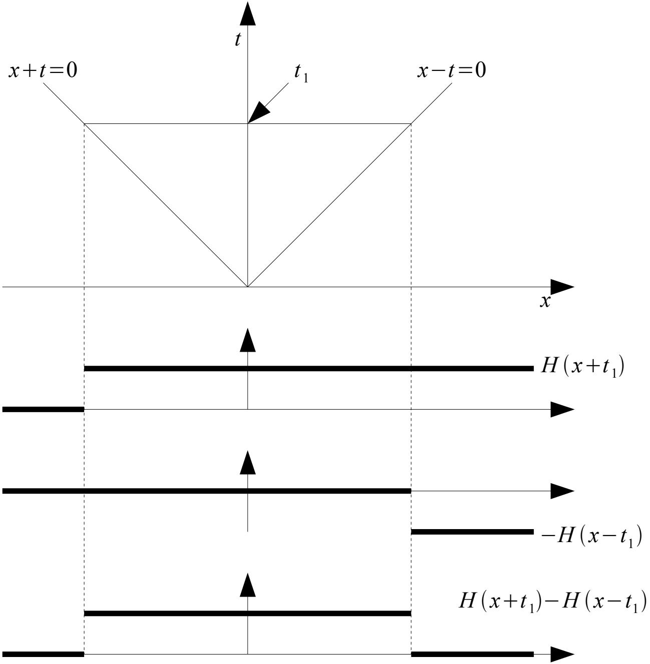
$$u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} \delta(z)dz = \frac{1}{2c}\left\{H(x+ct) - H(x-ct)\right\},$$

with H the Heaviside function,

$$H(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x > 0. \end{cases}$$

For simplicity, take $c = 1$. This gives

$$u(x,t) = \frac{1}{2}\left\{H(x+t) - H(x-t)\right\},$$


t
$x+t=0$
t_1
$x-t=0$
x
$H(x+t_1)$
$-H(x-t_1)$
$H(x+t_1)-H(x-t_1)$

As t increases, we move further up in the top graph in (x, t)-space, resulting in a wider and wider square pulse.