Summary of topics relevant for the final

Mathematical techniques

Regression

Discrete time systems

Systems of ODEs

Linear systems of ODE – Brief theory

PDEs

Some elementary probability

Markov chains

Outline

Regression

See Dr. Berry's notes.

Outline

Discrete-time systems

So far, we have seen continuous-time models, where $t \in \mathbb{R}_+$.
Another way to model natural phenomena is by using a discrete-time formalism, that is, to consider equations of the form

$$x_{t+1} = f(x_t),$$

where $t \in \mathbb{N}$ or $\mathbb{Z}$, that is, t takes values in a discrete valued (countable) set.

Time could for example be days, years, etc.

Some mathematical analysis

Suppose we have a system in the form

$$x_{t+1} = f(x_t),$$

with initial condition given for $t = 0$ by x_0. Then,

$$\begin{aligned} x_1 &= f(x_0) \\ x_2 &= f(x_1) = f(f(x_0)) \stackrel{\Delta}{=} f^2(x_0) \\ &\vdots \\ x_k &= f^k(x_0). \end{aligned}$$

The $f^k = \underbrace{f \circ f \circ \cdots \circ f}_{k \text{ times}}$ are called the *iterates* of f.

Fixed points

Definition 1 (Fixed point)

Let f be a function. A point p such that $f(p) = p$ is called a *fixed point* of f.

Theorem 2

Consider the closed interval $I = [a, b]$. If $f : I \to I$ is continuous, then f has a fixed point in I.

Theorem 3

Let I be a closed interval and $f : I \to \mathbb{R}$ be a continuous function. If $f(I) \supset I$, then f has a fixed point in I.

Periodic points

Definition 4 (Periodic point)

Let f be a function. If there exists a point p and an integer n such that

$$f^n(p) = p, \quad \text{but} \quad f^k(p) \neq p \text{ for } k < n,$$

then p is a periodic point of f with (least) period n (or a n-periodic point of f).

Thus, p is a n-periodic point of f iff p is a 1-periodic point of f^n.

Stability of fixed points, of periodic points

Theorem 5

Let f be a continuously differentiable function (that is, differentiable with continuous derivative, or C^1), and p be a fixed point of f.

1. *If $|f'(p)| < 1$, then there is an open interval $\mathcal{I} \ni p$ such that $\lim_{k\to\infty} f^k(x) = p$ for all $x \in \mathcal{I}$.*
2. *If $|f'(p)| > 1$, then there is an open interval $\mathcal{I} \ni p$ such that if $x \in \mathcal{I}$, $x \neq p$, then there exists k such that $f^k(x) \notin \mathcal{I}$.*

Definition 6

Suppose that p is a n-periodic point of f, with $f \in C^1$.

- If $|(f^n)'(p)| < 1$, then p is an *attracting* periodic point of f.
- If $|(f^n)'(p)| > 1$, then p is an *repelling* periodic point of f.

Parametrized families of functions

Consider a system

$$x_{t+1} = f(x_t)$$

which depends on a parameter r. We write

$$x_{t+1} = f_r(x_t).$$

The function f_r is called a *parametrized family* of functions.

Bifurcations

Definition 7 (Bifurcation)

Let f_μ be a parametrized family of functions. Then there is a *bifurcation* at $\mu = \mu_0$ (or μ_0 is a bifurcation point) if there exists $\varepsilon > 0$ such that, if $\mu_0 - \varepsilon < a < \mu_0$ and $\mu_0 < b < \mu_0 + \varepsilon$, then the dynamics of $f_a(x)$ are "different" from the dynamics of $f_b(x)$.

An example of "different" would be that f_a has a fixed point (that is, a 1-periodic point) and f_b has a 2-periodic point.

Outline

Steps of the analysis

1. Assess well-posedness of the system:
   1.1 Determine whether solutions exist and are unique.
   1.2 Determine whether solutions remain in a realistic region and are bounded.
2. Find the equilibria of the system.
3. Determine the local stability properties of the equilibria.
4. Determine the global stability properties of the equilibria (**much harder**, often not possible).

Existence and uniqueness of solutions

Theorem 8 (Cauchy-Lipschitz)

Consider the equation $x' = f(x)$, with $x \in \mathbb{R}^n$, and suppose that $f \in C^1$. Then there exists a unique solution of $x' = f(x)$ such that $x(t_0) = x_0$, where $t_0 \in \mathbb{R}$ and $x_0 \in \mathbb{R}^n$, defined on the largest interval $J \ni t_0$ on which $f \in C^1$.

Equilibria

Definition 9 (Equilibrium point)

Consider a differential equation

$$x' = f(x), \qquad (1)$$

with $x \in \mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}^n$. Then x^* is an equilibrium (solution) of (1) if $f(x^*) = 0$.

Linearization

Consider x^* an equilibrium of (1). For simplicity, assume here that $x^* = 0$ (it is always possible to do this, by considering $y = x - x^*$).

Taylor's theorem:

$$f(x) = Df(0)x + \frac{1}{2}D^2f(0)(x,x) + \cdots ,$$

where $Df(0)$ is the Jacobian matrix of f evaluated at 0.

What is stability?

Definition 10 (Stable and unstable EP)

Let ϕ_t be the flow of (1), assumed to be defined for all $t \in \mathbb{R}$. An equilibrium x^* of (1) is (locally) *stable* if for all $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x \in \mathcal{N}_\delta(x^*)$ and $t \geq 0$, there holds

$$\phi_t(x) \in \mathcal{N}_\varepsilon(x^*).$$

The equilibrium point is *unstable* if it is not stable.

Definition 11 (Asymptotically stable EP)

Let ϕ_t be the flow of (1) is (locally) *asymptotically stable* if there exists $\delta > 0$ such that for all $x \in \mathcal{N}_\delta(x^*)$ and $t \geq 0$, there holds

$$\lim_{t\to\infty} \phi_t(x) = x^*.$$

Clearly, Asymtotically Stable $\Rightarrow$ Stable.

Hyperbolic EPs, sinks, sources

Definition 12 (Sink)

An equilibrium point x^* of (1) is *hyperbolic* if none of the eigenvalues of the matrix $Df(x^*)$ (Jacobian matrix of f evaluated at x^*) have zero real parts.

Definition 13 (Sink)

An equilibrium point x^* of (1) is a *sink* if all the eigenvalues of the matrix $Df(x^*)$ have negative real parts.

Definition 14 (Source)

An equilibrium point x^* of (1) is a *source* if all the eigenvalues of the matrix $Df(x^*)$ have positive real parts.

Theorem 15

If x^* *is a sink of* (1) *and for all the eigenvalues* λ_j *of the matrix* $Df(x^*)$

$$\Re(\lambda_j) < -\alpha < 0,$$

where $\Re(\lambda)$ *denotes the real part of* λ*, then for a given* $\varepsilon > 0$*, there exists* $\delta > 0$ *such that for all* $x \in \mathcal{N}_\delta(x^*)$*, the flow* $\phi_t(x)$ *of* (1) *satisfies*

$$\|\phi_t(x) - x^*\| \leq \varepsilon e^{-\alpha t}$$

for all $t \geq 0$*.*

Theorem 16

If x^* *is a stable equilibrium point of* (1)*, no eigenvalue of* $Df(x^*)$ *has positive real part.*

Phase plane analysis

- In $\mathbb{R}^2$, nullclines are curves.
- Nullclines are the level set 0 of the vector field. If we have

$$x_1' = f_1(x_1, x_2)$$
$$x_2' = f_2(x_1, x_2)$$

then the nullclines for x_1 are the curves defined by

$$\{(x_1, x_2) \in \mathbb{R}^2 : f_1(x_1, x_2) = 0\}$$

those for x_2 are

$$\{(x_1, x_2) \in \mathbb{R}^2 : f_2(x_1, x_2) = 0\}$$

- On the nullcline associated to one state variable, this state variable has zero derivative.
- Equilibria lie at the intersections of nullclines for both state variables (in $\mathbb{R}^2$).

Outline

Linear ODEs

Definition 17 (Linear ODE)

A *linear* ODE is a differential equation taking the form

$$\frac{d}{dt}x = A(t)x + B(t), \tag{LNH}$$

where $A(t) \in \mathcal{M}_n(\mathbb{R})$ with continuous entries, $B(t) \in \mathbb{R}^n$ with real valued, continuous coefficients, and $x \in \mathbb{R}^n$. The associated IVP takes the form

$$\begin{aligned} \frac{d}{dt}x &= A(t)x + B(t) \\ x(t_0) &= x_0. \end{aligned} \tag{2}$$

Types of systems

- $x' = A(t)x + B(t)$ is linear nonautonomous ($A(t)$ depends on t) nonhomogeneous (also called *affine* system).
- $x' = A(t)x$ is linear nonautonomous homogeneous.
- $x' = Ax + B$, that is, $A(t) \equiv A$ and $B(t) \equiv B$, is linear autonomous nonhomogeneous (or affine autonomous).
- $x' = Ax$ is linear autonomous homogeneous.

Existence and uniqueness of solutions

Theorem 18 (Existence and Uniqueness)

Solutions to (2) *exist and are unique on the whole interval over which* A *and* B *are continuous.*
In particular, if A, B *are constant, then solutions exist on* $\mathbb{R}$.

Autonomous linear systems

Consider the autonomous affine system

$$\frac{d}{dt}x = Ax + B, \tag{A}$$

and the associated homogeneous autonomous system

$$\frac{d}{dt}x = Ax. \tag{L}$$

Exponential of a matrix

Definition 19 (Matrix exponential)

Let $A \in \mathcal{M}_n(\mathbb{K})$ with $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. The *exponential* of A, denoted e^{At}, is a matrix in $\mathcal{M}_n(\mathbb{K})$, defined by

$$e^{At} = \mathbb{I} + \sum_{k=1}^{\infty} \frac{t^k}{k!} A^k,$$

where $\mathbb{I}$ is the identity matrix in $\mathcal{M}_n(\mathbb{K})$.

Properties of the matrix exponential

- $e^{At_1}e^{At_2} = e^{A(t_1+t_2)}$ for all $t_1, t_2 \in \mathbb{R}$. 1
- $Ae^{At} = e^{At}A$ for all $t \in \mathbb{R}$.
- $(e^{At})^{-1} = e^{-At}$ for all $t \in \mathbb{R}$.
- The unique solution ϕ of (L) with $\phi(t_0) = x_0$ is given by

$$\phi(t) = e^{A(t-t_0)}x_0.$$

Computing the matrix exponential

Let P be a nonsingular matrix in $\mathcal{M}_n(\mathbb{R})$. We transform the IVP

$$\begin{aligned} \frac{d}{dt}x &= Ax \\ x(t_0) &= x_0 \end{aligned} \qquad \text{(L\_IVP)}$$

using the transformation $x = Py$ or $y = P^{-1}x$.

The dynamics of y is

$$\begin{aligned} y' &= (P^{-1}x)' \\ &= P^{-1}x' \\ &= P^{-1}Ax \\ &= P^{-1}APy \end{aligned}$$

The initial condition is $y_0 = P^{-1}x_0$.

We have thus transformed IVP (L_IVP) into

$$\begin{aligned} \frac{d}{dt}y &= P^{-1}APy \\ y(t_0) &= P^{-1}x_0 \end{aligned} \qquad \text{(L\_IVP\_y)}$$

From the earlier result, we then know that the solution of (L_IVP_y) is given by

$$\psi(t) = e^{P^{-1}AP(t-t_0)}P^{-1}x_0,$$

and since $x = Py$, the solution to (L_IVP) is given by

$$\phi(t) = Pe^{P^{-1}AP(t-t_0)}P^{-1}x_0.$$

So everything depends on $P^{-1}AP$.

The cases

- $P^{-1}AP$ is diagonal, the solution to (L_IVP) is given by

$$\phi(t) = P \begin{pmatrix} e^{\lambda_1 t} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n t} \end{pmatrix} P^{-1} x_0.$$

- $P^{-1}AP$ is not diagonal, then use Jordan form (slightly more complicated).

Theorem 20

For all $(t_0, x_0) \in \mathbb{R} \times \mathbb{R}^n$, *there is a unique solution* $x(t)$ *to* (L_IVP) *defined for all* $t \in \mathbb{R}$. *Each coordinate function of* $x(t)$ *is a linear combination of functions of the form*

$$t^k e^{\alpha t} \cos(\beta t) \quad and \quad t^k e^{\alpha t} \sin(\beta t)$$

where $\alpha + i\beta$ *is an eigenvalue of* A *and* k *is less than the algebraic multiplicity of the eigenvalue.*

Generalized eigenvectors, nilpotent matrix

Definition 21 (Generalized eigenvectors)

Let $A \in \mathcal{M}_r(\mathbb{R})$. Suppose λ is an eigenvalue of A with multiplicity $m \leq n$. Then, for $k = 1, \ldots, m$, any nonzero solution v of

$$(A - \lambda \mathbb{I})^k v = 0$$

is called a *generalized eigenvector* of A.

Definition 22 (Nilpotent matrix)

Let $A \in \mathcal{M}_n(\mathbb{R})$. A is *nilpotent* (of order k) if $A^j \neq 0$ for $j = 1, \ldots, k-1$, and $A^k = 0$.

Jordan normal form

Theorem 23 (Jordan normal form)

Let $A \in \mathcal{M}_n(\mathbb{R})$ have eigenvalues $\lambda_1, \ldots, \lambda_n$, repeated according to their multiplicities.

- *Then there exists a basis of generalized eigenvectors for $\mathbb{R}^n$.*
- *And if $\{v_1, \ldots, v_n\}$ is any basis of generalized eigenvectors for $\mathbb{R}^n$, then the matrix $P = [v_1 \cdots v_n]$ is invertible, and A can be written as*

$$A = S + N,$$

where

$$P^{-1}SP = diag(\lambda_j),$$

the matrix $N = A - S$ is nilpotent of order $k \leq n$, and S and N commute, i.e., $SN = NS$.

Theorem 24

Under conditions of the Jordan normal form Theorem, the linear system $x' = Ax$ with initial condition $x(0) = x_0$, has solution

$$x(t) = P\,diag\left(e^{\lambda_j t}\right) P^{-1} \left(\mathbb{I} + Nt + \cdots \frac{t^k}{k!} N^k\right) x_0.$$

The result is particularly easy to apply in the following case.

Theorem 25 (Case of an eigenvalue of multiplicity n)

Suppose that λ is an eigenvalue of multiplicity n of $A \in \mathcal{M}_n(\mathbb{R})$. Then $S = diag(\lambda)$, and the solution of $x' = Ax$ with initial value x_0 is given by

$$x(t) = e^{\lambda t} \left(\mathbb{I} + Nt + \cdots \frac{t^k}{k!} N^k\right) x_0.$$

In the simplified case, we do not need the matrix P (the basis of generalized eigenvectors).

A variation of constants formula

Theorem 26 (Variation of constants formula)

Consider the IVP

$$x' = Ax + B(t) \tag{3a}$$

$$x(t_0) = x_0, \tag{3b}$$

where $B : \mathbb{R} \to \mathbb{R}^n$ *a smooth function on* $\mathbb{R}$, *and let* $e^{A(t-t_0)}$ *be matrix exponential associated to the homogeneous system* $x' = Ax$. *Then the solution* ϕ *of* (3) *is given by*

$$\phi(t) = e^{A(t-t_0)}x_0 + \int_{t_0}^{t} e^{A(t-s)}B(s)ds. \tag{4}$$

Outline

Checking that a given function is solution to a PDE

Give a PDE, to check that a given function is solution to the PDE, you need to check that it satisfies the PDE.

For example, consider the wave equation

$$u_{tt} = c^2 u_{xx} \tag{5}$$

To check that

$$\xi(x,t) = F(x-ct) + G(x+ct)$$

satisfies (24), we need to compute ξ_{tt}, ξ_{xx}, and verify that

$$\xi_{tt} = c^2 \xi_{xx}$$

By the chain rule, we have

$$\frac{\partial}{\partial t}\xi(x,t) = -cF'(x-ct) + cG'(x+ct)$$

and thus

$$\frac{\partial^2}{\partial t^2}\xi(x,t) = c^2 F''(x-ct) + c^2 G''(x+ct)$$

Also, by the chain rule,

$$\frac{\partial}{\partial x}\xi(x,t) = F'(x-ct) + G'(x+ct)$$

and thus

$$\frac{\partial^2}{\partial t^2}\xi(x,t) = F''(x-ct) + G'(x+ct)$$

So we have

$$\begin{aligned}\xi_{tt} &= c^2 F''(x - ct) + c^2 G''(x + ct) \\ &= c^2 (F''(x - ct) + G''(x + ct)) \\ &= c^2 \xi_{xx}\end{aligned}$$

which implies that

$$\xi(x, t) = F(x - ct) + G(x + ct)$$

satisfies (24).

Outline

Probability, random variable

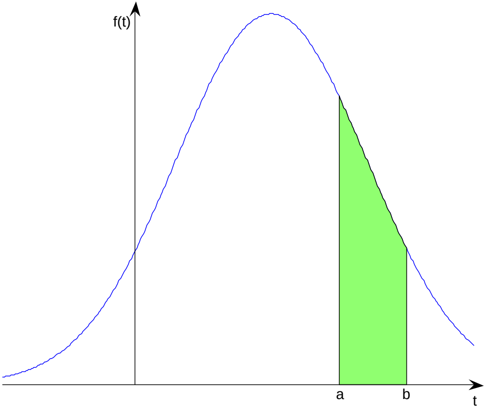
A *probability* is a function $\mathcal{P}$, with values in $[0, 1]$.

A random variable X is a variable taking random values. If the values are in a continuous space ($\mathbb{R}, \mathbb{R}^n$, etc.), then the variable is continuous. Otherwise ($\mathbb{N}, \mathbb{Z}$, etc.), the variable is discrete.

Probability density function

Suppose T is a continuous random variable. Then it has a continuous *probability density function*, f.

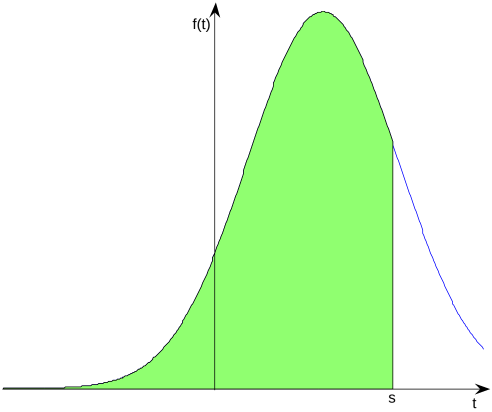
- $f \geq 0$,
- $\int_{-\infty}^{+\infty} f(s)ds = 1$.
- $\mathcal{P}\left(a \leq T \leq b\right) = \int_a^b f(t)dt$.



Cumulative distribution function

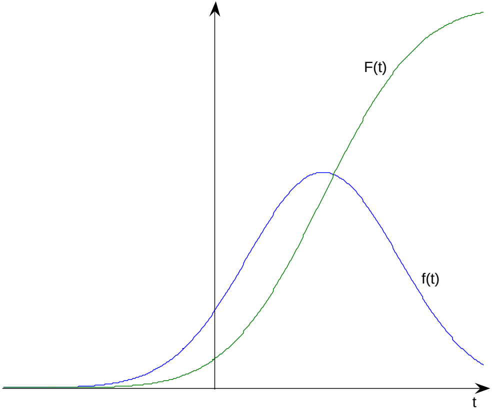
The cumulative distribution function (c.d.f.) is a function $F(t)$ that characterizes the distribution of T, and defined by

$$F(s) = \mathcal{P}\left(T \leq s\right) = \int_{-\infty}^{s} f(x)dx.$$



Properties of the c.d.f.

- Since f is a nonnegative function, F is nondecreasing.
- Since f is a probability density function, $\int_{-\infty}^{+\infty} f(s)ds = 1$, and thus $\lim_{t\to\infty} F(t) = 1$.



Mean value

For a continuous random variable T with probability density function f, the *mean* value of T, denoted $\bar{T}$ or $E(T)$, is given by

$$E(T) = \int_{-\infty}^{+\infty} tf(t)dt.$$

Survival function

Another characterization of the distribution of the random variable T is through the *survival* (or *sojourn*) function.

The survival function of state S_1 is given by

$$\mathcal{S}(t) = 1 - F(t) = \mathcal{P}\left(T > t\right) \tag{6}$$

This gives a description of the *sojourn time* of a system in a particular state (the time spent in the state).

$\mathcal{S}$ is a nonincreasing function (since $\mathcal{S} = 1 - F$ with F a c.d.f.), and $\mathcal{S}(0) = 1$ (since T is a positive random variable).

The *average sojourn time* τ in state S_1 is given by

$$\tau = E(T) = \int_0^{\infty} t f(t) dt$$

Assuming that $\lim_{t \to \infty} t\mathcal{S}(t) = 0$ (which is verified for most probability distributions),

$$\tau = \int_0^{\infty} \mathcal{S}(t) dt$$

Outline

We conduct an experiment with a set of r outcomes,

$$S = \{S_1, \ldots, S_r\}.$$

The experiment is repeated n times (with n large, potentially infinite).

The system has no memory: the next state depends only on the present state.

The probability of S_j occurring on the next step, given that S_i occurred on the last step, is

$$p_{ij} = p(S_j | S_i).$$

Markov chain

Definition 27

An experiment with finite number of possible outcomes $S_1, \ldots, S_r$ is repeated. The sequence of outcomes is a *Markov chain* if there is a set of r^2 numbers $\{p_{ij}\}$ such that the conditional probability of outcome S_j on any experiment given outcome S_i on the previous experiment is p_{ij}, i.e., for $1 \leq i, j \leq r$, $n = 1, \ldots,$

$$p_{ij} = \Pr(S_j \text{ on experiment } n+1 | S_i \text{ on experiment } n).$$

The outcomes $S_1, \ldots, S_r$ are the *states*, and the p_{ij} are the *transition probabilities*. The matrix $P = [p_{ij}]$ is the *transition matrix*.

Transition matrix

The matrix

$$P = \begin{pmatrix} p_{11} & p_{12} & \cdots & p_{1r} \\ p_{21} & p_{22} & \cdots & p_{2r} \\ \\ p_{r1} & p_{r2} & \cdots & p_{rr} \end{pmatrix}$$

has

- nonnegative entries, $p_{ij} \geq 0$
- entries less than 1, $p_{ij} \leq 1$
- row sum 1, which we write

$$\sum_{j=1}^{r} p_{ij} = 1, \quad i = 1, \ldots, r$$

or, using the notation $\mathbb{1}^T = (1, \ldots, 1)$,

$$P\mathbb{1} = \mathbb{1}$$

Repetition of the process

Let $p_i(n)$ be the probability that the state S_i will occur on the n^{th} repetition of the experiment, $1 \leq i \leq r$. Then

$$p(n+1) = p(n)P, \quad n = 1, 2, 3, \ldots \tag{7}$$

where $p(n) = (p_1(n), p_2(n), \ldots, p_r(n))$ is a (row) probability vector and $P = (p_{ij})$ is a $r \times r$ *transition matrix*,

$$P = \begin{pmatrix} p_{11} & p_{12} & \cdots & p_{1r} \\ p_{21} & p_{22} & \cdots & p_{2r} \\ & & & \\ p_{r1} & p_{r2} & \cdots & p_{rr} \end{pmatrix}$$

Stochastic matrices

Definition 28 (Stochastic matrix)

The nonnegative $r \times r$ matrix M is *stochastic* if $\sum_{j=1}^{r} a_{ij} = 1$ for all $i = 1, 2, \ldots, r$.

Theorem 29

Let M be a stochastic matrix M. Then all eigenvalues λ of M are such that $|\lambda| \leq 1$. Furthermore, $\lambda = 1$ is an eigenvalue of M.

To see that 1 is an eigenvalue, write the definition of a stochastic matrix, i.e., M has row sums 1. In vector form, $M\mathbb{1} = \mathbb{1}$. Now remember that λ is an eigenvalue of M, with associated eigenvector v, iff $Mv = \lambda v$. So, in the expression $M\mathbb{1} = \mathbb{1}$, we read an eigenvector, $\mathbb{1}$, and an eigenvalue, 1.

Long "time" behavior

Let $p(0)$ be the initial distribution (row) vector. Then

$$\begin{aligned} p(1) &= p(0)P \\ p(2) &= p(1)P \\ &= (p(0)P)P \\ &= p(0)P^2 \end{aligned}$$

Iterating, we get that for any n,

$$p(n) = p(0)P^n$$

Therefore,

$$\lim_{n\to+\infty} p(n) = \lim_{n\to+\infty} p(0)P^n = p(0) \lim_{n\to+\infty} P^n$$

Additional properties of stochastic matrices

Theorem 30

If M, N are stochastic matrices, then MN is a stochastic matrix.

Theorem 31

If M is a stochastic matrix, then for any $k \in \mathbb{N}$, M^k is a stochastic matrix.

Markov chains

Regular Markov chains

Absorbing Markov chains

Regular Markov chain

Definition 32 (Regular Markov chain)

A regular Markov chain is one in which P^k is positive for some integer $k > 0$, i.e., P^k has only positive entries, no zero entries.

Definition 33

A nonnegative matrix M is primitive if, and only if, there is an integer $k > 0$ such that M^k is positive.

Theorem 34

A Markov chain is regular if, and only if, the transition matrix P is primitive.

Important result for regular Markov chains

Theorem 35

If P is the transition matrix of a regular Markov chain, then

1. *the powers P^n approach a stochastic matrix W,*
2. *each row of W is the same (row) vector $w = (w_1, \ldots, w_r)$,*
3. *the components of w are positive.*

So if the Markov chain is regular,

$$\lim_{n \to +\infty} p(n) = p(0) \lim_{n \to +\infty} P^n = p(0)W$$

Left and right eigenvectors

Let M be an $r \times r$ matrix, u, v be two column vectors, $\lambda \in \mathbb{R}$. Then, if

$$Mu = \lambda u,$$

u is the (right) eigenvector corresponding to λ, and if

$$v^T M = \lambda v^T$$

then v is the left eigenvector corresponding to λ. Note that to a given eigenvalue there corresponds one left and one right eigenvector.

The vector w is in fact the left eigenvector corresponding to the eigenvalue 1 of P. (We already know that the (right) eigenvector corresponding to 1 is $\mathbb{1}$.)

To see this, remark that, if $p(n)$ converges, then $p(n+1) = p(n)P$, so w is a fixed point of the system. We thus write

$$wP = w$$

and solve for w, which amounts to finding w as the left eigenvector corresponding to the eigenvalue 1.

Alternatively, we can find w as the (right) eigenvector associated to the eigenvalue 1 for the transpose of P,

$$P^T w^T = w^T$$

Now remember that when you compute an eigenvector, you get a result that is the eigenvector, to a multiple.

So the expression you obtain for w might have to be normalized (you want a probability vector). Once you obtain w, check that the norm $\|w\|$ defined by

$$\|w\| = w_1 + \cdots + w_r$$

is equal to one. If not, use

$$\frac{w}{\|w\|}$$

Absorbing states, absorbing chains

Definition 36

A state S_i in a Markov chain is *absorbing* if whenever it occurs on the n^{th} generation of the experiment, it then occurs on every subsequent step. In other words, S_i is absorbing if $p_{ii} = 1$ and $p_{ij} = 0$ for $i \neq j$.

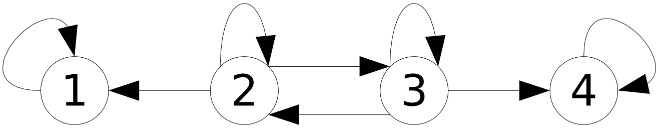
Definition 37

A Markov chain is said to be absorbing if it has at least one absorbing state, and if from every state it is possible to go to an absorbing state.

In an absorbing Markov chain, a state that is not absorbing is called *transient*.

Some questions on absorbing chains

Suppose we have a chain like the following:



1. Does the process eventually reach an absorbing state?
2. Average number of times spent in a transient state, if starting in a transient state?
3. Average number of steps before entering an absorbing state?
4. Probability of being absorbed by a given absorbing state, when there are more than one, when starting in a given transient state?

Reaching an absorbing state

Answer to question 1:

Theorem 38

In an absorbing Markov chain, the probability of reaching an absorbing state is 1.

Standard form of the transition matrix

For an absorbing chain with k absorbing states and $r - k$ transient states, the transition matrix can be written as

$$P = \begin{pmatrix} \mathbb{I}_k & \mathbf{0} \\ R & Q \end{pmatrix}$$

with following meaning,

	Absorbing states	Transient states
Absorbing states	$\mathbb{I}_k$	$\mathbf{0}$
Transient states	R	Q

with $\mathbb{I}_k$ the $k \times k$ identity matrix, $\mathbf{0}$ an $k \times (r-k)$ matrix of zeros, R an $(r-k) \times k$ matrix and Q an $(r-k) \times (r-k)$ matrix.

The matrix $\mathbb{I}_{r-k} - Q$ is invertible. Let

- $N = (\mathbb{I}_{r-k} - Q)^{-1}$ be the *fundamental matrix* of the Markov chain
- T_i be the sum of the entries on row i of N
- $B = NR$.

Answers to our remaining questions:

2. N_{ij} is the average number of times the process is in the jth transient state if it starts in the ith transient state.
3. T_i is the average number of steps before the process enters an absorbing state if it starts in the ith transient state.
4. B_{ij} is the probability of eventually entering the jth absorbing state if the process starts in the ith transient state.

Modelling topics

Single population dynamics and the logistic situation

Time of residence in a state – Exponential distribution

Epidemic models

The chemostat

Traffic flow

Shallow water waves

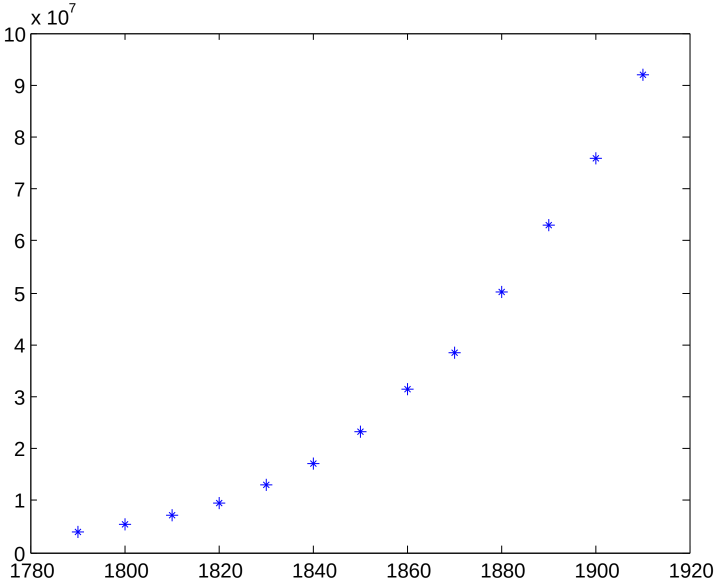
A simple genetic model

Outline

The US population from 1790 to 1910

Year	Population (millions)
1790	3.929
1800	5.308
1810	7.240
1820	9.638
1830	12.866
1840	17.069
1850	23.192

Year	Population (millions)
1860	31.443
1870	38.558
1880	50.156
1890	62.948
1900	75.995
1910	91.972


x 10^7
10
9
8
7
6
5
4
3
2
1
0
1780
1800
1820
1840
1860
1880
1900
1920

First idea

The curve looks like a piece of a parabola. So let us fit a curve of the form

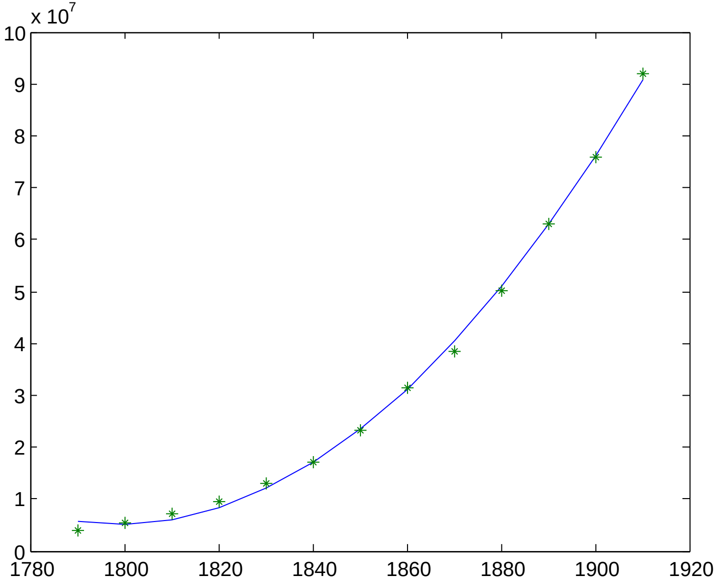
$$P(t) = a + bt + ct^2.$$

To do this, we want to minimize

$$S = \sum_{k=1}^{13} \left(P(t_k) - P_k\right)^2,$$

where t_k are the known dates, P_k are the known populations, and $P(t_k) = a + bt_k + ct_k^2$.

Our first guess, in pictures



which turned out to work quite well

How does our formula do for present times?

```
f(2006)
ans =      301468584.066013
```

301,468,584, compared to the 298,444,215 July 2006 estimate, overestimates the population by 3,024,369, a relative error of approximately 1%.

The logistic equation

The logistic curve is the solution to the ordinary differential equation

$$N' = rN\left(1 - \frac{N}{K}\right),$$

which is called the *logistic equation*. r is the *intrinsic growth rate*, K is the *carrying capacity*.

This equation was introduced by Pierre-François Verhulst (1804-1849), in 1844.

Reinterpreting the logistic equation

The equation

$$N' = bN - dN - cN^2$$

is rewritten as

$$N' = (b - d)N - cN^2.$$

- $b - d$ represents the rate at which the population increases (or decreases) in the absence of competition. It is called the *intrinsic growth rate* of the population.
- c is the rate of *intraspecific* competition. The prefix *intra* refers to the fact that the competition is occurring between members of the same species, that is, within the species.

Equivalent equations

$$
\begin{aligned}
N' &= (b-d)N - cN^2 \\
&= \big((b-d) - cN\big)N \\
&= \left(r - \frac{r}{r}cN\right)N, \quad \text{with } r = b-d \\
&= rN\left(1 - \frac{c}{r}N\right) \\
&= rN\left(1 - \frac{N}{K}\right),
\end{aligned}
$$

with

$$
\frac{c}{r} = \frac{1}{K},
$$

that is, $K = r/c$.

Studying the logistic equation qualitatively

We study

$$N' = rN\left(1 - \frac{N}{K}\right). \qquad \text{(ODE1)}$$

For this, write

$$f(N) = rN\left(1 - \frac{N}{K}\right).$$

Consider the initial value problem (IVP)

$$N' = f(N), \quad N(0) = N_0 > 0. \qquad \text{(IVP1)}$$

- f is C^1 (differentiable with continuous derivative) so solutions to (IVP1) exist and are unique.

Equilibria of (ODE1) are points such that $f(N) = 0$ (so that $N' = f(N) = 0$, meaning N does not vary). So we solve $f(N) = 0$ for N. We find two points:

- $N = 0$
- $N = K$.

By uniqueness of solutions to (IVP1), solutions cannot cross the lines $N(t) = 0$ and $N(t) = K$.

There are several cases.

- $N = 0$ for some t, then $N(t) = 0$ for all $t \geq 0$, by uniqueness of solutions.
- $N \in (0, K)$, then $rN > 0$ and $N/K < 1$ so $1 - N/K > 0$, which implies that $f(N) > 0$. As a consequence, $N(t)$ increases if $N \in (0, K)$.
- $N = K$, then $rN > 0$ but $N/K = 1$ so $1 - N/K = 0$, which implies that $f(N) = 0$. As a consequence, $N(t) = K$ for all $t \geq 0$, by uniqueness of solutions.
- $N > K$, the $rN > 0$ and $N/K > 1$, implying that $1 - N/K < 0$ and in turn, $f(N) < 0$. As a consequence, $N(t)$ decreases if $N \in (K, +\infty)$.

Therefore,

Theorem 39

Suppose that $N_0 > 0$. Then the solution $N(t)$ of (IVP1) is such that

$$\lim_{t\to\infty} N(t) = K,$$

so that K is the number of individuals that the environment can support, the carrying capacity of the environment.
If $N_0 = 0$, then $N(t) = 0$ for all $t \geq 0$.

Single population dynamics and the logistic situation

The delayed logistic equation

Consider the equation as

$$\frac{N'}{N} = (b - d) - cN,$$

that is, the per capita rate of growth of the population depends on the net growth rate $b - d$, and some density dependent inhibition cN (resulting of competition).

Suppose that instead of instantaneous inhibition, there is some delay τ between the time the inhibiting event takes place and the moment where it affects the growth rate. (For example, two individuals fight for food, and one later dies of the injuries sustained when fighting).

The delay logistic equation

In the of a time τ between inhibiting event and inhibition, the equation would be written as

$$\frac{N'}{N} = (b - d) - cN(t - \tau).$$

Using the change of variables introduced earlier, this is written

$$N'(t) = rN(t)\left(1 - \frac{N(t-\tau)}{K}\right). \qquad \text{(DDE1)}$$

Such an equation is called a *delay* differential equation. It is much more complicated to study than (ODE1). In fact, some things remain unknown about (DDE1).

Delayed initial value problem

The IVP takes the form

$$\begin{aligned} N'(t) &= rN(t)\left(1-\frac{N(t-\tau)}{K}\right), \\ N(t) &= \phi(t) \text{ for } t\in[-\tau,0], \end{aligned} \tag{IVP2}$$

where $\phi(t)$ is some continuous function. Hence, initial conditions (called initial data in this case) must be specific on an interval, instead of being specified at a point, to guarantee existence and uniqueness of solutions.

We will not learn how to study this type of equation (this is graduate level mathematics). I will give a few results.

To find equilibria, remark that delay should not play a role, since N should be constant. Thus, equilibria are found by considering the equation with no delay, which is (ODE1).

Theorem 40

Suppose that $r\tau < 22/7$. Then all solutions of (IVP2) *with positive initial data $\phi(t)$ tend to K. If $r\tau > \pi/2$, then K is an unstable equilibrium and all solutions of* (IVP2) *with positive initial data $\phi(t)$ on $[-\tau, 0]$ are oscillatory.*

Note that there is a gray zone between $22/7$ and $\pi/2$.. The first part of the theorem was proved in 1945 by Wright. Although there is very strong numerical evidence that this is in fact true up to $\pi/2$, nobody has yet managed to prove it.

The logistic map

The logistic *map* is, for $t \geq 0$,

$$N_{t+1} = rN_t \left(1 - \frac{N_t}{K}\right). \tag{DT1}$$

To transform this into an initial value problem, we need to provide an initial condition $N_0 \geq 0$ for $t = 0$.

Consider the simplified version (**??**),

$$x_{t+1} = rx_t(1 - x_t) \triangleq f_r(x_t).$$

Are solutions well defined? Suppose $x_0 \in [0, 1]$, do we stay in $[0, 1]$? f_r is continuous on $[0, 1]$, so it has a extrema on $[0, 1]$. We have

$$f_r'(x) = r - 2rx = r(1 - 2x),$$

which implies that f_r increases for $x < 1/2$ and decreases for $x > 1/2$, reaching a maximum at $x = 1/2$.

$f_r(0) = f_r(1) = 0$ are the minimum values, and $f(1/2) = r/4$ is the maximum. Thus, if we want $x_{t+1} \in [0, 1]$ for $x_t \in [0, 1]$, we need to consider $r \leq 4$.

- Note that if $x_0 = 0$, then $x_t = 0$ for all $t \geq 1$.
- Similarly, if $x_0 = 1$, then $x_1 = 0$, and thus $x_t = 0$ for all $t \geq 1$.
- This is true for all t: if there exists t_k such that $x_{t_k} = 1$, then $x_t = 0$ for all $t \geq t_k$.
- This last case might occur if $r = 4$, as we have seen.
- Also, if $r = 0$ then $x_t = 0$ for all t.

For these reasons, we generally consider

$$x \in (0, 1)$$

and

$$r \in (0, 4).$$

Fixed points: existence

Fixed points of (**??**) satisfy $x = rx(1-x)$, giving:

- $x = 0$;
- $1 = r(1-x)$, that is, $p \triangleq \dfrac{r-1}{r}$.

Note that $\lim_{r \to 0^+} p = 1 - \lim_{r \to 0^+} 1/r = -\infty$, $\frac{\partial}{\partial r} p = 1/r^2 > 0$ (so p is an increasing function of r), $p = 0 \Leftrightarrow r = 1$ and $\lim_{r \to \infty} p = 1$. So we come to this first conclusion:

- 0 always is a fixed point of f_r.
- If $0 < r < 1$, then p tales negative values so is not relevant.
- If $1 < r < 4$, then p exists.

Stability of the fixed points

Stability of the fixed points is determined by the (absolute) value f_r' at these fixed points. We have

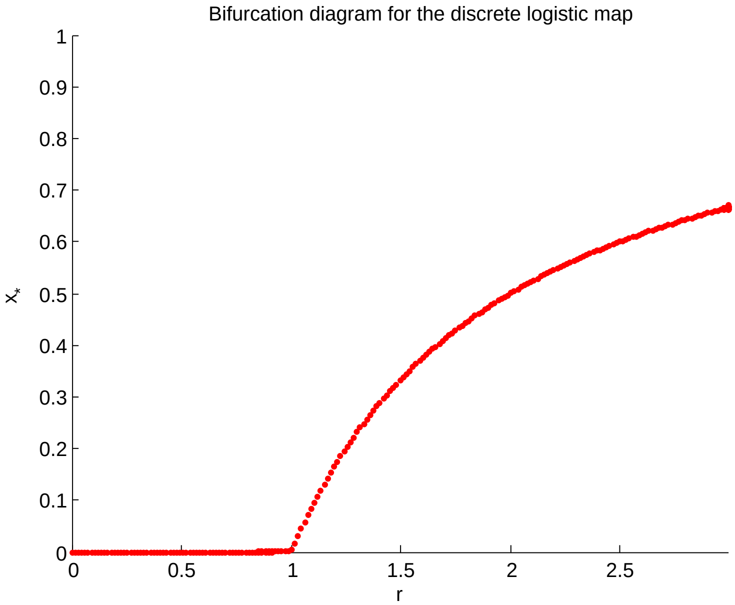
$$|f_r'(0)| = r,$$

and

$$\begin{aligned} |f_r'(p)| &= \left| r - 2r\frac{r-1}{r} \right| \\ &= |r - 2(r-1)| \\ &= |2 - r| \end{aligned}$$

Therefore, we have

- if $0 < r < 1$, then the fixed point $x = p$ does not exist and $x = 0$ is attracting,
- if $1 < r < 3$, then $x = 0$ is repelling, and $x = p$ is attracting,
- if $r > 3$, then $x = 0$ and $x = p$ are repelling.


Bifurcation diagram for the discrete logistic map
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
x*
0
0.5
1
1.5
2
2.5
r

Another bifurcation

Thus the points $r = 1$ and $r = 3$ are bifurcation points. To see what happens when $r > 3$, we need to look for period 2 points.

$$\begin{aligned} f_r^2(x) &= f_r(f_r(x)) \\ &= rf_r(x)(1 - f_r(x)) \\ &= r^2 x(1-x)(1 - rx(1-x)). \end{aligned} \tag{8}$$

0 and p are points of period 2, since a fixed point x^* of f satisfies $f(x^*) = x^*$, and so, $f^2(x^*) = f(f(x^*)) = f(x^*) = x^*$.
This helps localizing the other periodic points. Writing the fixed point equation as

$$Q(x) \stackrel{\Delta}{=} f_r^2(x) - x = 0,$$

we see that, since 0 and p are fixed points of f_μ^2, they are roots of $Q(x)$. Therefore, Q can be factorized as

$$Q(x) = x(x-p)(-r^3x^2 + Bx + C),$$

Substitute the value $(r-1)/r$ for p in Q, develop Q and (8) and equate coefficients of like powers gives

$$Q(x) = x\left(x - \frac{r-1}{r}\right)\left(-r^3x^2 + r^2(r+1)x - r(r+1)\right). \quad (9)$$

We already know that $x = 0$ and $x = p$ are roots of (9). So we search for roots of

$$R(x) := -r^3x^2 + r^2(r+1)x - r(r+1).$$

Discriminant is

$$\begin{aligned}\Delta &= r^4(r+1)^2 - 4r^4(r+1)\\ &= r^4(r+1)(r+1-4)\\ &= r^4(r+1)(r-3).\end{aligned}$$

Therefore, R has distinct real roots if $r > 3$. Remark that for $r = 3$, the (double) root is $p = 2/3$. For $r > 3$ but very close to 3, it follows from the continuity of R that the roots are close to $2/3$.

We use Descartes' rule of signs.

- R has signed coefficients $- + -$, so 2 sign changes imlying 0 or 2 positive real roots.
- $R(-x)$ has signed coefficients $- - -$, so no negative real roots.
- Since $\Delta > 0$, the roots are real, and thus it follows that both roots are positive.

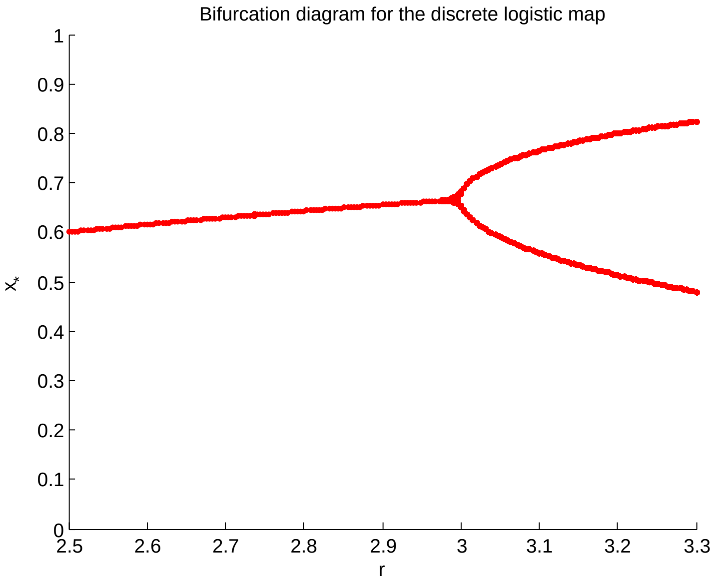
To show that the roots are also smaller than 1, consider the change of variables $z = x - 1$. The polynomial R is transformed into

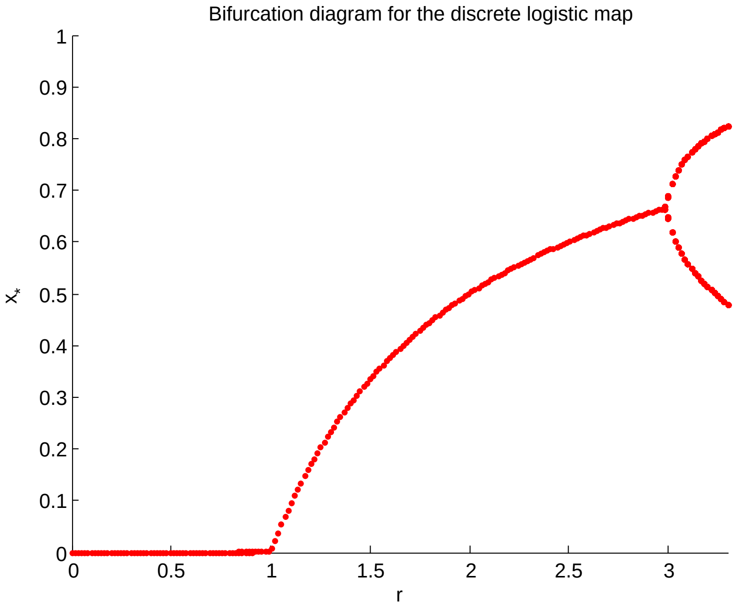
$$\begin{aligned} R_2(z) &= -r^3(z+1)^2 + r^2(r+1)(z+1) - r(r+1) \\ &= -r^3z^2 + r^2(1-r)z - r. \end{aligned}$$

For $r > 1$, the signed coefficients are $- - -$, so R_2 has no root $z > 0$, implying in turn that R has no root $x > 1$.

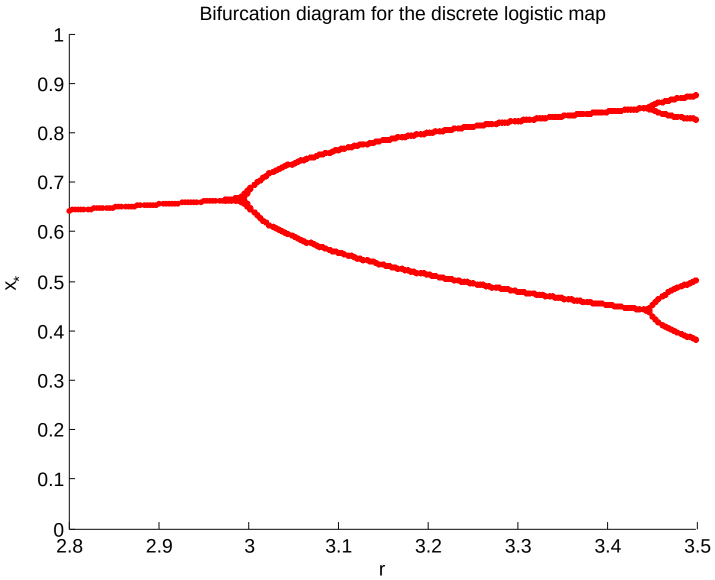
Summing up

- If $0 < r < 1$, then $x = 0$ is attracting, p does not exist and there are no period 2 points.
- At $r = 1$, there is a bifurcation (called a *transcritical* bifurcation).
- If $1 < r < 3$, then $x = 0$ is repelling, p is attracting, and there are no period 2 points.
- At $r = 3$, there is another bifurcation (called a *period-doubling* bifurcation).
- For $r > 3$, both $x = 0$ and $x = p$ are repelling, and there is a period 2 point.


Bifurcation diagram for the discrete logistic map
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
x_*
2.5
2.6
2.7
2.8
2.9
3
3.1
3.2
3.3
r


Bifurcation diagram for the discrete logistic map
x_*
r
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
0.5
1.5
2
2.5
3

This process continues



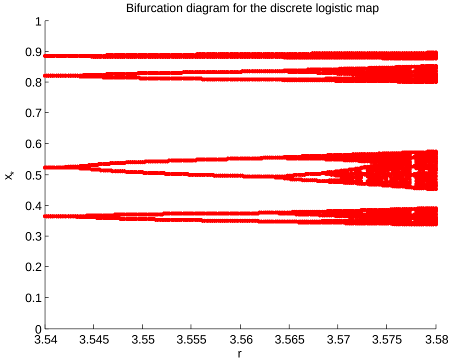
The period-doubling cascade to chaos

The logistic map undergoes a sequence of period doubling bifurcations, called the *period-doubling cascade*, as r increases from 3 to 4.

- Every successive bifurcation leads to a doubling of the period.
- The bifurcation points form a sequence, $\{r_n\}$, that has the property that

$$\lim n \to \infty \frac{r_n - r_{n-1}}{r_{n+1} - r_n}$$

  exists and is a constant, called the Feigenbaum constant, equal to $4.669202\ldots$
- This constant has been shown to exist in many of the maps that undergo the same type of cascade of period doubling bifurcations.


Bifurcation diagram for the discrete logistic map
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
x_*
3.54
3.545
3.55
3.555
3.56
3.565
3.57
3.575
3.58
r

Chaos

After a certain value of r, there are periodic points with all periods. In particular, there are periodic points of period 3.

By a theorem (called the Sarkovskii theorem), the presence of period 3 points implies the presence of points of all periods.

At this point, the system is said to be in a *chaotic regime*, or *chaotic*.

Outline

The exponential distribution

The random variable T has an *exponential* distribution if its probability density function takes the form

$$f(t) = \begin{cases} 0 & \text{if } t < 0, \\ \theta e^{-\theta t} & \text{if } t \geq 0, \end{cases} \tag{10}$$

with $\theta > 0$. Then the survival function for state S_1 is of the form $\mathcal{S}(t) = e^{-\theta t}$, for $t \geq 0$, and the average sojourn time in state S_1 is

$$\tau = \int_0^\infty e^{-\theta t} dt = \frac{1}{\theta}$$

Time of residence in a state – Exponential distribution

A cohort model

Sojourn times in an SIS disease transmission model

A model for a cohort with one cause of death

We consider a population consisting of individuals born at the same time (a *cohort*), for example, the same year.

We suppose

- At time $t = 0$, there are initially $N_0 > 0$ individuals.
- All causes of death are compounded together.
- The time until death, for a given individual, is a random variable T, with continuous probability density distribution $f(t)$ and survival function $P(t)$.

The model

Denote $N(t)$ the population at time $t \geq 0$. Then

$$N(t) = N_0 P(t). \tag{11}$$

- $N_0 P(t)$ gives the proportion of N_0, the initial population, that is still alive at time t.

Case where T is exponentially distributed

Suppose that T has an exponential distribution with mean $1/d$ (or parameter d), $f(t) = de^{-dt}$. Then the survival function is $P(t) = e^{-dt}$, and (11) takes the form

$$N(t) = N_0 e^{-dt}. \tag{12}$$

Now note that

$$\begin{aligned}\frac{d}{dt}N(t) &= -dN_0 e^{-dt} \\ &= -dN(t),\end{aligned}$$

with $N(0) = N_0$.

$\Rightarrow$ The ODE $N' = -dN$ makes the assumption that the life expectancy at birth is exponentially distributed.

Time of residence in a state – Exponential distribution

A cohort model

Sojourn times in an SIS disease transmission model

An SIS model

Consider a disease that confers no immunity. In this case, individuals are either

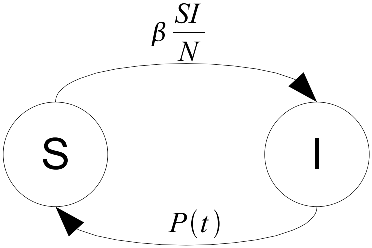
- *susceptible* to the disease, with the number of such individuals at time t denoted by $S(t)$,
- or *infected* by the disease (and are also *infective* in the sense that they propagate the disease), with the number of such individuals at time t denoted by $I(t)$.

Assumptions:

- Individuals typically recover from the disease.
- The disease does not confer immunity.
- There is no birth or death.
- Infection is of *standard incidence* type

A flow diagram for the model

This is the *flow diagram* of our model:



Reducing the dimension of the problem

To formulate our model, we would in principle require an equation for S and an equation for I.

But we have

$$S(t) + I(t) = N, \text{ or equivalently, } S(t) = N - I(t).$$

N is constant (equal total population at time $t = 0$), so we can deduce the value of $S(t)$, once we know $I(t)$, from the equation $S(t) = N - I(t)$.

We only need to consider 1 equation. **Do this when possible!** (nonlinear systems are hard, one less equation can make a lot of difference)

Model for infectious individuals

Integral equation for the number of infective individuals:

$$I(t) = I_0(t) + \int_0^t \beta \frac{(N - I(u))I(u)}{N} P(t-u)du \qquad (13)$$

- $I_0(t)$ number of individuals who were infective at time $t = 0$ and still are at time t.
  - $I_0(t)$ is nonnegative, nonincreasing, and such that $\lim_{t\to\infty} I_0(t) = 0$.
- $P(t-u)$ proportion of individuals who became infective at time u and who still are at time t.
- $\beta(N - I(u))S(u)/N$ is $\beta S(u)I(u)/N$ with $S(u) = N - I(u)$, from the reduction of dimension.

Case of an exponentially distributed time to recovery

Suppose that $P(t)$ is such that the sojourn time in the infective state has an exponential distribution with mean $1/\gamma$, *i.e.*, $P(t) = e^{-\gamma t}$.

Then the initial condition function $I_0(t)$ takes the form

$$I_0(t) = I_0(0)e^{-\gamma t},$$

with $I_0(0)$ the number of infective individuals at time $t = 0$. This is obtained by considering the cohort of initially infectious individuals, giving a model such as (11).

Equation (13) becomes

$$I(t) = I_0(0)e^{-\gamma t} + \int_0^t \beta \frac{(N - I(u))I(u)}{N} e^{-\gamma(t-u)} du. \qquad (14)$$

Taking the time derivative of (14) yields

$$I'(t) = \beta \frac{(N - I(t))I(t)}{N} - \gamma I(t),$$

which is the classical logistic type ordinary differential equation (ODE) for I in an SIS model without vital dynamics (no birth or death).

Conclusion

- The time of sojourn in classes (compartments) plays an important role in determining the type of model that we deal with.
- All ODE models, when they use terms of the form κX, make the assumption that the time of sojourn in compartments is exponentially distributed.
- At the other end of the spectrum, delay delay differential with discrete delay make the assumption of a constant sojourn time, equal for all individuals.

- Both can be true sometimes.. but reality is often somewhere in between.

Outline

Epidemic models

SIS model without vital dynamics

SIR model of Kermack and McKendrick

SIRS model with demography

A SIS model

Consider a disease that confers no immunity. In this case, individuals are either

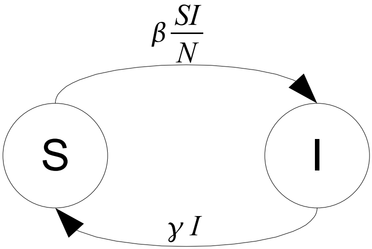
- *susceptible* to the disease, with the number of such individuals at time t denoted by $S(t)$,
- or *infected* by the disease (and are also *infective* in the sense that they propagate the disease), with the number of such individuals at time t denoted by $I(t)$.

We want to model the evolution with time of S and I (t is omitted unless necessary).

Hypotheses

- Individuals recover from the disease at the *per capita* rate γ.
- The disease does not confer immunity.
- There is no birth or death.
- Infection is of *standard incidence* type, $\beta = SI/N$.

Flow diagram of the model



The evolution of $I(t)$ is described by the following equation (see slides on *residence time*):

$$I' = \beta \frac{(N-I)I}{N} - \gamma I.$$

Develop and reorder the terms, giving

$$I' = (\beta - \gamma)I - \frac{\beta}{N}I^2 \tag{15}$$

The basic reproduction number

Define the *basic reproduction number* (the average number of people that an infectious individual will infect, when introduced in a population of susceptibles) as

$$\mathcal{R}_0 = \frac{\beta}{\gamma}$$

We have

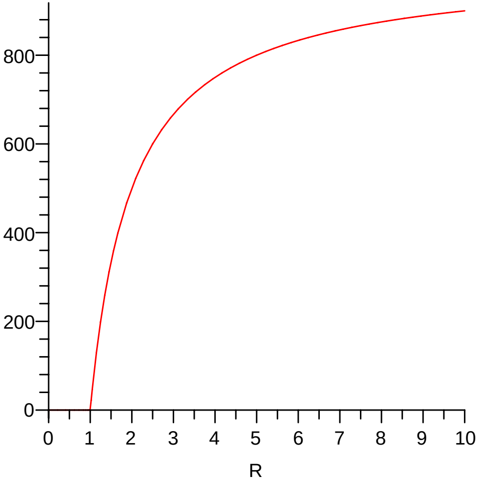
$$(\mathcal{R}_0 < 1 \Leftrightarrow (\beta - \gamma) < 0) \text{ and } (\mathcal{R}_0 > 1 \Leftrightarrow (\beta - \gamma) > 0)\,.$$

Then

- If $\mathcal{R}_0 < 1$, then $\lim_{t\to\infty} I(t) = 0$.
- If $\mathcal{R}_0 > 1$, then

$$lim_{t\to\infty} I(t) = \left(1 - \frac{1}{\mathcal{R}_0}\right) N.$$

(the case $\mathcal{R}_0 = 1$ is usually omitted)


800
600
400
200
0
0
1
2
3
4
5
6
7
8
9
10
R

Epidemic models

SIS model without vital dynamics

SIR model of Kermack and McKendrick

SIRS model with demography

Kermack and McKendrick

In 1927, Kermack and McKendrick started publishing a series of papers on epidemic models. In the first of their papers, they have this model as a particular case:

$$
\begin{aligned}
S' &= -\beta S I \\
I' &= \beta S I - \gamma I \\
R' &= \gamma I
\end{aligned}
\tag{16}
$$

Analyzing the system

First, note (as KMK) that the total population in the system is constant. This is deduced from the fact that

$$N' = (S + I + R)' = -\beta SI + \beta SI - \gamma I + \gamma I = 0.$$

Since this is true for all values of t, we have N constant.

Let us ignore the R equation for now. We can compute

$$\frac{dI}{dS} = \frac{dI}{dt}\frac{dt}{dS} = \frac{I'}{S'} = \frac{\gamma}{\beta S} - 1$$

This gives

$$I(S) = S - \frac{\gamma}{\beta}\ln S + K,$$

which, considering the initial condition (S_0, I_0), is,

$$I(S) = S - \frac{\gamma}{\beta}\ln S + I_0 - (S_0 - \frac{\gamma}{\beta}\ln S_0).$$

This gives a curve in the (S, I) plane.

$$I(S) = S - \frac{\gamma}{\beta} \ln S + I_0 - (S_0 - \frac{\gamma}{\beta} \ln S_0).$$

Typically, assume $S \approx N$ and $I > 0$ small. Let us denote $S_\infty = \lim_{t\to\infty} S(t)$.
We want to find the value of S when $I \to 0$. Then

$$I_0 - \frac{\gamma}{\beta} \ln S_0 = S_\infty - \frac{\gamma}{\beta} \ln S_\infty$$

Epidemic models

SIS model without vital dynamics

SIR model of Kermack and McKendrick

SIRS model with demography

The SIRS model – Assumptions (1/2)

- Like KMK, individuals are S, I or R.
- Infection is βSI (mass action) or $\beta SI/N$ (proportional incidence).
- Different interpretation of the R class: R stands for "removed", individuals who are immune to the disease following recovery.
- Recovery from the disease (movement from I class to R class) occurs at the per capita rate γ.
  (Time spent in I before recovery is exponentially distributed.)
- Immunity can be lost: after some time, R individuals revert back to S individuals.
- Time spent in R class before loss of immunity is exponentially distributed, with mean $1/\nu$.

The SIRS model – Assumptions (2/2)

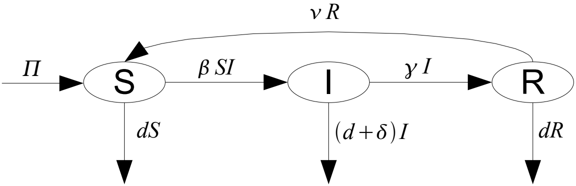
- There is birth and death of individuals:
  - No vertical transmission of the disease (mother to child) or of immunity, so all birth is into the S class.
    Birth occurs at the rate Π.
  - Individuals in all classes die of at the per capita rate d, i.e., the average life duration is exponentially distributed with mean $1/d$.
  - The disease is lethal: infected individuals are subject to additional mortality at the per capita rate δ.

Note that birth and death can have different interpretations:

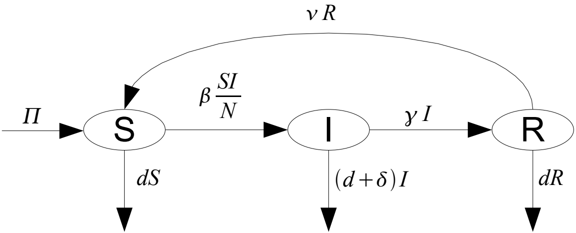
- birth and death in the classical sense,
- but also, entering the susceptible population and leaving it.

Flow diagrams for the models

Mass action



Standard incidence



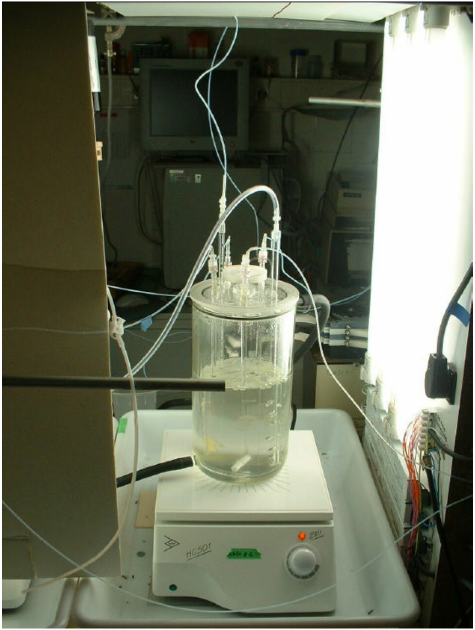
SIRS model with mass action incidence

Consider the model with mass action incidence,

$$\begin{aligned}
S' &= \Pi + \nu R - \beta SI - dS \\
I' &= \beta SI - (d + \delta + \gamma)I \\
R' &= \gamma I - (d + \nu)R
\end{aligned}$$

Outline

A chemostat



Principle

- One main chamber (vessel), in which some microorganisms (bacteria, plankton), typically unicellular, are put, together with liquid and nutrient.
- Contents are stirred, so nutrient and organisms are well-mixed.
- Organisms consume nutrient, grow, multiply.
- Two major modes of operation:
  - *Batch* mode: let the whole thing sit.
  - *Continuous flow* mode: there is an input of fresh water and nutrient, and an outflow the comprises water, nutrient and organisms, to keep the volume constant.

The chemostat

Batch mode

Continous flow mode

Model for batch mode – No organism death

First, assume no death of organisms. Model is

$$S' = -\mu(S)x \tag{17a}$$

$$x' = \mu(S)x \tag{17b}$$

with initial conditions $S(0) \geq 0$ and $x(0) > 0$, and where $\mu(S)$ is such that

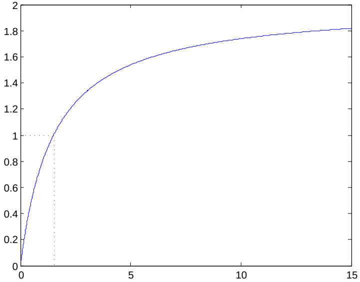
- $\mu(0) = 0$ (no substrate, no growth)
- $\mu(S) \geq 0$ for all $S \geq 0$
- $\mu(S)$ bounded for $S \geq 0$

The Michaelis-Menten curve

Typical form for $\mu(S)$ is the *Michaelis-Menten* (MM) curve,

$$\mu(S) = \mu_{max}\frac{S}{K_S + S} \tag{18}$$

- μ_{max} maximal growth rate
- K_S half-saturation constant ($\mu(K_S) = \mu_{max}/2$).



From now on, assume MM function.

Equilibria

To compute the equilibria, suppose $S' = x' = 0$, giving

$$\mu(S)x = -\mu(S)x = 0$$

This implies $\mu(S) = 0$ or $x = 0$. Note that $\mu(S) = 0 \Leftrightarrow S = 0$, so the system is at equilibrium if $S = 0$ or $x = 0$.

This is a complicated situation, as it implies that there are lines of equilibria ($S = 0$ and any x, and $x = 0$ and any S), so that the equilibria are not *isolated* (arbitrarily small neighborhoods of one equilibrium contain other equilibria), and therefore, studying the linearization is not possible.

Here, some analysis is however possible. Consider

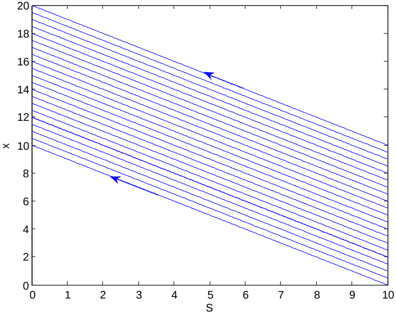
$$\frac{dx}{dS} = \frac{dx}{dt}\frac{dt}{dS} = -\frac{\mu(S)x}{\mu(S)x} = -1$$

This implies that we can find the solution

$$x(S) = C - S,$$

or, supposing the initial condition is $(S(0), x(0)) = (S_0, x_0)$, that is, $x(S_0) = x_0$,

$$x(S) = S_0 + x_0 - S$$



Model for batch mode – Organism death

Assume death of organisms at per capita rate d. Model is

$$S' = -\mu(S)x \tag{19a}$$

$$x' = \mu(S)x - dx \tag{19b}$$

Equilibria

$S' = 0 \Leftrightarrow \mu(S)x = 0$
$x' = 0 \Leftrightarrow (\mu(S) - d)x = 0.$
So we have $x = 0$ or $\mu(S) = d$. So $x = 0$ and any value of S, and S such that $\mu(S) = d$ and $x = 0$. One such particular value is $(S, x) = (0, 0)$.

This is once again a complicated situation, since there are lines of equilibria. Intuitively, most solutions will go to $(0, 0)$. This is indeed the case (we will not show it).

The chemostat

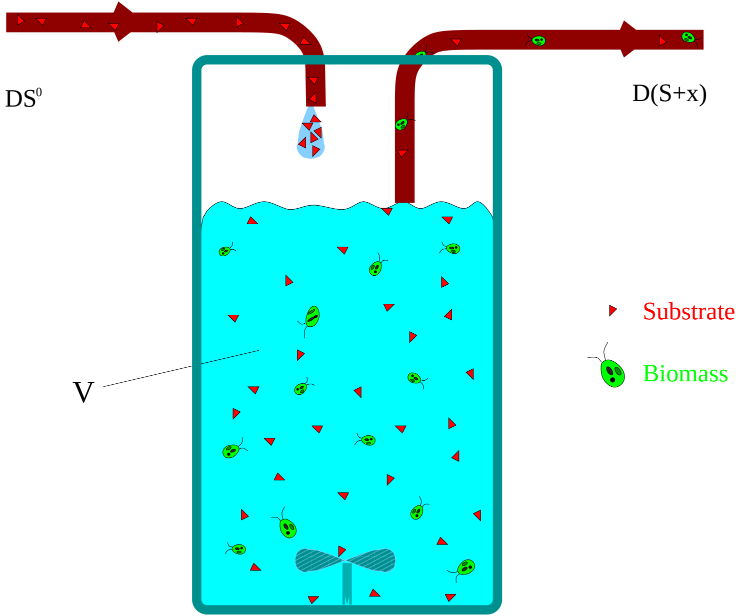
Batch mode

Continous flow mode

Modelling principles – Continuous flow mode

- Organisms (concentration denoted x) are in the main vessel.
- Limiting substrate (concentration in the vessel denoted S) is input (at rate D and concentration S^0).
- There is an outflow of both nutrient and organisms (at same rate D as input).
- Homogeneous mixing.
- Residence time in device is assumed small compared to lifetime (or time to division) $\Rightarrow$ no death considered.

Schematic representation



Model for continuous flow mode

Model is

$$S' = D(S^0 - S) - \mu(S)x \tag{20a}$$

$$x' = \mu(S)x - Dx \tag{20b}$$

with initial conditions $S(0) \geq 0$ and $x(0) \geq 0$, and $D, S^0 > 0$.

Equilibria

Existence: done in class using nullclines.

Stability: done in class using Jacobian matrix.

Outline

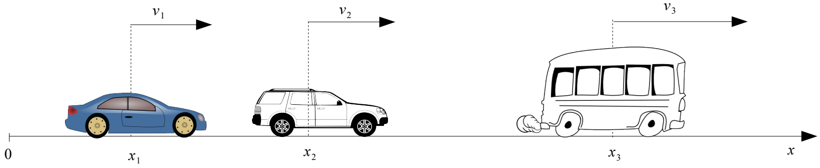
Traffic flow

ODE model

DDE model

Hypotheses

- N cars in total.
- Road is the x-axis.
- $x_n(t)$ position of the nth car at time t.
- $v_n(t) \stackrel{\Delta}{=} x_n'(t)$ velocity of the nth car at time t.



- All cars start with the same initial speed v_0 before time $t = 0$.

Moving frame coordinates

To make computations easier, express velocity of cars in a reference frame moving at speed u_0.

Remark that here, speed=velocity, since movement is 1-dimensional.

Let

$$u_n(t) = v_n(t) - u_0.$$

Then $u_n(t) = 0$ for $t \leq 0$, and u_n is the speed of the nth car in the moving frame coordinates.

Modeling driver behavior

Assume that

- Driver adjusts his/her speed according to relative speed between his/her car and the car in front.
- This adjustment is a linear term, equal to λ for all drivers.

- First car: evolution of speed remains to be determined.
- Second car:
$$u_2' = \lambda(u_1 - u_2).$$
- Third car:
$$u_3' = \lambda(u_2 - u_3)$$
- Thus, for $n = 1, \ldots, N-1$,
$$u_{n+1}' = \lambda(u_n - u_{n+1}). \tag{21}$$

This can be solved using *linear cascades*: if $u_1(t)$ is known, then

$$u_2' = \lambda(u_1(t) - u_2)$$

is a linear first-order nonhomogeneous equation. Solution (integrating factors, or variation of constants) is

$$u_2(t) = \lambda e^{-\lambda t} \int_0^t u_1(s) e^{\lambda s} ds$$

Then use this function $u_2(t)$ in u_3' to get $u_3(t)$,

$$u_3(t) = \lambda e^{-\lambda t} \int_0^t u_2(s) e^{\lambda s} ds$$

Example

Suppose driver of car 1 follows this function

$$u_1(t) = \alpha \sin(\omega t)$$

that is, ω-periodic, 0 at $t = 0$ (we want all cars to start with speed relative to the moving reference equal to 0), and with amplitude α.

Then

$$u_2(t) = \frac{\lambda\alpha}{\lambda^2 + \omega^2} \left(\omega e^{-\lambda t} + \lambda \sin(\omega t) - \omega \cos(\omega t) \right).$$

When $t \to \infty$, first term goes to 0, we are left with a ω-periodic term.

Using the theory of linear systems

Consider the case of 3 cars. Let

$$X = \begin{pmatrix} u_2 \\ u_3 \end{pmatrix}$$

Then the system can be written as

$$X' = \begin{pmatrix} -\lambda & 0 \\ \lambda & -\lambda \end{pmatrix} U + \begin{pmatrix} \lambda u_1(t) \\ 0 \end{pmatrix}$$

which we write for short as $X' = AX + B(t)$.

The matrix A has the eigenvalue $-\lambda$ with multiplicity 2. Its Jordan form is

$$J = \begin{pmatrix} -\lambda & 1 \\ 0 & -\lambda \end{pmatrix}$$

with matrix of change of basis

$$P = \begin{pmatrix} 0 & 1 \\ \lambda & 0 \end{pmatrix}$$

which is such that $P^{-1}AP = J$.

Because $-\lambda$ is an eigenvalue with multiplicity 2 (same as the size of the matrix), we can use the simplified theorem, and only need N.

We have

$$
\begin{aligned}
N &= A - S \\
&= \begin{pmatrix} -\lambda & 0 \\ \lambda & -\lambda \end{pmatrix} - \begin{pmatrix} -\lambda & 0 \\ 0 & -\lambda \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 \\ \lambda & 0 \end{pmatrix}
\end{aligned}
$$

Clearly, $N^2 = 0$, so, by the theorem in the simplified case,

$$x(t) = e^{-\lambda t} (\mathbb{I} + Nt) \, x_0$$

But we know that solutions are unique, and that the solution to the differential equation is given by $x(t) = e^{At} x_0$. This means that

$$\begin{aligned} e^{At} &= e^{-\lambda t} (\mathbb{I} + Nt) \\ &= e^{-\lambda t} \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ \lambda t & 0 \end{pmatrix} \right) \\ &= e^{-\lambda t} \begin{pmatrix} 1 & 0 \\ \lambda t & 1 \end{pmatrix} \\ &= \begin{pmatrix} e^{-\lambda t} & 0 \\ \lambda t e^{-\lambda t} & e^{-\lambda t} \end{pmatrix} \end{aligned}$$

Now notice that the solution to

$$X' = AX$$

is trivially established here, since

$$X(0) = \begin{pmatrix} u_2(0) \\ u_3(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

and thus

$$X(t) = e^{At}0 = 0.$$

e^{At} does however play a role in the solution (fortunately), since it is involved in the variation of constants formula:

$$X(t) = e^{At}X_0 + \int_0^t e^{A(t-s)}B(s)ds$$

Let

$$\Psi(t) = \int_0^t e^{\lambda s} u_1(s) ds$$

and

$$\Phi(t) = \int_0^t s e^{\lambda s} u_1(s) ds$$

These can be computed when we choose a function $u_1(t)$. Then, finally, we have

$$\begin{aligned} X(t) &= \int_0^t e^{A(t-s)} B(s) ds \\ &= \begin{pmatrix} \lambda e^{-\lambda t} \Psi(t) \\ \lambda^2 e^{-\lambda t} \left(t\Psi(t) - \Phi(t) \right) \end{pmatrix} \end{aligned}$$

Case of the $\alpha \sin(\omega t)$ driver

We set

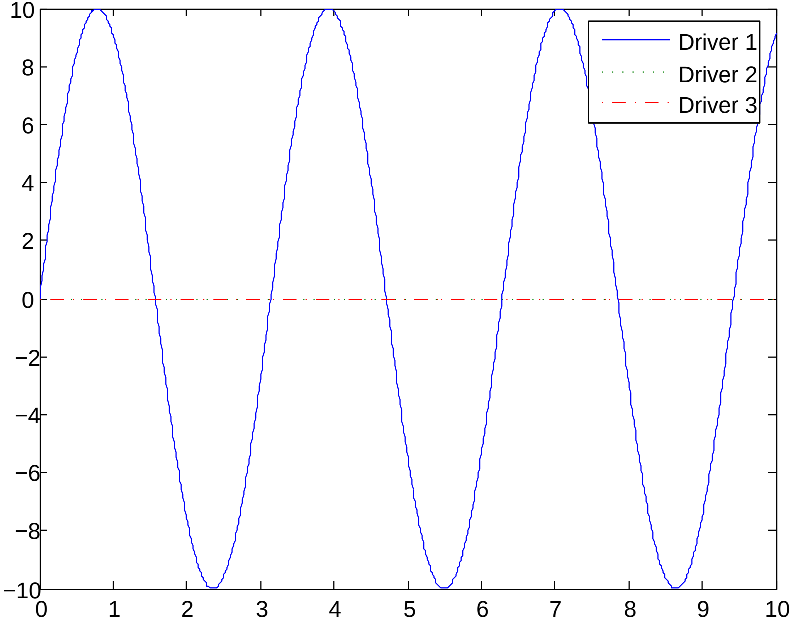
$$u_1(t) = \alpha \sin(\omega t).$$

Then

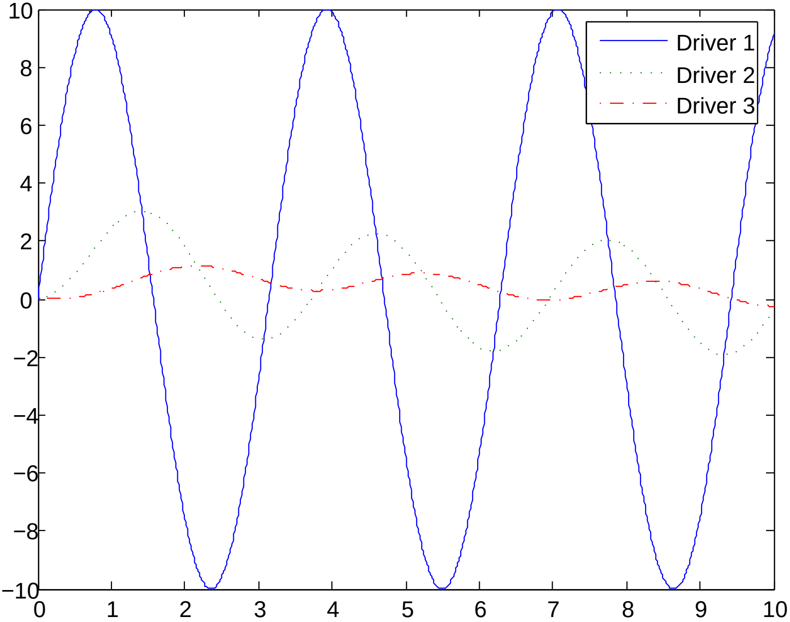
$$\Psi(t) = \frac{\alpha(\omega - \omega e^{\lambda t} \cos(\omega t) + \lambda e^{\lambda t} \sin(\omega t))}{\lambda^2 + \omega^2}$$

and

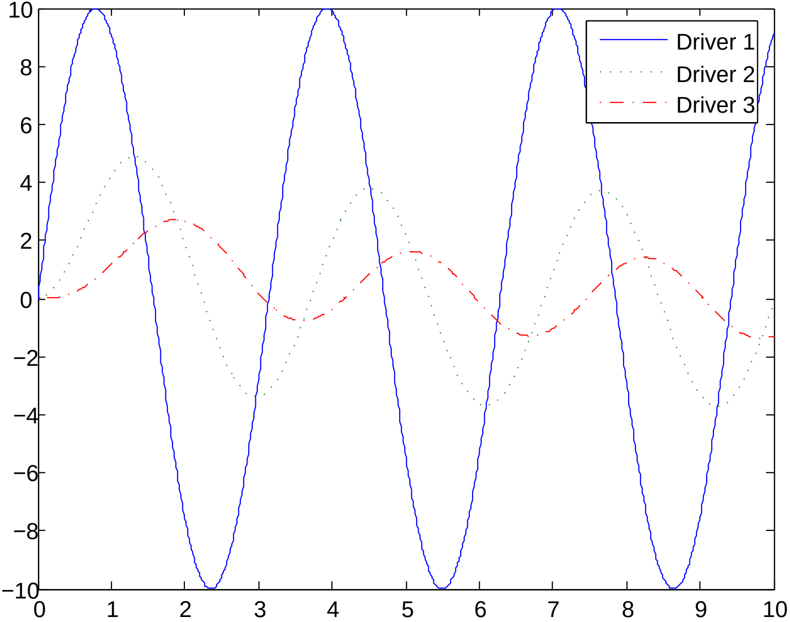
$$\Phi(t) = \frac{\alpha(\lambda^3 t + \lambda t \omega^2 - \lambda^2 + \omega^2) \sin(\omega t) e^{\lambda t}}{(\lambda^2 + \omega^2)^2} - \frac{\alpha\omega \cos(\omega t)(t\lambda^2 + t\omega^2 - 2\lambda) e^{\lambda t} + 2\alpha\lambda\omega}{(\lambda^2 + \omega^2)^2}$$



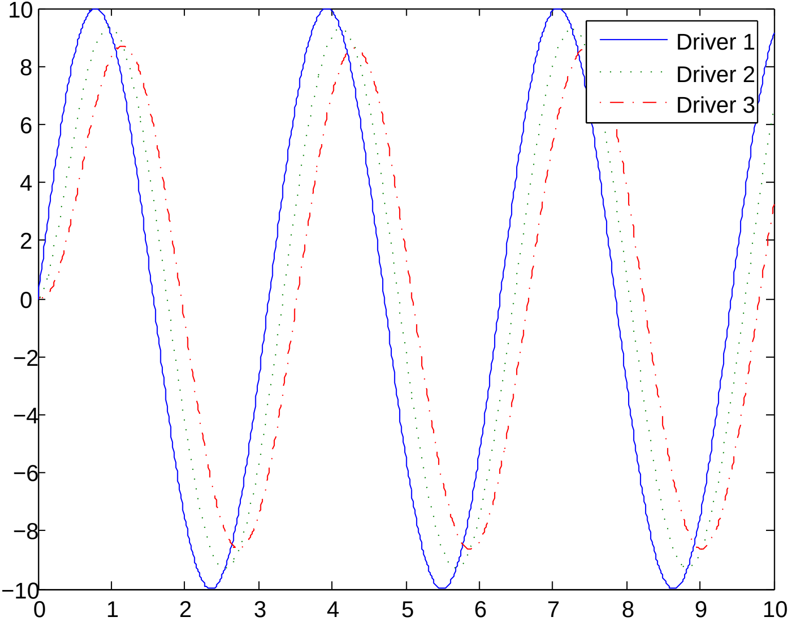
$\lambda = 0$



$\lambda = 0.4$



$\lambda = 0.8$



$\lambda = 5$

Traffic flow

ODE model

DDE model

A delayed model of traffic flow

We consider the same setting as previously, except that now, for $t > 0$,

$$u'_{n+1}(t) = \lambda(u_n(t-\tau) - u_{n+1}(t-\tau)), \qquad (22)$$

for $n = 1, \ldots, N-1$. Here, $\tau \geq 0$ is called the *time delay* (or *time lag*), or for short, *delay* (or *lag*).

If $\tau = 0$, we are back to the previous model.

Initial data

For a delay equation such as (22), the initial conditions become *initial data*. This initial data must be specified on an interval of length τ, left of zero.

This is easy to see by looking at the terms: $u(t-\tau)$ involves, at time t, the state of u at time $t-\tau$. So if $t<\tau$, we need to know what happened for $t \in [-\tau, 0]$.

So, normally, we specify initial data as

$$u_n(t) = \phi(t) \text{ for } t \in [-\tau, 0],$$

where ϕ is some function, that we assume to be continuous. We assume $u_1(t)$ is known.

Here, we assume, for $n = 1, \dots, N$,

$$u_n(t) = 0, \qquad t \leq (n-1)\tau$$

Important remark

Although (22) looks very similar to (21), you must keep in mind that it is in fact much more complicated.

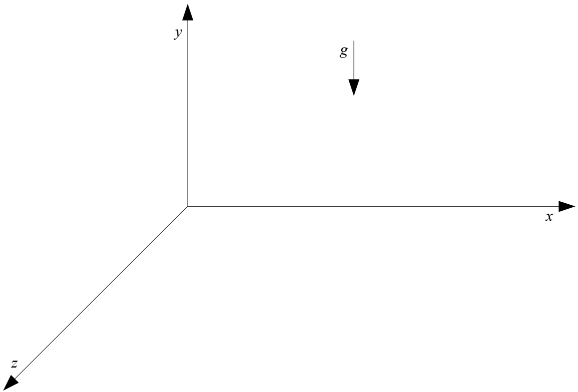
- A solution to (21) is a continuous function from $\mathbb{R}$ to $\mathbb{R}$ (or to $\mathbb{R}^n$ if we consider the system).
- A solution to (22) is a continuous function in the space of continuous functions.
- The space $\mathbb{R}^n$ has dimension n. The space of continuous functions has dimension ∞.

We then computed the Laplace transform of the system, but this was not very helpful, since, after solving the problem in s-space, we were not able to transform back into the original t-space.

Outline

Spatial domain

We consider the motion of a body of water that is infinite in the z direction, with or without boundary in the x direction, and the vertical direction of gravity taken as the y direction.



From now on, suppose z direction uniform (the same for all z), so ignore z except for the sake of argument.

The one-dimensional wave equation (1)

Following a long and complex argument, the following was derived.

The partial differential equation

$$\zeta_{tt} = c^2 \zeta_{xx} \tag{23}$$

with $c^2 = Hg$, is the one-dimensional wave equation. Initial conditions are given by

$$\begin{aligned} \zeta(x,0) &= h_0(x) - H \equiv \zeta_0(x) \\ \zeta_t(x,0) &= -Hu_x(x,0) = -H[u_0(x)]_x \equiv \nu_0(x) \end{aligned}$$

The one-dimensional wave equation (2)

Things can also be expressed in terms of u. Using the same type of simplification used before for ζ, we get

$$u_{tt} = c^2 u_{xx} \tag{24}$$

with $c^2 = Hg$. Initial conditions are given by

$$\begin{aligned} u(x,0) &= u_0(x) \\ u_t(x,0) &= -g\zeta_x(x,0) = -g[h_0(x)]_x \equiv v_0(x) \end{aligned}$$

Shallow water waves

Traveling wave solutions

Traveling wave solutions

This was obtained by d'Alembert. Consider

$$u_{tt} = c^2 u_{xx} \tag{24}$$

Note that this can be written as

$$\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right) u = 0$$

This implies that for any F, G, the sum

$$u(x,t) = F(x - ct) + G(x + ct)$$

satisfies (24).

Set

$$u(x,0) = f(x) \qquad u_t(x,0) = g(x)$$

Then d'Alembert's formula gives

$$u(x,t) = \frac{f(x-ct)+f(x+ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)ds$$

Case of a Dirac delta initial condition

Suppose $u_0(x) = 0$ and $v_0(x) = \delta(x)$, for $-\infty < x < \infty$, with δ the Dirac delta,

$$\delta(x) = \begin{cases} \infty & \text{if } x = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Therefore,

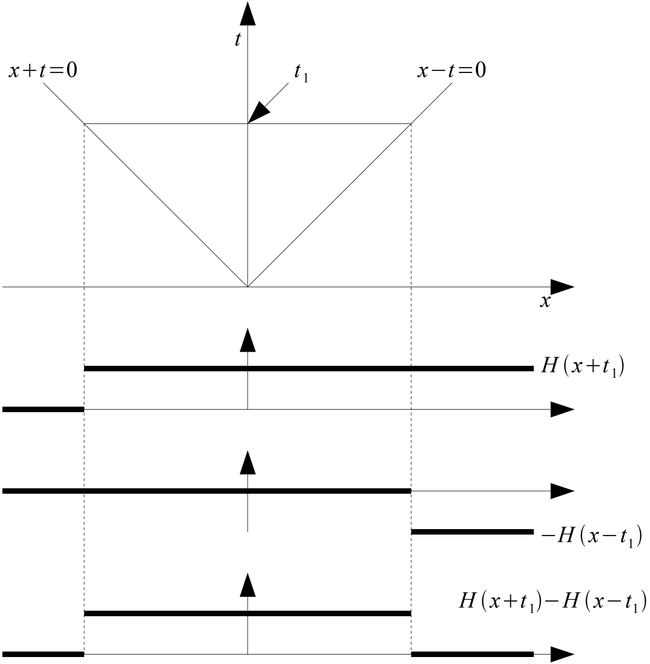
$$u(x,t) = \frac{1}{2c} \int_{x-ct}^{x+ct} \delta(z) dz = \frac{1}{2c} \left\{ H(x+ct) - H(x-ct) \right\},$$

with H the Heaviside function,

$$H(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x > 0. \end{cases}$$

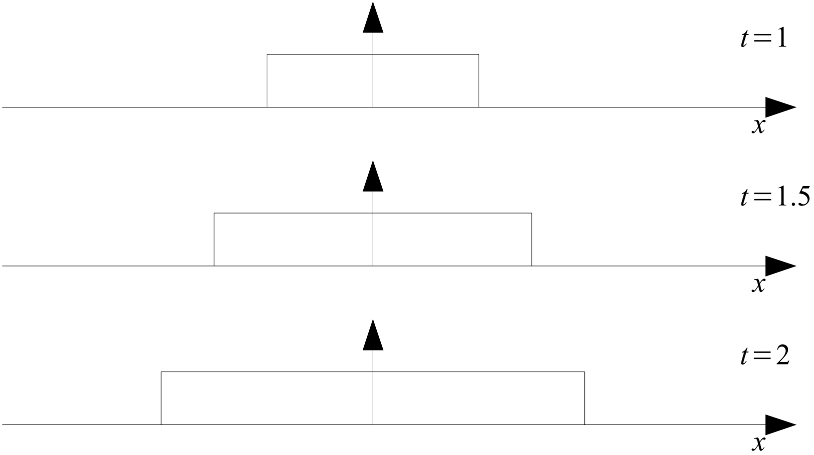
For simplicity, take $c = 1$. This gives

$$u(x,t) = \frac{1}{2}\left\{H(x+t) - H(x-t)\right\},$$



t
x+t=0
t1
x−t=0
x
H(x+t1)
−H(x−t1)
H(x+t1)−H(x−t1)

As t increases, we move further up in the top graph in (x, t)-space, resulting in a wider and wider square pulse.



Outline

Simple Mendelian inheritance

A certain trait is determined by a specific pair of genes, each of which may be two types, say G and g.

One individual may have:

- GG combination (*dominant*)
- Gg or gG, considered equivalent genetically (*hybrid*)
- gg combination (*recessive*)

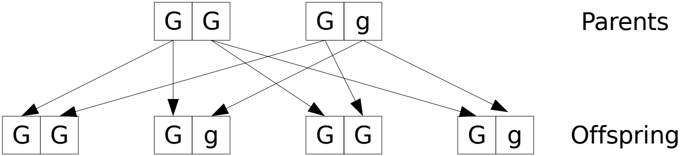
In sexual reproduction, offspring inherit one gene of the pair from each parent.

Basic assumption of Mendelian genetics

Genes inherited from each parent are selected at random, independently of each other. This determines probability of occurrence of each type of offspring. The offspring

- of two GG parents must be GG,
- of two gg parents must be gg,
- of one GG and one gg parent must be Gg,
- other cases must be examined in more detail.

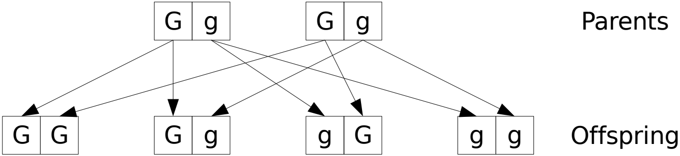
GG and Gg parents



Offspring has probability

- $\frac{1}{2}$ of being GG
- $\frac{1}{2}$ of being Gg

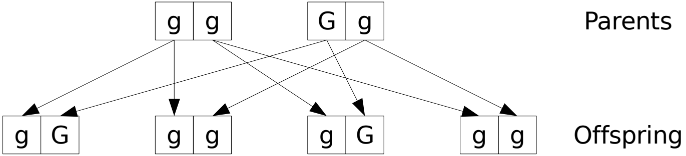
Gg and *Gg* parents



Offspring has probability

- $\frac{1}{4}$ of being GG
- $\frac{1}{2}$ of being Gg
- $\frac{1}{4}$ of being gg

gg and *Gg* parents



Offspring has probability

- $\frac{1}{2}$ of being *Gg*
- $\frac{1}{2}$ of being *gg*

A simple genetic model

Continued matings with a Gg individual – Regular chain

Continued matings with a GG individual – Absorbing chain

Continued matings

Consider a process of continued matings.

- Start with an individual of known or unknown genetic character and mate it with a hybrid.
- Assume that there is at least one offspring; choose one of them at random and mate it with a hybrid.
- Repeat this process through a number of generations.

The genetic type of the chosen offspring in successive generations can be represented by a Markov chain, with states GG, Gg and gg. So there are 3 possibles states $S_1 = GG$, $S_2 = Gg$ and $S_3 = gg$.

We have

$\nearrow$	GG	Gg	gg
GG	0.5	0.5	0
Gg	0.25	0.5	0.25
gg	0	0.5	0.5

The transition probabilities are thus

$$P = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

The Markov chain is here regular. Indeed, take the matrix P,

$$P = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

and compute P^2:

$$P^2 = \begin{pmatrix} \frac{3}{8} & \frac{1}{2} & \frac{1}{8} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{8} & \frac{1}{2} & \frac{3}{8} \end{pmatrix}$$

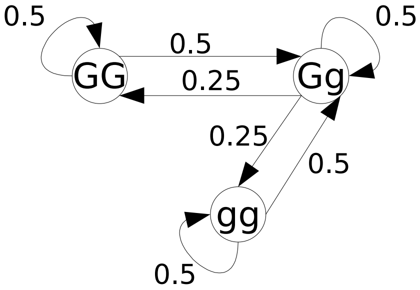
As all entries are positive, P is primitive and the Markov chain is regular.

Another way to check regularity:

Theorem 41

A matrix M is primitive if the associated connection graph is strongly connected, i.e., that there is a path between any pair (i, j) of states, and that there is at least one positive entry on the diagonal of M.

This is checked directly on the transition graph



Compute the left eigenvector associated to 1 by solving

$$(w_1, w_2, w_3) \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix} = (w_1, w_2, w_3)$$

$$\frac{1}{2}w_1 + \frac{1}{4}w_2 = w_1 \tag{25a}$$

$$\frac{1}{2}w_1 + \frac{1}{2}w_2 + \frac{1}{2}w_3 = w_2 \tag{25b}$$

$$\frac{1}{4}w_2 + \frac{1}{2}w_3 = w_3 \tag{25c}$$

From (25a), $w_1 = w_2/2$, and from (25c), $w_3 = w_2/2$. Substituting these values into (25b),

$$\frac{1}{4}w_2 + \frac{1}{2}w_2 + \frac{1}{4}w_2 = w_2,$$

that is, $w_2 = w_2$, i.e., w_2 can take any value. So $w = (\frac{1}{4}, \frac{1}{2}, \frac{1}{4})$.

A simple genetic model

Continued matings with a Gg individual – Regular chain

Continued matings with a GG individual – Absorbing chain

Mating with a *GG* individual

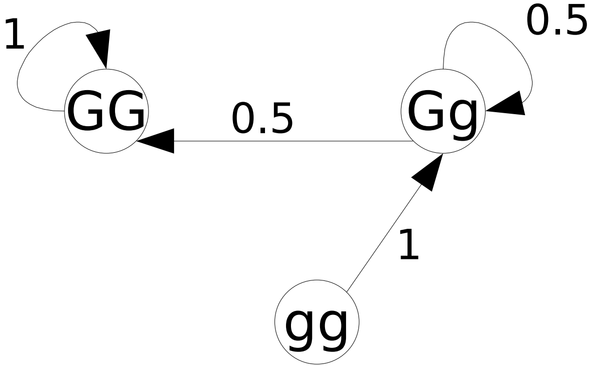
Suppose now that we conduct the same experiment, but mate each new generation with a *GG* individual instead of a *Gg* individual. Transition table is

$\nearrow$	GG	Gg	gg
GG	1	0	0
Gg	0.5	0.5	0
gg	0	1	0

The transition probabilities are thus

$$P = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

New transition graph



Clearly:

- we leave *gg* after one iteration, and can never return,
- as soon as we leave *Gg*, we can never return,
- can never leave *GG* as soon as we get there.

The matrix is already in standard form,

$$P = \left(\begin{array}{ccc} 1 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & 0 \end{array} \right) = \begin{pmatrix} \mathbb{I}_1 & \mathbf{0} \\ R & Q \end{pmatrix}$$

with $\mathbb{I}_1 = 1$, $\mathbf{0} = (0 \;\; 0)$ and

$$R = \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix} \qquad Q = \begin{pmatrix} \frac{1}{2} & 0 \\ 1 & 0 \end{pmatrix}$$

We have

$$\mathbb{I}_2 - Q = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} \frac{1}{2} & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 \\ -1 & 1 \end{pmatrix}$$

so

$$N = (\mathbb{I}_2 - Q)^{-1} = 2 \begin{pmatrix} 1 & 0 \\ 1 & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 2 & 1 \end{pmatrix}$$

Then

$$T = N\mathbb{1} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

and

$$B = NR = \begin{pmatrix} 2 & 0 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$