The discrete logistic equation

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Abstract

This details the analysis of the logistic equation as done in class, and adds additional considerations.

We consider the logistic map

$$f_\mu(x) = \mu x(1-x), \tag{1}$$

used to define the discrete time logistic equation

$$x_{t+1} = f_\mu(x_t), \tag{2}$$

the latter being considered with initial condition $x_0 \in [0,1]$. It is assumed that $\mu > 0$.

1 Well-posedness

We have seen that the set

$$\Lambda = \{x : \forall n \in \mathbb{N},\ f_\mu^n(x) \in [0,1]\}$$

is a Cantor set. We therefore consider the simpler case of $0 < \mu < 4$, where we know for certain that the iterates of f_μ remain in the set $[0,1]$.

2 Fixed points of f_μ

Fixed points of (1) are found by solving the fixed point equation

$$f_\mu(x) = x,$$

that is,

$$\mu x(1-x) = x.$$

It is clear that there are two points that satisfy this equation, namely $x = 0$ and $x = (\mu-1)/\mu$. We denote from now on $p = (\mu-1)/\mu$.

Note that $x = 0$ always exists. On the other hand, p has the following properties:

- $\lim_{\mu\to 0^+} p = -\infty$.
- $\dfrac{\partial}{\partial\mu} p = \dfrac{1}{\mu^2} > 0$, so p is an increasing function of μ.
- $p = 0$ if and only if $\mu = 1$ (unique since p is increasing).
- $\lim_{\mu\to\infty} p = 1$.

Remember that we are modelling a population, so we want $p > 0$ (or at least, nonnegative). If $p > 0$, we say that p is *biologically relevant*. For this, we need $\mu > 1$. In the case that $\mu < 1$, then p does exist, but we do not consider it, as it is not biologically relevant, and by abuse of language, say that p does not exist.

Conclusion 1 We conclude, at this point, that the situation is as follows. The fixed point $x = 0$ always exists, and

- if $\mu \in (0, 1)$, then p does not exist,
- if $\mu > 1$, then p exists.

3 Stability of the fixed points

To determine the stability of f_μ at a fixed point x^*, we need to compare $|f'_\mu(x^*)|$ with the value 1. We have

$$f'_\mu(x) = \mu - 2\mu x = \mu(1 - 2x), \tag{3}$$

and therefore

$$|f'_\mu(0)| = |\mu| = \mu,$$

and

$$\begin{aligned} |f'_\mu(p)| &= \left|\mu\left(1 - 2\frac{\mu-1}{\mu}\right)\right| \\ &= |1 - 2\mu|. \end{aligned}$$

As a consequence, $x = 0$ is attracting if $\mu < 1$ and repelling otherwise, and $p = (\mu - 1)/\mu$ is attracting if $1 - 2\mu < 1$, that is, $\mu < 3$, and repelling otherwise.

Conclusion 2 Building upon **Conclusion 1**, we therefore deduce that

- if $\mu \in (0, 1)$, then $x = 0$ is attracting, and the fixed point $x = p$ does not exist,
- if $\mu \in (1, 3)$, then $x = 0$ is repelling, and the fixed point $x = p$ exists and is attracting,
- if $\mu > 3$, then $x = 0$ is repelling, and the fixed point $x = p$ exists and is repelling.

Remark – The case $\mu = 1$ is called *non hyperbolic*, and is harder to treat. The probability that $\mu = 1$ is zero (the set $\mu = \{1\}$ has measure zero in the parameter space $\mathbb{R}_+$), explaining why, most of the time, the case $\mu = 1$ is omitted. ∘

4 Stable sets of the fixed points

Conclusion 2 establishes that $x = 0$ and $x = p$ are attracting when, respectively, $\mu \in (0, 1)$ and $\mu \in (1, 3)$. This is not sufficient to characterize the behavior of all solutions. Remember that attractiveness of a fixed point x^* implies that there is a neighborhood of x^* that belongs to $W^s(x^*)$, i.e., there exists a neighborhood $\mathcal{N} \ni x^*$ such that $\forall x \in \mathcal{N}$, x is forward asymptotic to x^*.

If we want to make sure that we have the "complete picture", we need to show that $W^s(x^*) = [0, 1]$, i.e., that all solutions go to x^*. There are several ways to tackle this problem, only one is shown here.

4.1 Case of the fixed point $x = 0$ (i.e., case $0 < \mu < 1$)

Since $\mu < 1$, it follows that $f_\mu(x) = \mu x(1 - x) < x(1 - x)$ Also, $x \in [0, 1]$ implies that $1 - x \in [0, 1]$, and therefore $f_\mu(x) < x(1 - x) \leq x$. Therefore, for any $x_0 \in [0, 1]$,

$$\begin{aligned} x_1 &= f_\mu(x_0) \\ &< x_0 \\ x_2 &= f_\mu(x_1) \\ &< x_1. \end{aligned}$$

Therefore we have a strictly decreasing sequence. Since $[0, 1]$ is invariant, the sequence is bounded below by 0. Therefore $\lim_{k\to\infty} f^k(x_0) = 0$, and $W^s(0) = [0, 1]$ when $0 < \mu < 1$. Therefore we can strengthen **Conclusion 2**.

Conclusion 3′: If $0 < \mu < 1$, then for all $x_0 \in [0, 1]$, $\lim_{k\to\infty} f^k(x_0) = 0$, or, equivalently, $\lim_{t\to\infty} x_t = 0$.

4.2 Case of the fixed point $x = p$ (i.e., case $1 < \mu < 3$)

Here, the first idea would be to show the following:

- if $x_0 \in (0, p)$, then $\{x_k\}$ is an increasing sequence,
- if $x_0 \in (p, 1)$, then $\{x_k\}$ is a decreasing sequence.

Note, however, that this is not sufficient: clearly, nothing forbids the sequence from "jumping" from one side of p to the other. This is easy to see using cobwebbing: choosing x_0 such that $f_\mu(x_0) > p$ is not difficult (if suffices to choose x_0 such that $f_\mu(x_0)$ is above the line $y = p$). We thus want to show that the jumps take us closer and closer to p.

To do this, we consider the interval $I_0 = (\varepsilon, 1)$, with $\varepsilon > 0$ small. We have $f'_\mu(0) = \mu > 1$, and by continuity of f'_μ, it is therefore possible to choose ε small enough that $f'_\mu(\varepsilon) = \mu(1 - 2\varepsilon) > 1$.

5 Points of period 2

We now study the existence of periodic points with least period 2, that is, fixed points of the map $f_\mu^2(x)$. We have

$$\begin{aligned} f_\mu^2(x) &= f_\mu(f_\mu(x)) \\ &= \mu f_\mu(x)(1-f_\mu(x)) \\ &= \mu^2 x(1-x)(1-\mu x(1-x)). \end{aligned} \tag{4}$$

Remark that 0 and p are points of period 2. Indeed, a fixed point x^* of f satisfies $f(x^*) = x^*$, and as a consequence, $f^2(x^*) = f(f(x^*)) = f(x^*) = x^*$. This is extremely helpful in localizing the other periodic points, if there are any. Indeed, writing the fixed point equation as

$$f_\mu^2(x) - x = 0,$$

and defining $Q(x) := f_\mu^2(x) - x$, we see that, since 0 and p are fixed points of f_μ^2, they are roots of $Q(x)$. Therefore, Q can be factorized as

$$Q(x) = x(x-p)(-\mu^3 x^2 + Bx + C),$$

since it is clear from (4) that f_μ^2 is a polynomial of degree 4 with leading coefficient equal to $-\mu^3$. Substituting the value $(\mu-1)/\mu$ for p in Q, developping Q and (4) and equating coefficients of like powers gives

$$Q(x) = x\left(x - \frac{\mu-1}{\mu}\right)\left(-\mu^3 x^2 + \mu^2(\mu+1)x - \mu(\mu+1)\right). \tag{5}$$

The roots of (5) are the fixed points of f_μ^2. Since $x = 0$ and $x = p$ are already known, we can concentrate on the roots of the polynomial

$$R(x) := -\mu^3 x^2 + \mu^2(\mu+1)x - \mu(\mu+1).$$

The discriminant is $\Delta = \mu^4(\mu+1)^2 - 4\mu^4(\mu+1) = \mu^4(\mu+1)(\mu+1-4) = \mu^4(\mu+1)(\mu-3)$. Therefore, R has distinct real roots if $\mu > 3$, and a double real root if $\mu = 3$. In the latter case, the root is

$$\frac{-\mu^2(\mu+1)}{-2\mu^3} = \frac{\mu+1}{2\mu} = \frac{2}{3},$$

which is the value of p when evaluated at $\mu = 3$: the fixed point p and the fixed point deduced from R coincide at $\mu = 3$.

So we now consider the case $\mu > 3$. In this case, R has two distinct real roots (that is, f_μ^2 has two distinct real fixed points). More than their actual value, what is of interest is

i) are these roots positive,

ii) are these roots smaller than 1.

If one or both of them is not in $(0,1)$, it will indeed have to be considered as non biologically relevant and ignored.

First, remark that for $\mu > 3$ but very close to 3, it follows from the continuity of R that the roots are in $(0,1)$. To show that this is still the case as we move away from 3, we use Descartes' rule of signs.

Theorem 1 (Descartes' rule of signs). *Let $p(x) = \sum_{i=0}^{m} a_i x^i$ be a polynomial with real coefficients such that $a_m \neq 0$. Define v to be the number of variations in sign of the sequence of coefficients $a_m, \ldots, a_0$. By 'variations in sign' we mean the number of values of n such that the sign of a_n differs from the sign of a_{n-1}, as n ranges from m down to 1.*

Then the number of positive real roots of $p(x)$ is $v - 2N$ for some integer N satisfying $0 \leq N \leq \dfrac{v}{2}$. The number of negative roots of $p(x)$ may be obtained by the same method by applying the rule of signs to $p(-x)$.

Example – Let $p(x) = x^3 + 3x^2 - x - 3$. Looking at the list of coefficients, we have $1, 3, -1, -3$, so there is only one variation in sign (from 3 to -1).

Thus $v = 1$. Since $0 \leq N \leq \frac{1}{2}$ then we must have $N = 0$. Thus $v - 2N = 1$ and so there is exactly one positive real root of $p(x)$.

To find the negative roots, we examine $p(-x) = -x^3 + 3x^2 + x - 3$. The coefficient list is $-1, 3, 1, -3$, so here are are two variations in sign (from -1 to 3 and 1 to -3. Thus $v = 2$ and so $0 \leq N \leq \frac{2}{2} = 1$.

Thus we have two possible solutions, $N = 0$ and $N = 1$, and two possible values of $v - 2N$. Therefore there are either two negative real roots or none at all.

We note that $p(-1) = (-1)^3 + 3 \cdot (-1)^2 - (-1) - 3 = 0$, hence there is at least one negative root. Therefore there must be exactly two. $\diamond$

To use this, note that R has signed coefficients $- + -$, giving two sign changes and the possibility of 0 or 2 positive real roots. On the other hand, $R(-x)$ has signed coefficients $- - -$, hence there are no negative real roots. As we are in the case where the roots are real, it follows that both roots are positive.

To show that the roots are also smaller than 1, consider the change of variables $z = x - 1$. The polynomial R is transformed into

$$\begin{aligned} R_2(z) &= -\mu^3(z+1)^2 + \mu^2(\mu+1)(z+1) - \mu(\mu+1) \\ &= -\mu^3 z^2 + \mu^2(1-\mu)z - \mu. \end{aligned}$$

For $\mu > 1$, all the coefficients of this polynomial are negative, implying that R_2 has no root $z > 0$, implying in turn that R has no root $x > 1$.

6 Attractiveness of the periodic orbits

$$\left(f_\mu^2\right)'(x) = -\mu^2(2x-1)(2\mu x^2 - 2\mu x + 1)$$