THE UNIVERSITY OF MANITOBA

April 15, 2005

FINAL EXAMINATION
TITLE PAGE

DEPARTMENT & COURSE NO: 136.480
TIME: 2.5 HOURS
EXAMINATION: Dynamical Systems: Theory & Applications
EXAMINERS: A. Gumel

LAST NAME: (Print in ink) ______________________________

FIRST NAME (Print in ink) ______________________________

STUDENT NUMBER: ________________

SIGNATURE (in ink): ______________________________
(I understand that cheating is a serious offense)

INSTRUCTIONS TO STUDENTS:

The duration of the exam is 2.5 hours.

Attempt ALL questions.

No texts, cell phones, notes or calculators are permitted

Justify your answers. Show all your work and provide explanations where necessary.

DO NOT WRITE IN THIS COLUMN	
1.	/6
2.	/4
3.	/5
4.	/10
5.	/7
6.	/8
7.	/10
TOTAL	/50

THE UNIVERSITY OF MANITOBA

April 15, 2005 **FINAL EXAMINATION**

DEPARTMENT & COURSE NO: 136.480 TIME: 2.5 HOURS
EXAMINATION: Dynamical Systems: Theory & Applications
EXAMINERS: A. Gumel

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[6] 1. Find the fixed points of the following discrete maps:

(a) $U_{n+1} = 3U_n - U_n^3$;

(b) $U_{n+1} = \mu U_n(1 - U_n)$;

(c) $X_{n+1} = \frac{3}{4} - X_n^2 - Y_n^2, \quad Y_{n+1} = X_n Y_n$.

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[4] 2. Consider the map $U_{n+1} = U_n + \alpha U_n (1 - U_n)$, where α is a parameter. Determine whether or not the map has a 2-cycle. If applicable, give the range of α for which a 2-cycle exists.

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[5] 3. Classify the type of bifurcation the following single parameter differential equation undergoes. Determine the value(s) of x and μ where such bifurcation occur(s).

$$\dot{x} = \mu + \frac{1}{2}x - \frac{x}{1+x}.$$

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[10] 4. By considering the flow in the rectangle with corners $(\pm 1, \pm 2)$, prove that the following system has at least one limit cycle. Sketch a phase diagram for the flow across the rectangle.

$$\dot{x} = y - 8x^3,$$
$$\dot{y} = 2y - 4x - 2y^3.$$

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[7] 5. For the system below, use an appropriate Lyapunov function to prove that the origin is globally asymptotically stable

$$\dot{x} = -x - y^2 + x\,z - x^3,$$
$$\dot{y} = -y + z^2 + x\,y - y^3,$$
$$\dot{z} = -z + x^2 + y\,z - z^3.$$

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[8] 6. Show that the system

$$\dot{x} = \mu x - y + x y^2$$
$$\dot{y} = x + \mu y + y^3$$

undergoes a Hopf bifurcation at the origin for some values of μ. Is the bifurcation degenerate, subcritical or supercritical?
Hint: You many use deductive reasoning and elimination.

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[10] 7. Consider the epidemic model

$$\frac{dS}{dt} = \Pi - \beta SI - \mu S,$$

$$\frac{dE}{dt} = \beta SI - \mu E - \sigma E,$$

$$\frac{dI}{dt} = \sigma E - \mu I,$$

where $S = S(t)$, $E = E(t)$ and $I = I(t)$ are the populations of susceptible, exposed (asymptomatically-infected) and infectious individuals. The parameter $\Pi > 1$ and $0 < \beta, \mu, \sigma \leq 1$. Find the condition(s) for the local and global asymptotic stability of the disease-free equilibrium $\left(S^*, E^*, I^*\right) = \left(\frac{\Pi}{\mu}, 0, 0\right)$.

THE UNIVERSITY OF MANITOBA

DATE: April 2004

FINAL EXAMINATION

TITLE PAGE

DEPARTMENT & COURSE #: 136.480

TIME: 3 HOURS

EXAMINATION: Dynamical Systems : Theory and Applications

EXAMINERS: A. Gumel

NAME: (PRINT) ____________________________

STUDENT NUMBER: __________________

SIGNATURE: ____________________________
(I understand that cheating is a serious offense)

INSTRUCTIONS TO CANDIDATES:

This exam is 3 hours long.

Attempt ALL questions.

No texts, notes or calculators are permitted

Justify your answers. Show all your work and provide explanations where necessary.

Answer questions in the space provided beneath the questions. If space is insufficient, you may continue your answer on the reverse side of the page. If you do continue your answer, please indicate that it is continued.

DO NOT WRITE IN THIS COLUMN	
1.	/15
2.	/9
3.	/10
4.	/8
5.	/8
6.	/12
7.	/18
TOTAL	/80

THE UNIVERSITY OF MANITOBA

DATE: April 2004 FINAL EXAMINATION

DEPARTMENT & COURSE #: 136.480 TIME: 3 HOURS

EXAMINATION: Dynamical Systems : Theory and Applications EXAMINER: A. Gumel

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Values

(1) Consider the map $x_{n+1} = x_n + \mu\ell\, x_n(1 - x_n)$, where μ is a parameter and $\ell > 0$ is the step size.

(i) Determine the stability of each of the fixed points of the map;

(ii) Determine whether or not the map has a 2-cycle (and, if applicable, give the range of $\mu\ell$ for which this happens);

(iii) If a 2-cycle exists, determine its stability range in parameter space.

Values

(2A) Find the Lyapunov exponent of the tent map given by

$$x_{n+1} = \begin{cases} rx_n & \text{for } 0 \le x_n \le 0.5, \\ r - rx_n & \text{for } 0.5 \le x_n \le 1, \end{cases}$$

with $0 \le r \le 2$ and $0 \le x_n \le 1$.

(2B) Show that the "Newton Map" $x_{n+1} = f(x_n)$ when $f(x_n) = x_n - \dfrac{g(x_n)}{g'(x_n)}$ for finding the roots of $g(x) = x^2 - 16 = 0$ has super-stable fixed points at $x^* = \pm 4$.

Values

(3) Classify the type of bifurcation the following differential equations undergo and draw their bifurcation diagrams:

(i) $\dot{x} = x(\mu + e^x)$

(ii) $\dot{x} = x - \dfrac{\mu x}{1 + x^2}$.

Values

(4) Use the Dulac criterion in (i) and a suitable Lyapunov function in (ii) to show that the following planar systems have no closed orbits in the positive quadrant (note that "dot" represents differentiation with respect to t).

$$(i)\quad \begin{aligned} \dot{x} &= x(2-x-y) \\ \dot{y} &= y(4x-x^2-3) \end{aligned}$$

$$(ii)\quad \begin{aligned} \dot{x} &= -8x-xy^2-3y^3 \\ \dot{y} &= 2x^2y+2xy^2 \end{aligned}$$

Values

(5) Show that the system

$$\begin{aligned}\dot{x} &= \mu x - y + xy^2 \\ \dot{y} &= x + \mu y + y^3\end{aligned}$$

undergoes a Hopf bifurcation at the origin as μ varies. Is the bifurcation subcritical, supercritical or degenerate?

Values

(6) Consider the Lorenz system given by

$$\begin{aligned} \dot{x} &= \sigma(y-x) \\ \dot{y} &= \beta x - y - xz \\ \dot{z} &= xy - bz. \end{aligned}$$

(i) Show that the origin is globally-asymptotically stable for $0 < \beta < 1$.

(ii) What type of bifurcation occurs at $\beta = 1$? Briefly justify your answer.

Values

(7) Consider the epidemic model

$$\begin{aligned}\frac{dS}{dt} &= \Pi - \beta SI - \mu S\\ \frac{dE}{dt} &= \beta SI - \mu E - \sigma E\\ \frac{dI}{dt} &= \sigma E - \mu I\end{aligned}$$

where $S = S(t)$, $E = E(t)$ and $I = I(t)$ are the populations of susceptible, exposed (asymptomatically-infected) and infectious individuals and $\Pi >> 1$, β, μ, and σ are positive parameters.

(i) Find condition(s) for the local asymptotic stability of the disease-free equilibrium $(S^*, E^*, I^*) = \left(\frac{\Pi}{\mu}, 0, 0\right)$.

(ii) Use the Perron-Frobenius theorem to establish the global asymptotic stability of this equilibrium solution.

THE UNIVERSITY OF MANITOBA

DATE: April 23, 2003 | FINAL EXAMINATION

DEPARTMENT & COURSE NO: 136.480 | TIME: 3 hours

EXAMINATION: Dynamical Systems: Theory & Applications | EXAMINER: A. Gumel

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Attempt **ALL** questions

(1*A*) Show that the logistic map, given by $x_{n+1} = \mu\, x_n(1 - x_n)$, has a stable 2-cycle for $3 < \mu < 1 + \sqrt{6}$.

(1*B*) Suppose that f has a super-stable p-cycle containing the point x_0. Show that the Liapunov exponent $\lambda = -\infty$.

(1C) Consider the map $x_{n+1} = \sin x_n$. Show that the stability of its unique fixed point is not determined by linearization. Use graphical (cobweb) approach to show that this fixed point is stable.

(2) Consider the differential equation $\dot{x} = x + \dfrac{rx}{1+x^2}$. Classify the type of bifurcation it undergoes, and find the bifurcation point (r_c). Sketch the bifurcation diagram in the (x, r) plane.

(3) Obtain, and classify, the critical points of the following system

$$\dot{x} = x\left(1 - \frac{x}{2} - y\right),$$

$$\dot{y} = y\left(x - 1 - \frac{y}{2}\right).$$

Use this information to draw the phase portrait for the system.

(4) Show that the system

$$\dot{x} = y - x^3$$
$$\dot{y} = -x - y^3$$

has no closed orbits, by constructing a Liapunov function of the form $V = c_1x^2 + c_2y^2$ (where c_1 and c_2 are to be suitably chosen).

(5) Let $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ be a continuously differentiable vector field defined on a simply connected subset R of the plane. Prove that there are closed orbits lying entirely in R if there exists a continuously differentiable, real-valued function $g(\mathbf{x})$ such that $\nabla.(g\dot{\mathbf{x}})$ has one sign throughout R.

(6) Consider the Lorenz system

$$\dot{x} = \sigma(y - x)$$
$$\dot{y} = rx - y - xz$$
$$\dot{z} = xy - \beta z.$$

(a) Use the Liapunov function $V = \frac{1}{2}(x^2 + \sigma y^2 + \sigma z^2)$ to find conditions on σ, r, β which are sufficient for asymptotic stability of the origin.

(b) Investigate the existence and stability of other possible equilibria of the system (determine, in particular, whether the system undergoes a certain type of Hopf bifurcation). What would you expect if you simulate the model with $\sigma = 10$, $\beta = \frac{8}{3}$ and $r = 28$?

THE UNIVERSITY OF MANITOBA

Dec. 14, 2001

FINAL EXAMINATION

TITLE PAGE

DEPARTMENT & COURSE #: 136.480

TIME: 2.5 HOURS

EXAMINATION: Dynamical Systems : Theory and Applications

EXAMINERS: A. Gumel

NAME: (PRINT) ____________________

STUDENT NUMBER: ____________________

SIGNATURE: ____________________
(I understand that cheating is a serious offense)

INSTRUCTIONS TO CANDIDATES:

Attempt ALL four questions. Each question carries 10 marks.

This exam is 2.5 hours long.

No texts, notes or calculators are permitted

Justify your answers. Show all your work and provide explanations where necessary.

Answer questions in the space provided beneath the questions. If space is insufficient, you may continue your answer on the reverse side of the page. If you do continue your answer, please indicate that it is continued.

QUESTION	MARK
1	/10
2	/10
3	/10
4	/10
TOTAL	/40

THE UNIVERSITY OF MANITOBA

Dec. 14, 2001 | FINAL EXAMINATION

DEPARTMENT & COURSE #: 136.480 | TIME: 2.5 HOURS

EXAMINATION: Dynamical Systems : Theory and Applications | EXAMINERS: A. Gumel

Values

5 $Q1A$ Consider the "Newton map"

$$x_{n+1} = f(x_n) \text{ with } f(x_n) = x_n - \frac{g(x_n)}{g'(x_n)}.$$

Show that the Newton map for finding the roots of $g(x) = x^2 - 4$ has superstable fixed points at $x^* = \pm 2$.

2 $Q1B$ Calculate the Liapunov exponent of the map $x_{n+1} = rx_n$. Does the map exhibit chaotic dynamics?

3 Q1C Consider the map $x_{n+1} = \sin x_n$. Show that the stability of its unique fixed point is not determined by linearization. Use graphical (cobweb) approach to show that this fixed point is stable.

THE UNIVERSITY OF MANITOBA

Values

10 Q2 Consider differential equation $\dot{x} = x + \dfrac{rx}{1+x^2}$. Classify the type of bifurcation it undergoes, and find the critical bifurcation value (r_c). Sketch the bifurcation diagram in the (x, r) plane.

Values

Q3A Classify the fixed points of the system

4

$$\begin{aligned} \dot{x} &= xy - 1 \\ \dot{y} &= x - y^3. \end{aligned} \tag{1}$$

Q3B Use the phase portrait of

6

$$\begin{aligned} \frac{dx}{dt} &= -10^{-6}xy + 0.0003x \\ \frac{dy}{dt} &= 10^{-6}xy - 0.1y \end{aligned} \tag{2}$$

to determine the fate of a trajectory $(x, y) = (20,000, 100)$

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EXAMINATION: Dynamical Systems : Theory and Applications | EXAMINERS: A. Gumel

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Values

10 $Q4$ Given the system

$$\begin{aligned}\dot{x} &= -y - \mu(x^3 + xy^2)\\ \dot{y} &= x - \mu(y^3 + x^2y)\end{aligned} \tag{3}$$

where μ is a parameter. Show that the linearized system incorrectly predicts the origin to be a centre for all values of μ.