First-order difference equation

A difference equation takes the form

$$x(n+1) = f(x(n)),$$

which is also denoted

$$x_{n+1} = f(x_n).$$

Starting from an initial point x_0, we have

$$\begin{aligned} x_1 &= f(x_0) \\ x_2 &= f(x_1) = f(f(x_0)) = f^2(x_0) \\ x_3 &= f(x_2) = f(f(f(x_0))) = f^3(x_0) \\ &\dots \end{aligned}$$

Definition 1 (Iterates)

$f(x_0)$ is the *first iterate* of x_0 under f; $f^2(x_0)$ is the *second iterate* of x_0 under f. More generally, $f^n(x_0)$ is the nth iterate of x_0 under f. By convention, $f^0(x_0) = x_0$.

Definition 2 (Orbits)

The set

$$\{f^n(x_0) : n \geq 0\}$$

is called the *forward orbit* of x_0 and is denoted $O^+(x_0)$. The *backward* orbit $O^-(x_0)$ is defined, if f is invertible, by the negative iterates of f. Lastly, the (*whole*) orbit of x_0 is

$$\{f^k(x_0) : -\infty < k < \infty\}.$$

The forward orbit is also called the *positive* orbit. The function f is always assumed to be continuous. If its derivative or second derivative is used in a result, then the assumption is made that $f \in C^1$ or $f \in C^2$..

Periodic points

Definition 3 (Periodic point)

A point p is a *periodic point* of (*least*) *period* n if

$$f^n(p) = p \quad \text{and} \quad f^j(p) \neq p \text{ for } 0 < j < n.$$

Definition 4 (Fixed point)

A periodic point with period $n = 1$ is called a *fixed point*.

Definition 5 (Eventually periodic point)

A point p is an eventually periodic point of period n if there exists $m > 0$ such that

$$f^{m+n}(p) = f^m(p),$$

so that $f^{j+n}(p) = f^j(p)$ for all $j \geq m$ and $f^m(p)$ is a periodic point.

Finding fixed points and periodic points

- A fixed point is such that $f(x) = x$, so it lies at the intersection of the first bisectrix $y = x$ with the graph of $f(x)$.
- A periodic point is such that $f^n(x) = x$, it is thus a fixed point of the nth iterate of f, and so lies at the intersection of the first bisectrix $y = x$ with the graph of $f^n(x)$.

Stable set

Definition 6 (Forward asymptotic point)

q is *forward asymptotic* to p if

$$|f^j(q) - f^j(p)| \to 0 \text{ as } j \to \infty.$$

If p is n-periodic, then q is asymptotic to p if

$$|f^{jn}(q) - p| \to 0 \text{ as } j \to \infty.$$

Definition 7 (Stable set)

The *stable set* of p is

$$W^s(p) = \{q : q \text{ forward asymptotic to } p\}.$$

Unstable set

Definition 8 (Backward asymptotic point)

If f is invertible, then q is *backward asymptotic* to p if

$$|f^j(q) - f^j(p)| \to 0 \text{ as } j \to -\infty.$$

Definition 9 (Unstable set)

The *unstable set* of p is

$$W^u(p) = \{q : q \text{ backward asymptotic to } p\}.$$

Stability

Definition 10 (Stable fixed point)

A fixed point p is *stable* (or Lyapunov stable) if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that $|x_0 - p| < \delta$ implies $|f^n(x_0) - p| < \varepsilon$ for all $n > 0$. If a fixed point p is not stable, then it is *unstable*.

Definition 11 (Attracting fixed point)

A fixed point p is *attracting* if there exists $\eta > 0$ such that

$$|x(0) - p| < \eta \quad \text{implies} \quad \lim_{n\to\infty} x(n) = p.$$

If $\eta = \infty$, then p is a *global attractor* (or is *globally attracting*).

Definition 12 (Asymptotically stable point)

A fixed point p is *asymptotically stable* if it is stable and attracting. It is *globally asymptotically stable* if $\eta = \infty$.

The point does not have to be a fixed point to be stable.

Definition 13

A point p is stable if for every $\varepsilon > 0$, there exists $\delta > 0$ such that if $|x - p| < \delta$, then $|f^k(x) - f^k(p)| < \varepsilon$ for all $k \geq 0$.

Another characterization of asymptotic stability:

Definition 14

A point p is asymptotically stable if it is stable and $W^s(p)$ contains a neighborhood of p.

Can be used with periodic point, in which case we talk of *attracting periodic point* (or *periodic sink)*. A periodic point p for which $W^u(p)$ is a neighborhood of p is a *repelling periodic point* (or *periodic source*).

Condition for stability/instability

Theorem 15

Let $f : \mathbb{R} \to \mathbb{R}$ be C^1.

1. *If p is a n-periodic point of f such that $|(f^n)'(p)| < 1$, then p is an attracting periodic point.*
2. *If p is a n-periodic point of f such that $|(f^n)'(p)| > 1$, then p is repelling.*

ω-limit points and sets

Definition 16

A point y is an ω-limit point of x for f is there exists a sequence $\{n_k\}$ going to infinity as $k \to \infty$ such that

$$\lim_{k\to\infty} d(f^{n_k}(x), y) = 0.$$

The set of all ω-limit points of x is the ω-limit set of x and is denoted $\omega(x)$.

α-limit points and sets

Definition 17

Suppose that f is invertible. A point y is an α-limit point of x for f is there exists a sequence $\{n_k\}$ going to minus infinity as $k \to \infty$ such that

$$\lim_{k\to\infty} d(f^{n_k}(x), y) = 0.$$

The set of all α-limit points of x is the α-limit set of x and is denoted $\alpha(x)$.

Invariant sets

Definition 18

Let $S \subset X$ be a set. S is *positively invariant* (under the flow of f) if $f(x) \in S$ for all $x \in S$, i.e., $f(S) \subset S$. S is *negatively invariant* if $f^{-1}(S) \subset S$. S is *invariant* if $f(S) = S$.

Theorem 19

Let $f : X \to X$ be continuous on a complete metric space X. Then

1. *For any x, $\omega(x) = \bigcap_{N \geq 0} \overline{\bigcup_{n \geq N} \{f^n(x)\}}$.*
2. *If $f^j(x) = y$ for some j, then $\omega(x) = \omega(y)$.*
3. *For any x, $\omega(x)$ is closed and positively invariant. If $O^+(x)$ is contained in some compact subset of X (e.g., the forward orbit is bounded in some Euclidian space) or if f is one-to-one, then $\omega(x)$ is invariant.*
4. *If $O^+(x)$ is contained in some compact subset of X, then $\omega(x)$ is nonempty and compact and $d(f^n(x), \omega(x)) \to 0$ as $n \to \infty$.*
5. *If $D \subset X$ is closed and positively invariant, and $x \in D$, then $\omega(x) \subset D$.*
6. *If $y \in \omega(x)$, then $\omega(y) \subset \omega(x)$.*

Minimal set

Definition 20

A set S is a *minimal* set for f if (i) S is a closed, nonempty, invariant set and (ii) if B is a closed nonempty invariant subset of S then $B = S$.

Clearly, any periodic orbit is a minimal set.

Proposition 1

Let X be a metric space, $f : X \to X$ a continuous map, and $S \subset X$ a nonempty compact subset. Then S is a minimal set if and only if $\omega(x) = S$ for all $x \in S$.

Cantor sets

Let X be a topological space and $S \subset X$ a subset.

Definition 21 (Nowhere dense set)

S is *nowhere dense* if $\mathrm{int}(\mathrm{cl}(S)) = \emptyset$.

Definition 22 (Totally disconnected set)

S is *totally disconnected* if the connected components of S are single points.

Definition 23 (Perfect set)

S is *perfect* if it is closed and that every point $p \in S$ is the limit of points $q_n \in S$ with $q_n \neq p$.

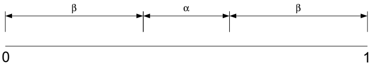
Definition 24 (Cantor set)

S is a *Cantor set* if it is totally disconnected, perfect and compact.

Construction of the middle-α Cantor set – Step 0

Let $\alpha \in (0, 1)$ and β such that $2\beta + \alpha = 1$.
Step 0 : Consider the interval $S_0 = [0, 1]$.

S_0 can be decomposed as two subintervals of length β and one subinterval of length α:



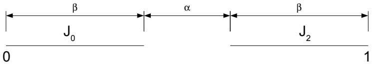
Note that α and β are proportions of the length of S_0.

Step 1

Remove the middle open interval of length α, $G = (\beta, 1-\beta)$, and define

$$S_1 = S_0 \setminus G = J_1 \cup J_2,$$

where J_0 and J_2 are the left and right closed intervals, respectively, resulting from the cut:



We get the lengths $L(J_0) = L(J_2) = \beta$.

Step 2

Apply the same procedure to each of J_0 and J_2: remove the middle $\alpha L(J_k) = \alpha\beta$ sized open interval.
Add a suffix 0 to the interval that is left of this middle interval, 2 to the interval on the right:

There are 2^2 intervals, each of length $L(J_{k_1,k_2}) = \beta L(J_{k_1}) = \beta^2$ $(k_1, k_2 = 0, 2)$.

Step 3

Apply the same procedure to each of J_{k_1,k_2}, $k_1, k_2 = 0, 2$: remove the middle $\alpha L(J_{k_1,k_2}) = \alpha\beta^2$ sized open interval.
Add a suffix 0 to the interval that is left of this middle interval, 2 to the interval on the right:

There are 2^3 intervals, each of length $L(J_{k_1,k_2,k_3}) = \beta L(J_{k_1,k_2}) = \beta^3$ $(k_1, k_2, k_3 = 0, 2)$.

Step k

After proceeding to the kth cut, we have

$$S_k = \bigcup_{j_1,\ldots,j_k=0,2} J_{j_1,\ldots,j_k},$$

where each of the 2^k closed intervals $J_{j_1,\ldots,j_k}$ has length $L(J_{j_1,\ldots,j_k}) = \beta^k$.

The Cantor set

Finally, we let

$$C = \bigcap_{k=0}^{\infty} S_k.$$

Theorem 25

C is a Cantor set.

This is proved by showing that C is nowhere dense and perfect.

S_k has a total length of $2^k\beta^k = (2\beta)^k$, so, since $2\beta < 1$, the total length of S_k goes to zero as $k \to \infty$.

Consider the logistic map

$$f_\mu(x) = \mu x(1-x), \tag{1}$$

in the case $\mu > 4$. For each $n \in \mathbb{N}$, define

$$\Lambda_n = \{x : f_\mu^n(x) \in [0,1]\}. \tag{2}$$

The set

$$\Lambda = \bigcap_{n=1}^{\infty} \Lambda_n$$

describes the points that remain in $[0,1]$ forever under iteration of f.

Theorem 26

Λ is a Cantor set for $\mu > 4$.

This implies that there are infinitely many points in $[0,1]$ for which all iterates remain in $[0,1]$.. although these points are hard to find.

Parametrized families of functions

Consider the logistic map

$$x_{t+1} = \mu x_t(1-x_t), \tag{3}$$

where μ is a parameter in $\mathbb{R}_+$, and x will typically be taken in $[0,1]$. Let

$$f_\mu(x) = \mu x(1-x). \tag{1}$$

The function f_μ is called a *parametrized family* of functions.

Bifurcations

Definition 27 (Bifurcation)

Let f_μ be a parametrized family of functions. Then there is a *bifurcation* at $\mu = \mu_0$ (or μ_0 is a bifurcation point) if there exists $\varepsilon > 0$ such that, if $\mu_0 - \varepsilon < a < \mu_0$ and $\mu_0 < b < \mu_0 + \varepsilon$, then the dynamics of $f_a(x)$ are "different" from the dynamics of $f_b(x)$.

An example of "different" would be that f_a has a fixed point (that is, a 1-periodic point) and f_b has a 2-periodic point.

Formally, f_a and f_b are *topologically conjugate* to two different functions.

Topological conjugacy

Definition 28

Let $f : D \to D$ and $g : E \to E$ be functions. Then f *topologically conjugate* to g if there exists a homeomorphism $\tau : D \to E$, called a *topological conjugacy*, such that $\tau \circ f = g \circ \tau$.

Proposition 2

Let D and E be subsets of $\mathbb{R}$, and $\phi : D \to E$ be an homeomorphism. Then

1. *The set $U \subset D$ is open iff $\phi(U)$ is open in E.*
2. *The sequence $\{x_k\}$ converges in D converges to x in D iff the sequence $\{\phi(x_k)\}$ converges to $\phi(x)$ in E.*
3. *The set F is closed in D iff the set $\phi(F)$ is closed in E.*
4. *The set A is dense in D iff the set $\phi(A)$ is dense in E.*

Theorem 29

Let D and E be subsets of $\mathbb{R}$, $f : D \to D$, $g : E \to E$, and $\tau : D \to E$ be a topological conjugacy of f and g. Then

1. *$\tau^{-1} : E \to D$ is a topological conjugacy.*
2. *$\tau \circ f^n = g^n \circ \tau$ for all $n \in \mathbb{N}$.*
3. *p is a periodic point of f with least period n iff $\tau(p)$ is a periodic point of g with least period n.*
4. *If p is a periodic point of f with stable set $W^s(p)$, then the stable set of $\tau(p)$ is $\tau(W^s(p))$.*
5. *The periodic points of f are dense in D iff the periodic points of g are dense in E.*
6. *f is topologically transitive on D iff g is topologically transitive on E.*
7. *f is chaotic on D iff g is chaotic on E.*

The result of Li and Yorke

Theorem 30

Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. Assume that there exists a point a such that either

- $f^3(a) \leq a < f(a) < f^2(a)$

or

- $f^3(a) \geq a > f(a) > f^2(a)$.

Then f has points of all periods.

Sharkovskii's ordering of the natural integers

Consider the set of integers, and order them with the Sharkovskii ordering $\triangleright$. To do this, first consider all odd integers,

$$3 \triangleright 5 \triangleright 7 \triangleright 9 \triangleright 11 \triangleright \cdots$$

followed by all odd integers multiplied by 2

$$\triangleright 2 \cdot 3 \triangleright 2 \cdot 5 \triangleright 2 \cdot 7 \triangleright 2 \cdot 7 \triangleright 2 \cdot 9 \triangleright 2 \cdot 11 \triangleright \cdots$$

followed by all odd integers multiplied by 2^2

$$\triangleright 2^2 \cdot 3 \triangleright 2^2 \cdot 5 \triangleright 2^2 \cdot 7 \triangleright 2^2 \cdot 7 \triangleright 2^2 \cdot 9 \triangleright 2^2 \cdot 11 \triangleright \cdots$$

continue..

$$\triangleright 2^n \cdot 3 \triangleright 2^n \cdot 5 \triangleright \cdots \triangleright 2^{n+1} \cdot 3 \triangleright 2^{n+1} \cdot 5 \triangleright \cdots$$

and finally, add all the powers of 2 in decreasing powers,

$$\triangleright 2^{n+1} \triangleright 2^n \triangleright \cdots 2^2 \triangleright 2 \triangleright 1.$$

Sharkovskii's theorem

The Sharkovskii ordering gives an ordering between all positive integers.

Theorem 31 (Sharkovskii)

Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a continuous function. Assume that f has a point of least period n, and that $n \triangleright k$. Then f has a point of least period k.

A function that has a periodic point of period 3 has good chances of being "agitated"..
Note that this says nothing about the stability of the periodic points.

Topologically transitive function

Definition 32

The function $f : D \to D$ is *topologically transitive* on D if for any open sets U and V that interset D, there exists $z \in U \cap D$ and $n \in \mathbb{N}$ such that $f^n(z) \in D$.
Equivalently, f is topologically transitive on D if for any two points $x, y \in D$ and any $\varepsilon > 0$, there exists $z \in D$ such that $|z - x| < \varepsilon$ and $|f^n(x) - y| < \varepsilon$ for some n.

Sensitive dependence on initial conditions

Definition 33

The function $f : D \to D$ exhibits *sensitive dependence on initial conditions* if there exists $\delta > 0$ such that for any $x \in D$ and any $\varepsilon > 0$, there exists $y \in D$ and $n \in \mathbb{N}$ such that $|x - y| < \varepsilon$ and $|f^n(x) - f^n(y)| > \delta$.

Chaos

The following in due to Devaney. There are other definitions.

Definition 34

The function $f : D \to D$ is *chaotic* if

1. the periodic points of f are dense in D,
2. f is topologically transitive,
3. and f exhibits sensitive dependence on initial conditions.

Reminder: dense sets

Definition 35

Let A be a subset of B. Then A is dense in B if every point in B is an accumulation point of A, a point of A, or both.

Proposition 3

Let A be a subset of B. Then the following statements are equivalent.

1. *A is dense in B.*
2. *For each point $x \in B$ and each $\varepsilon > 0$, there exists $y \in A$ such that $|x - y| < \varepsilon$.*
3. *For every point $x \in B$, there exists a sequence of points contained in A that converges to x.*

Point 2 "says" that every circle centered at a point in B contains a point of A.

Theorem 36

Let $f_\mu(x) = \mu x(1-x)$, $\mu > 4$ and

$$\Lambda = \{x : \forall n \in \mathbb{N},\ f_\mu^n(x) \in [0,1]\}.$$

Then

1. *If $x \in \mathbb{R}$ does not belong to Λ, then x is in the stable set of infinity.*
2. *The set Λ is a Cantor set.*
3. *The function $f : \Lambda \to \Lambda$ is chaotic.*

Sensitive dependence, expansive maps

Definition 37 (Sensitive dependence on IC)

A map f on a metric space X has *sensitive dependence on initial conditions* provided there exists $r > 0$ (independent of the point) such that for each $x \in X$ and for each $\varepsilon > 0$, there exists a point $y \in X$ with $d(x, y) < \varepsilon$ and a $k \geq 0$, such that $d(f^k(x), f^k(y)) \geq r$.

Definition 38 (Expansive map)

A map f on a metric space X is *expansive* provided there exists $r > 0$ (independent of the points) such that for each pair of points $x, y \in X$, there exists $k \geq 0$ such that $d(f^k(x), f^k(y)) \geq r$.

On a perfect metric space (every point $p \in X$ is limit of a sequence of elements $q_n \in X$, $p \neq q_n$), expansiveness implies sensitive dependence on IC.

Transitive map

Definition 39 (Transitive map)

A map $f : X \to X$ is *(topologically) transitive* on an invariant set Y provided the (forward) orbit of some point p is dense in Y.

f is transitive if, given any two open sets U and V in Y, there exists $n \in \mathbb{N}$, $n > 0$, such that $f^n(U) \cap V \neq \emptyset$.

f transitive means that f mixes up the points of Y.

Chaos

Definition 40

A map f on a metric space X is *chaotic on an invariant set* Y (or exhibits chaos) provided

1. f is transitive on Y.
2. f has sensitive dependence on initial conditions on Y.

Lyapunov exponents

Definition 41 (Lyapunov exponents)

Let $f : \mathbb{R} \to \mathbb{R}$ be a C^1 function. The Lyapunov exponent of $x_0 \in \mathbb{R}$, $\lambda(x_0)$, is defined by

$$\begin{aligned} \lambda(x_0) &= \limsup_{n\to\infty} \frac{1}{n} \log\left(\left|(f^n)'(x_0)\right|\right) \\ &= \limsup_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log\left(\left|f'(x_j)\right|\right). \end{aligned}$$