Bifurcations

The general context of bifurcations

Consider the discrete time system

$$x_{t+1} = f(x_t)$$

or the continuous time system

$$x' = f(x).$$

We start with a function $f : \mathbb{R}^2 \to \mathbb{R}$, C^r when a map is considered, C^1 when continuous time is considered.

In both cases, the function f can depend on some parameters. We are interested in the differences of qualitative behavior, as one of these parameters, which we call μ, varies.

So we write

$$x_{t+1} = f(x_t, \mu) = f_\mu(x_t) \tag{1}$$

and

$$x' = f(x, \mu) = f_\mu(x) \tag{2}$$

for $\mu \in \mathbb{R}$.

Bifurcations

Definition (Bifurcation)

Let f_μ be a parametrized family of functions. Then there is a *bifurcation* at $\mu = \mu_0$ (or μ_0 is a bifurcation point) if there exists $\varepsilon > 0$ such that, if $\mu_0 - \varepsilon < a < \mu_0$ and $\mu_0 < b < \mu_0 + \varepsilon$, then the dynamics of $f_a(x)$ are "different" from the dynamics of $f_b(x)$.

An example of "different" would be that f_a has a fixed point (that is, a 1-periodic point) and f_b has a 2-periodic point.

Formally, f_a and f_b are *topologically conjugate* to two different functions.

Topological conjugacy

Definition

Let $f : D \to D$ and $g : E \to E$ be functions. Then f *topologically conjugate* to g if there exists a homeomorphism $\tau : D \to E$, called a *topological conjugacy*, such that $\tau \circ f = g \circ \tau$.

Types of bifurcations (discrete time)

Saddle-node (or tangent):

$$x_{t+1} = \mu + x_t + x_t^2$$

Transcritical:

$$x_{t+1} = (\mu + 1)x_t + x_t^2$$

Pitchfork:

$$x_{t+1} = (\mu + 1)x_t - x_t^3$$

Period doubling (or flip):

$$x_{t+1} = \mu - x_t - x_t^2$$

Discrete-time saddle-node

$$x_{t+1} = \mu + x_t + x_t^2$$

Fixed points (FP)

$$\begin{aligned} x = \mu + x + x^2 &\Leftrightarrow x^2 = -\mu \\ &\Leftrightarrow x = \pm\sqrt{-\mu} \end{aligned}$$

So no real valued FP if $\mu > 0$, 2 if $\mu < 0$.

Stability of $\sqrt{-\mu}$ $f'(x) = 1 + 2x$, so, assuming $\mu < 0$,

$$f'(\sqrt{-\mu}) = 1 + 2\sqrt{-\mu}$$

Thus

$$|f'(\sqrt{-\mu})| < 1 \Leftrightarrow -1 < 1 + 2\sqrt{-\mu} < 1 \Leftrightarrow -1 < \sqrt{-\mu} < 0$$

which is impossible. Therefore, $\sqrt{-\mu}$ is always repelling.

Stability of $\sqrt{-\mu}$ assuming $\mu < 0$,

$$f'(-\sqrt{-\mu}) = 1 - 2\sqrt{-\mu}$$

Thus

$$\begin{aligned} |f'(-\sqrt{-\mu})| < 1 &\Leftrightarrow -1 < 1 - 2\sqrt{-\mu} < 1 \\ &\Leftrightarrow -1 < -\sqrt{-\mu} < 0 \\ &\Leftrightarrow 0 < \sqrt{-\mu} < 1 \\ &\Leftrightarrow -1 < \mu < 0 \end{aligned}$$

So, for $-1 < \mu < 0$, the FP $-\sqrt{-\mu}$ is attracting.

Summary: discrete-time saddle-node

Discrete-time period doubling

$$x_{t+1} = \mu - x_t - x_t^2$$

<u>FP</u>:

$$x = \mu - x - x^2 \Leftrightarrow x^2 + 2x - \mu = 0$$

Discriminant: $\Delta = 4 + 4\mu = 4(1+\mu)$. So we get

$$x_{1,2} = \frac{-2 \pm 2\sqrt{1+\mu}}{2} = -1 \pm \sqrt{1+\mu}$$

Types of bifurcations (continuous time)

- Saddle-node
$$x' = \mu - x^2$$
- Transcritical
$$x' = \mu x - x^2$$
- Pitchfork
  - supercritical
  $$x' = \mu x - x^3$$
  - subcritical
  $$x' = \mu x + x^3$$

Saddle-node for maps

Theorem

Assume $f \in C^r$ with $r \geq 2$, for both x and μ. Suppose that

1. $f(x_0, \mu_0) = x_0$,
2. $f'_{\mu_0}(x_0) = 1$,
3. $f''_{\mu_0}(x_0) \neq 0$ *and*
4. $\dfrac{\partial f}{\partial \mu}(x_0, \mu_0) \neq 0$.

Then $\exists I \ni x_0$ and $N \ni \mu_0$, and $m \in C^r(I, N)$, such that

1. $f_{m(x)}(x) = x$,
2. $m(x_0) = \mu_0$,
3. *the graph of m gives all the fixed points in $I \times N$.*

Theorem (cont.)

Moreover, $m'(x_0) = 0$ and

$$m''(x_0) = \frac{-\dfrac{\partial^2 f}{\partial x^2}(x_0, \mu_0)}{\dfrac{\partial f}{\partial \mu}(x_0, \mu_0)} \neq 0.$$

These fixed points are attracting on one side of x_0 and repelling on the other.

Saddle-node for continuous equations

Consider the system $x' = f(x, \mu)$, $x \in \mathbb{R}$. Suppose that $f(x_0, \mu_0) = 0$. Further, assume that the following nondegeneracy conditions hold:

1. $a_0 = \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(x_0, \mu_0) \neq 0$,
2. $\frac{\partial f}{\partial \mu}(x_0, \mu_0) \neq 0$.

Then, in a neighborhood of (x_0, μ_0), the equation $x' = f(x, \mu)$ is topologically equivalent to the normal form

$$x' = \gamma + \mathrm{sign}(a_0)x^2$$

Saddle-node for continuous systems

Theorem

Consider the system $x' = f(x,\mu)$, $x \in \mathbb{R}^n$. *Suppose that* $f(x,0) = x_0 = 0$. *Further, assume that*

1. *The Jacobian matrix* $A_0 = Df(0,0)$ *has a simple zero eigenvalue,*
2. $a_0 \neq 0$, *where*

$$a_0 = \frac{1}{2}\langle p, B(q,q)\rangle = \frac{1}{2}\frac{d^2}{d\tau^2}\langle p, f(\tau q, 0)\rangle\bigg|_{\tau=0}$$

3. $f_\mu(0,0) \neq 0$.

B is the bilinear function with components

$$B_j(x,y) = \sum_{k,\ell=1}^{n} \frac{\partial^2 f_j(\xi,0)}{\partial \xi_k \partial \xi_\ell}\bigg|_{\xi=0} x_k y_\ell, \quad j = 1,\ldots,n$$

and $\langle p,q\rangle = p^T q$ *the standard inner product.*

Theorem (cont.)

Then, in a neighborhood of the origin, the system $x' = f(x, \mu)$ is topologically equivalent to the suspension of the normal form by the standard saddle,

$$\begin{aligned} y' &= \gamma + \mathrm{sign}(a_0) y^2 \\ y_S' &= -y_S \\ y_U' &= y_U \end{aligned}$$

with $y \in \mathbb{R}$, $y_S \in \mathbb{R}^{n_S}$ and $y_U \in \mathbb{R}^{n_U}$, where $n_S + n_U + 1 = n$ and n_S is number of eigenvalues of A_0 with negative real parts.

Pitchfork bifurcation

The ODE $x' = f(x, \mu)$, with the function $f(x, \mu)$ satisfying

$$-f(x, \mu) = f(-x, \mu)$$

(f is odd),

$$\frac{\partial f}{\partial x}(0, \mu_0) = 0, \frac{\partial^2 f}{\partial x^2}(0, \mu_0) = 0, \frac{\partial^3 f}{\partial x^3}(0, \mu_0) \neq 0,$$
$$\frac{\partial f}{\partial r}(0, \mu_0) = 0, \frac{\partial^2 f}{\partial r \partial x}(0, \mu_0) \neq 0.$$

has a pitchfork bifurcation at $(x, \mu) = (0, \mu_0)$. The form of the pitchfork is determined by the sign of the third derivative:

$$\frac{\partial^3 f}{\partial x^3}(0, \mu_0) \begin{cases} < 0, & \text{supercritical} \\ > 0, & \text{subcritical} \end{cases}$$

Theorem (Period doubling bifurcation)

Assume f is C^r in x and μ, with $r \geq 3$, and that

1. *x_0 is a fixed point for $\mu = \mu_0$, i.e., $f(x_0, \mu_0) = x_0$,*
2. *$f'_{\mu_0}(x_0) = -1$ (so, since $\neq 1$, there is a curve of fixed points $x(\mu)$ for μ close to μ_0),*
3. *the derivative of $f'_\mu(x(\mu))$ with respect to μ is nonzero,*

$$\alpha = \left[\frac{\partial^2 f}{\partial \mu \partial x} + \frac{1}{2} \left(\frac{\partial f}{\partial \mu} \right) \left(\frac{\partial^2 f}{\partial x^2} \right) \right] \Bigg|_{(x_0, \mu_0)} \neq 0,$$

4. *the graph of $f^2_{\mu_0}$ has nonzero cubic terms in its tangency with the diagonal (the quadratic term is zero):*

$$\beta = \left(\frac{1}{3!} \frac{\partial^3 f}{\partial x^3}(x_0, \mu_0) \right) + \left(\frac{1}{2!} \frac{\partial^2 f}{\partial x^2}(x_0, \mu_0) \right)^2 \neq 0$$

Theorem (Period doubling bifurcation (cont.))

Then there is a period doubling bifurcation at (x_0, μ_0). *More specifically,*

1. *there is a differentiable curve of fixed points,* $x(\mu)$, *passing through* (x_0, μ_0), *and the stability of the fixed point changes at* μ_0;
2. *which side of* μ_0 *is attracting depends on the sign of* α;
3. *there is a differentiable curve* γ *passing through* (x_0, μ_0), *such that* $\gamma \setminus \{(x_0, \mu_0)\}$ *is the union of hyperbolic period 2 orbits;*
4. γ *is tangent to* $\mathbb{R} \times \{\mu_0\}$ *at* (x_0, μ_0), *so* γ *is the graph of a function* $\mu = m(x)$, *with* $m'(x_0) = 0$ *and* $m''(x_0) = -2\beta/\alpha \neq 0$;
5. *the stability of the period 2 orbit depends on* β: *if* $\beta > 0$, *it is attracting, if* $\beta < 0$, *it is repelling.*

Poincaré map

Consider

$$x' = f(x) \tag{3}$$

If Γ is a periodic orbit of (3) through x_0, and Σ is a hyperplane perpendicular to Γ at x_0, then for any point $x \in \Sigma$ close enough to x_0, the solution through x at $t = 0$, $\phi_t(x)$, crosses Σ again at a point $P(x)$ near x_0.

The mapping $x \mapsto P(x)$ is the *Poincaré map*.

Theorem

Let E be an open subset of $\mathbb{R}^n$ and $f \in C^1(E)$. Suppose that $\phi_t(x_0)$ is a periodic solution of (3) *of period T, and that*

$$\Gamma = \{x \in \mathbb{R}^n : \quad x = \phi_t(x_0), \quad 0 \le t \le T\}$$

is contained in E. Let Σ be the hyperplane orthogonal to Γ at x_0, i.e.,

$$\Sigma = \{x \in \mathbb{R}^n : \quad (x - x_0) \cdot f(x_0) = 0\}.$$

Then there exists $\delta > 0$ and a unique function $\tau(x)$, defined and continuously differentiable for $x \in \mathcal{N}_\delta(x_0)$, such that $\tau(x_0) = T$ and

$$\phi_{\tau(x)}(x) \in \Sigma$$

for all $x \in \mathcal{N}_\delta(x_0)$. For $x \in \mathcal{N}_\delta(x_0) \cap \sigma$,

$$P(x) = \phi_{\tau(x)}(x)$$

is the Poincaré map for Γ at x_0.

Example

Consider the system

$$\begin{aligned} x' &= -y + x(\mu - x^2 - y^2) \\ y' &= x + y(\mu - x^2 - y^2) \end{aligned}$$

Transform to polar coordinates:

$$\begin{aligned} r' &= r(\mu - r^2) \\ \theta' &= 1 \end{aligned}$$

$$
\begin{aligned}
r' &= r(\mu - r^2) \\
\theta' &= 1
\end{aligned}
$$

has solution

$$
r(t) = \frac{\sqrt{(1 + e^{-2\mu t} C\mu)\mu}}{1 + e^{-2\mu t} C\mu}
$$

$$
\theta(t) = t + \theta_0
$$

Hopf bifurcation

Theorem (Hopf bifurcation theorem)

Consider the system

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a_{11}(\mu) & a_{12}(\mu) \\ a_{21}(\mu) & a_{22}(\mu) \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} f_1(x,y,\mu) \\ f_2(x,y,\mu) \end{pmatrix}, \qquad (4)$$

with $\mu \in \mathbb{R}$ *a parameter. Suppose* $f_1, f_2 \in C^3$, *that the origin is an equilibrium of* (4), *and that the matrix*

$$J(\mu) = \begin{pmatrix} a_{11}(\mu) & a_{12}(\mu) \\ a_{21}(\mu) & a_{22}(\mu) \end{pmatrix}$$

is valid in a neighborhood of the origin. Additionally, suppose that the eigenvalues of $J(\mu)$ *are* $\alpha(\mu) + i\beta(\mu)$, *with* $\alpha(0) = 0$ *and* $\beta(r) \neq 0$, *satisfying the transversality condition*

$$\left.\frac{d\alpha}{d\mu}\right|_{\mu=0} \neq 0.$$

Theorem (Hopf bifurcation theorem, cont.)

Then, in any open set $\mathcal{U} \ni (0,0)$ *in* $\mathbb{R}^2$ *and for any* $\mu_0 > 0$*, there exists* $\bar{\mu}$*,* $|\bar{\mu}| < \mu_0$*, such that* (4) *has a periodic solution for* $\mu = \bar{\mu}$ *in* $\mathcal{U}$ *with approximate period* $T = 2\pi/\beta(0)$*.*

Another formulation

Theorem (Hopf bifurcation)

Let $x' = A(\mu)x + F(\mu, x)$ be a C^k planar vector field, with $k \geq 0$, depending on the scalar parameter μ such that $F(\mu, 0) = 0$ and $D_x F(\mu, 0) = 0$ for all μ sufficiently close enough to the origin. Assume that the linear part $A(\mu)$ at the origin has the eigenvalue $\alpha(\mu) \pm i\beta(\mu)$, with $\alpha(0) = 0$ and $\beta(0) \neq 0$. Furthermore, assume the eigenvalues cross the imaginary axis with nonzero speed, i.e.,

$$\frac{d}{d\mu}\alpha(\mu)\bigg|_{\mu=0} \neq 0.$$

Then, in any neighborhood $\mathcal{U} \ni (0,0)$ in $\mathbb{R}^2$ and any given $\mu_0 > 0$, there exists a $\bar{\mu}$ with $|\bar{\mu}| < \mu_0$ such that the differential equation $x' = A(\bar{\mu})x + F(\bar{\mu}, x)$ has a nontrivial periodic orbit in $\mathcal{U}$.

Supercritical or subcritical Hopf?

Transform the system into

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \alpha(\mu) & \beta(\mu) \\ -\beta(\mu) & \alpha(\mu) \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} f_1(x,y,\mu) \\ g_1(x,y,\mu) \end{pmatrix} = \begin{pmatrix} f(x,y,\mu) \\ g(x,y,\mu) \end{pmatrix}$$

The Jacobian at the origin is

$$J(\mu) = \begin{pmatrix} \alpha(\mu) & \beta(\mu) \\ -\beta(\mu) & \alpha(\mu) \end{pmatrix}$$

and thus eigenvalues are $\alpha(\mu) \pm i\beta(\mu)$, and $\alpha(0) = 0$ and $\beta(0) > 0$.

Supercritical or subcritical Hopf? (cont.)

Define

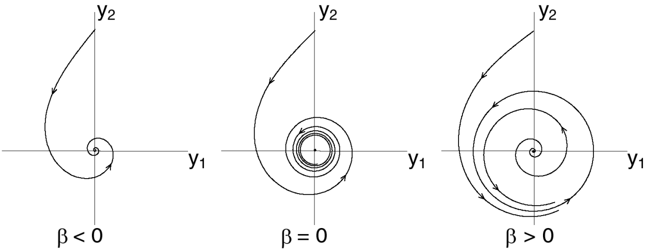
$$C = f_{xxx} + f_{xyy} + g_{xxy} + g_{yyy} \\ + \frac{1}{\beta(0)} \left(-f_{xy} \left(f_{xx} + f_{yy} \right) + g_{xy} \left(g_{xx} + g_{yy} \right) + f_{xx} g_{xx} - f_{yy} g_{yy} \right),$$

evaluated at $(0, 0)$ and for $\mu = 0$. Then, if $d\alpha(0)/d\mu > 0$,

1. If $C < 0$, then for $\mu < 0$, the origin is a stable spiral, and for $\mu > 0$, there exists a stable periodic solution and the origin is unstable (**supercritical Hopf**).
2. If $C > 0$, then for $\mu < 0$, there exists an unstable periodic solution and the origin is unstable, and for $\mu > 0$, the origin is unstable (**subcritical Hopf**).
3. If $C = 0$, the test is inconclusive.

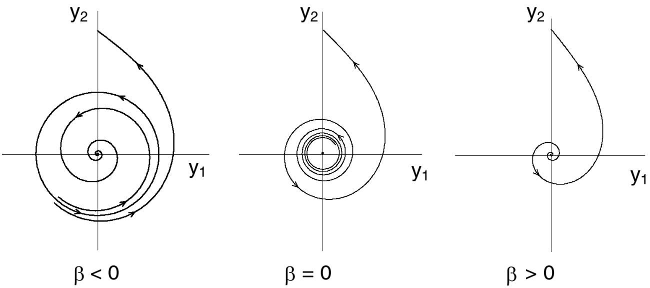
Supercritical Hopf

Here, y_1, y_2 are the variables, β is the bifurcation parameter.



Subcritical Hopf

Here, y_1, y_2 are the variables, β is the bifurcation parameter.



Example: predator-prey system

Predator-prey system:

$$x' = ax - bxy \tag{5a}$$

$$y' = cxy - dy \tag{5b}$$

where $a, b, c, d > 0$.

Example: a general chemostat

Chemostat:

$$S' = D(S^0 - S) - u(S)x \tag{6a}$$
$$x' = (g(S) - D_1)x \tag{6b}$$

with $u, g \in C^1$ such that

1. $u(0) = g(0) = 0$,
2. $\exists M_u, M_g \in \mathbb{R}$ such that for all $S \in \mathbb{R}_+$, $u(S) \leq M_u$ and $g(S) \leq M_g$.

Equilibria

From (6b), at equilibrium, $x = 0$ or $g(S) = D_1$.

Set $x = 0$ in (6a): $S = S^0$. So one EP (the *trivial* or *washout* EP) is $(S, x) = (S^0, 0)$.

Let $\lambda, \mu \in \bar{\mathbb{R}}$, where $\bar{\mathbb{R}} = \mathbb{R} \cup \{-\infty\} \cup \{+\infty\}$. We suppose $\lambda < \mu$, and that λ and μ are the solutions of $g(S) = D_1$. We set λ and/or μ equal to $+\infty$ if no solution to the equation exists.

Call S^* these equilibria (so $S^* = \lambda$ or $S^* = \mu$). Then, substituting into (6a),

$$x^* = \frac{D(S^0 - S^*)}{u(S^*)}$$

This EP is relevant only if $S^* < S^0$.

Equilibria

So, in conclusion, there are potentially two EPs:

- the washout equilibrium,

$$E_0 := (S, x) = (S^0, 0)$$

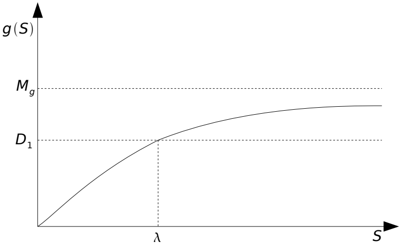
- one or two nontrivial equilibria,

$$E^* := (S, x) = \left(S^*, \frac{D(S^0 - S^*)}{u(S^*)}\right)$$

  where $S^* = \lambda$ or $S^* = \mu$, solution of $g(S) = D_1$. When needed, we write E^*_λ and E^*_μ. These EPs exist if $S^0 > S^*$.

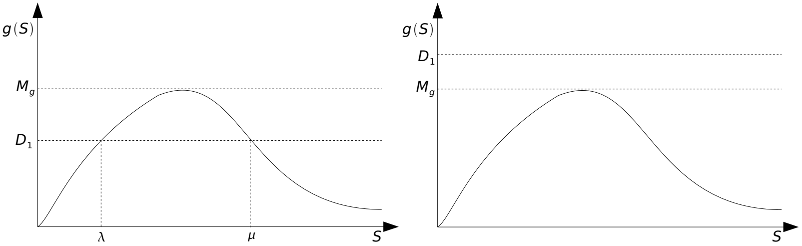
Note in particular that if λ or μ is equal to ∞, then $S^0 - S^* = -\infty$ and the E^* are not relevant.

Case of monotone growth



Left: $\mu = \infty$. Right: $\lambda = \mu = \infty$.

Case of one humped growth



Right: $\lambda = \mu = \infty$.

Stability of washout EP

Jacobian matrix at point (S, x) is

$$J = \begin{pmatrix} -D - u'(S)x & -u(S) \\ g'(S)x & g(S) - D_1 \end{pmatrix}$$

So at E_0,

$$J_{E_0} = \begin{pmatrix} -D & -u(S^0) \\ 0 & g(S^0) - D_1 \end{pmatrix}$$

Eigenvalues are $-D$ and $g(S^0) - D_1$. So local asymptotic stability depends on the sign of $g(S^0) - D_1$.

Theorem

If $g(S^0) < D_1$, then E_0 is locally asymptotically stable.
If $g(S^0) > D_1$, then E_0 is unstable.

Stability of nontrivial EP

At E^*,

$$J_{E^*} = \begin{pmatrix} -D - u'(S^*) & -u(S^*) \\ g'(S^*)x^* & 0 \end{pmatrix}$$

since $g(S^*) = D_1$.

Two ways to study stability:

- Study the characteristic polynomial using Descartes' rule of signs.
- Study the eigenvalues using some properties of eigenvalues of 2×2 matrices.

Properties of 2×2 matrices

Consider the matrix

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

The characteristic polynomial of M is

$$\begin{aligned} P(\lambda) &= (a-\lambda)(d-\lambda) - bc \\ &= \lambda^2 - (a+d)\lambda + (ad - bc) \\ &= \lambda^2 - \mathrm{tr}(M)\lambda + \det(M) \end{aligned}$$

If $\det(M) > 0$ and $\mathrm{tr}(M) < 0$, then by Descartes' rule of signs, there are no positive real roots to this polynomial. Then, computing $P(-\lambda)$, we obtain the sign pattern $+-+$, implying two or zero negative real roots.

With an additional result called the Routh-Hurwitz criterion, we obtain finally that

Theorem

The matrix M has eigenvalues with negative real parts if, and only if, $\det(M) > 0$ *and* $\mathrm{tr}(M) < 0$.

Hopf bifurcation

Theorem

Select one parameter as a bifurcation parameter, and call it α. If there exists a critical value α_c of α such that $x^*_{\alpha_c} u'(\lambda_{\alpha_c}) + D = 0$, then the system undergoes a Hopf bifurcation at $E^*_{\lambda_{\alpha_c}} = (\lambda_{\alpha_c}, x_{\alpha_c})$ if

- $g'(\lambda_{\alpha_c}) > 0$, *and*
- $\left.\dfrac{d}{d\alpha}(-Dx^*(\alpha)u'(S^*(\alpha)))\right|_{\alpha=\alpha_c} \neq 0.$

Let

$$C_H = -u(\lambda_{\alpha_c})g'(\lambda_{\alpha_c})u'''(\lambda_{\alpha_c}) \\ + u''(\lambda_{\alpha_c})(u'(\lambda_{\alpha_c})g'(\lambda_{\alpha_c}) + u(\lambda_{\alpha_c})g''(\lambda_{\alpha_c}))$$

The bifurcation is supercritical if $C_H < 0$, and subcritical if $C_H > 0$.

Let $\omega_0 = \sqrt{x^* u(S^*) g'(S^*)}$ be the imaginary part of the eigenvalue at the critical value α_c. Take

$$T = \begin{pmatrix} 0 & -1 \\ \frac{\omega_0}{u(S^*)} & 0 \end{pmatrix}$$

Then

$$\begin{pmatrix} r \\ v \end{pmatrix} = T^{-1} \begin{pmatrix} S \\ x \end{pmatrix} \Rightarrow \begin{cases} r = x \frac{u(S^*)}{\omega_0} \\ v = -S \end{cases}$$

Then in canonical form, the system is

$$\begin{aligned} r' &= r(-D_1 + g(-v)) && \equiv f(r,v) \\ v' &= -(S^0 + v)D + r\frac{\omega_0}{u(S^*)}u(-v) && \equiv g(r,v) \end{aligned}$$

Global stability

We can show

Theorem

For any $\varepsilon > 0$, there exists $T_\varepsilon \geq 0$ such that $S(t) \leq S^0 + \varepsilon$ for all $t \geq T_\varepsilon$. If in addition $\lambda < S^0$, $g(S) > D_1$ for all $S \in (\lambda, S^0]$, and $x(0) > 0$, then there exists T such that $S(t) < S^0$ for all $t > T$.

Theorem

If $S^0 \leq \lambda$, then E_0 is GAS.

A Lyapunov function

Consider the function

$$V(S,x) = \int_{\lambda}^{S} \frac{(g(\xi) - D_1)(S^0 - \lambda)}{u(\lambda)(S^0 - \xi)} d\xi + x - x^* \ln\left(\frac{x}{x^*_\lambda}\right) \qquad (7)$$

For clarity, define

$$\Psi(S) = \frac{u(S)}{S^0 - S}$$

Then

$$\begin{aligned} V' &= x(g(S) - D_1)\left(1 - \frac{u(S)(S^0 - \lambda)}{u(\lambda)(S^0 - S)}\right) \\ &= x(g(S) - D_1)\left(1 - \frac{\Psi(S)}{\Psi(\lambda)}\right) \end{aligned}$$

We have $V' = 0$ if and only if $S = \lambda$ or $S = \mu = S^0$.

Theorem

If

- $\lambda < S^0$,
- $g'(\lambda) > 0$,
- $g(\lambda) > D_1$,
- $u'(\lambda) > -\frac{u(\lambda)}{S^0-\lambda}$ *and*
- $1 - \frac{u(S)(S^0-\lambda)}{u(\lambda)(S^0-S)}$ *has exactly one sign change for* $S \in (0, S^0)$,

then the equilibrium E^*_λ *is globally asymptotically stable with respect to the interior of the positive cone.*