Ordinary differential equations
A few general results

Ordinary differential equations

Definition (ODE)

An *ordinary differential equation* (ODE) is an equation involving one independent variable (often called time), t, and a dependent variable, $x(t)$, with $x \in \mathbb{R}^n$, $n \geq 1$, and taking the form

$$\frac{d}{dt}x = f(t,x),$$

where $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is a function, called the *vector field*.

Definition (IVP)

An *initial value problem* (IVP) consists in an ODE and an initial condition,

$$\begin{aligned} \frac{d}{dt}x &= f(t,x) \\ x(t_0) &= x_0, \end{aligned} \tag{1}$$

where $t_0 \in \mathbb{R}$ and $x_0 \in \mathbb{R}^n$ is the initial condition.

Flow

Consider an autonomous IVP,

$$\begin{aligned} \frac{d}{dt}x &= f(t,x) \\ x(0) &= x_0, \end{aligned} \tag{2}$$

that is, where f does not depend explicitly on t.
Let $\phi^t(x_0)$ (the notations $\phi_t(x_0)$ and $\phi(t,x_0)$ are also used) be the solution of (2) with given initial condition. We have

$$\phi^0(x_0) = x_0$$

and

$$\frac{d}{dt}\phi^t(x_0) = f(\phi^t(x_0))$$

for all t for which it is defined.

$\phi^t(x_0)$ is the *flow* of the ODE.

Lipschitz function

Definition (Lipschitz function)

Let $f : U \subset \mathbb{R}^n \to \mathbb{R}^n$. If there exists $K > 0$ such that

$$|f(x) - f(y)| \leq K|x - y|$$

for all $x, y \in U$, then f is called a *Lipschitz* function with Lipschitz constant K. The smallest K for which the property holds is denoted $\mathrm{Lip}(f)$.

Remark: $f \in C^1 \Rightarrow f$ is Lipschitz.

Existence and uniqueness

Theorem (Existence and Uniqueness)

Let $U \subset \mathbb{R}^n$ be an open set, and $f : U \to \mathbb{R}^n$ be a Lipschitz function. Let $x_0 \in U$ and $t_0 \in \mathbb{R}$. Then there exists

- *$\alpha > 0$, and*
- *a unique solution $x(t)$ to the differential equation $x' = f(x)$ defined on $t_0 - \alpha \leq t \leq t_0 + \alpha$,*

such that $x(t_0) = x_0$.

Continuous dependence on IC

Theorem (Continuous dependence on initial conditions)

Let $U \subset \mathbb{R}^n$ be an open set, and $f : U \to \mathbb{R}^n$ be a Lipschitz function. Then the solution $\phi^t(x_0)$ depends continuously on the initial condition x_0.

Interval of existence of solutions

Theorem

Let U be an open subset of $\mathbb{R}^n$, anf $f : U \to \mathbb{R}^n$ be C^1.

- *Given $x \in U$, let (t_-, t_+) be the maximal interval of definition for $\phi^t(x)$. If $t_+ < \infty$, then given any compact subset $C \subset U$, there exists t_C with $0 \leq t_C < t_+$ such that $\phi^{t_C}(x) \notin C$.*
- *Similarly, if $t_- > -\infty$, then there exists t_{C-} with $t_- < t_{C-} \leq 0$ such that $\phi^{t_{C-}}(x) \notin C$.*
- *In particular, if $f : \mathbb{R}^n \to \mathbb{R}^n$ is defined on all of $\mathbb{R}^n$ and $|f(x)|$ is bounded, then the solutions exist for all t.*

Group property for flows

Theorem

Let (t_-, t_+) be the maximal interval of definition for the initial condition x_0. Then the flow ϕ^t satisfies the group property

$$\phi^t(\phi^s(x_0)) = \phi^{t+s}(x_0),$$

for all t, s such that $t, s, t + s \in (t_-, t_+)$.