Analysis near fixed points

First derivative

Consider the map $f : U \subset \mathbb{R}^k \to \mathbb{R}^n$, with U open subset of $\mathbb{R}^k$.

Then the derivative of f is an $n \times k$ matrix,

$$Df(p) = \left(\frac{\partial f_i}{\partial x_j}(p) \right)$$

for $i = 1, \ldots, n$ and $j = 1, \ldots, k$, at a point $p \in \mathbb{R}^k$. This is an element of $\mathcal{L}(\mathbb{R}^k, \mathbb{R}^n)$, the set of linear maps from $\mathbb{R}^k$ to $\mathbb{R}^n$.

Second derivative

Second derivative, $D^2f(p)$, is more complicated. Element of $\mathcal{L}(\mathbb{R}^k, \mathcal{L}(\mathbb{R}^k, \mathbb{R}^n))$ (or, an element of $\mathcal{L}^2(\mathbb{R}^k, \mathbb{R}^n)$, the set of bilinear maps from $\mathbb{R}^k$ to $\mathbb{R}^n$).

Let u, v be two vectors in $\mathbb{R}^k$, $u = (u_1, \ldots, u_k)$ and $v = (v_1, \ldots, v_k)$, then

$$D^2f(p)(u, v) = \sum_{i,j} \left(\frac{\partial^2 f}{\partial x_i \partial x_j} \right)(p) u_i v_j,$$

where

$$\left(\frac{\partial^2 f}{\partial x_i \partial x_j} \right)(p)$$

is a vector.

Implicit function theorem

Theorem

Let $U \subset \mathbb{R}^{n+1}$ be an open set, and $F : U \to \mathbb{R}$ be C^r ($F \in C^r(U, \mathbb{R})$), $r \geq 1$. Write $p = (x, y)$, $x \in \mathbb{R}^n$ and $y \in \mathbb{R}$. Assume $(x_0, y_0) \in U$ such that

$$\frac{\partial F}{\partial y}(x_0, y_0) \neq 0.$$

Let $C = F(x_0, y_0) \in \mathbb{R}$. Then there exists open sets $V \ni x_0$ and $W \ni y_0$ with $V \times W \subset U$, and $h : C^r(V, W)$ such that

$$h(x_0) = y_0$$
$$F(x, h(x)) = C \quad \forall x \in V.$$

Furthermore, $\forall x \in V$, $h(x)$ is the unique $y \in W$ such that $F(x, y) = C$.

Implicit function theorem for higher dimensions

Theorem

Let $U \subset \mathbb{R}^n \times \mathbb{R}^k$ be an open set, and $F : U \to \mathbb{R}^k$ be C^r ($F \in C^r(U, \mathbb{R})$), $r \geq 1$. Write $U \ni p = (x, y)$, $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^k$, and $F = (f_1, \ldots, f_k)$. Assume $(x_0, y_0) \in U$ such that

$$\left(\frac{\partial f_i}{\partial y_j}(x_0, y_0) \right)_{1 \leq i,j \leq k}$$

is an invertible $k \times k$ matrix. Let $C = F(x_0, y_0) \in \mathbb{R}^k$. Then there exists open sets $V \ni x_0$ and $W \ni y_0$ with $V \times W \subset U$, and $h : C^r(V, W)$ such that

$$h(x_0) = y_0$$
$$F(x, h(x)) = C \quad \forall x \in V.$$

Furthermore, $\forall x \in V$, $h(x)$ is the unique $y \in W$ such that $F(x, y) = C$.

Inverse function theorem

Theorem

Assume that $U \subset \mathbb{R}^n$ is an open set, and $f \in C^(U, \mathbb{R}^n)$ for $r \geq 1$. Let $x_0 \in U$. Assume that $|Df(x_0)| \neq 0$. Then there exist $V \ni x_0$ and $W \ni y_0 = f(x_0)$, and $g \in C^r(W, V)$ such that g is the inverse of f on V, i.e.,

$$g \circ f(x) = x \textit{ for } x \in V \textit{ and } f \circ g(y) = y \textit{ for } y \in W.$$

Also,

$$Dg(f(x)) = (Df(x))^{-1}.$$

Definition (Contraction mapping)

Let (X, d) be a metric space, and let $S \subset X$. A mapping $f : S \to S$ is a *contraction* on S if there exists $K < 1$ such that, for all $x, y \in S$,

$$d(f(x), f(y)) \leq Kd(x, y)$$

Every contraction is uniformly continuous on X.

Theorem (Contraction mapping principle)

Consider the complete metric space (X, d). Every contraction mapping $f : X \to X$ has one and only one $x \in X$ such that $f(x) = x$.

Objective

Consider the autonomous nonlinear system in $\mathbb{R}^n$

$$x' = f(x) \tag{1}$$

The object here is to show two results which link the behavior of (1) near a hyperbolic equilibrium point x^* to the behavior of the linearized system

$$x' = Df(x^*)(x - x^*) \tag{2}$$

about that same equilibrium.

Homeomorphism

Definition (Homeomorphism)

Let X be a metric space and let A and B be subsets of X. A *homeomorphism* $h : A \to B$ of A onto B is a continuous one-to-one map of A onto B such that $h^{-1} : B \to A$ is continuous. The sets A and B are called *homeomorphic* or *topologically equivalent* if there is a homeomorphism of A onto B.

Differentiable manifold

Definition (Differentiable manifold)

An n-dimensional *differentiable manifold* M (or a manifold of class C^k) is a connected metric space with an open covering $\{U_\alpha\}$ (*i.e.*, $M = \cup_\alpha U_\alpha$) such that

1. for all α, U_α is homeomorphic to the open unit ball in $\mathbb{R}^n$, $B = \{x \in \mathbb{R}^n : \ |x| < 1\}$, *i.e.*, for all α there exists a homeomorphism of U_α onto B, $h_\alpha : U_\alpha \to B$,
2. if $U_\alpha \cap U_\beta \neq \emptyset$ and $h_\alpha : U_\alpha \to B$, $h_\beta : U_\beta \to B$ are homeomorphisms, then $h_\alpha(U_\alpha \cap U_\beta)$ and $h_\beta(U_\alpha \cap U_\beta)$ are subsets of $\mathbb{R}^n$ and the map

$$h = h_\alpha \circ h_\beta^{-1} : \ h_\beta(U_\alpha \cap U_\beta) \to h_\alpha(U_\alpha \cap U_\beta)$$

   is differentiable (or of class C^k) and for all $x \in h_\beta(U_\alpha \cap U_\beta)$, the determinant of the Jacobian, $\det Dh(x) \neq 0$.

Stable manifold theorem

Theorem (Stable manifold theorem)

Let E be an open subset of $\mathbb{R}^n$ containing the origin, let $f \in C^1(E)$, and let ϕ_t be the flow of the nonlinear system (1). Suppose that $f(0) = 0$ and that $Df(0)$ has k eigenvalues with negative real part and $n - k$ eigenvalues with positive real part. Then there exists a k-dimensional differentiable manifold S tangent to the stable subspace E^s of the linear system (2) at 0 such that for all $t \geq 0$, $\phi_t(S) \subset S$ and for all $x_0 \in S$,

$$\lim_{t \to \infty} \phi_t(x_0) = 0$$

and there exists an $(n - k)$-dimensional differentiable manifold U tangent to the unstable subspace E^u of (2) at 0 such that for all $t \leq 0$, $\phi_t(U) \subset U$ and for all $x_0 \in U$,

$$\lim_{t \to -\infty} \phi_t(x_0) = 0$$

HG theorem – Formulation 1

Theorem (Hartman-Grobman)

Suppose that 0 is an equilibrium point of the nonlinear system (1). Let φ_t *be the flow of (1), and* ψ_t *be the flow of the linearized system* $x' = Df(0)x$*. If 0 is a hyperbolic equilibrium, then there exists an open subset* $\mathcal{D}$ *of* $\mathbb{R}^n$ *containing 0, and a homeomorphism* G *with domain in* $\mathcal{D}$ *such that* $G(\varphi_t(x)) = \psi_t(G(x))$ *whenever* $x \in \mathcal{D}$ *and both sides of the equation are defined.*

HG theorem – Formulation 2

Theorem (Hartman-Grobman)

Let E be an open subset of $\mathbb{R}^n$ containing the origin, let $f \in C^1(E)$, and let ϕ_t be the flow of the nonlinear system (1). Suppose that $f(0) = 0$ and that the matrix $A = Df(0)$ has no eigenvalue with zero real part.
Then there exists a homeomorphism H of an open set U containing the origin onto an open set V containing the origin such that for each $x_0 \in U$, there is an open interval $\mathcal{I}_0 \subset \mathbb{R}$ containing 0 such that for all $x_0 \in U$ and $t \in \mathcal{I}_0$,

$$H \circ \phi_t(x_0) = e^{At} H(x_0);$$

i.e., H maps trajectories of (1) near the origin onto trajectories of $x' = Df(0)x$ near the origin and preserves the parametrization by time.

Lyapunov function

We consider $x' = f(x)$, $x \in \mathbb{R}^n$, with flow $\phi_t(x)$. Let p be a fixed point.

Definition (Weak Lyapunov function)

The function $V \in C^1(U, \mathbb{R})$ is a *weak Lyapunov function* for ϕ_t on the open neighborhood $U \ni p$ if $V(x) > V(p)$ and $\frac{d}{dt}V(\phi_t(x)) \leq 0$ for all $x \in U \setminus \{p\}$.

Definition (Lyapunov function)

The function $V \in C^1(U, \mathbb{R})$ is a (*strong*) *Lyapunov function* for ϕ_t on the open neighborhood $U \ni p$ if $V(x) > V(p)$ and $\frac{d}{dt}V(\phi_t(x)) < 0$ for all $x \in U \setminus \{p\}$.

Theorem

Suppose that p is a fixed point of $x' = f(x)$, U is a neighborhood of p, and $V : U \to \mathbb{R}$.

1. *If V is a weak Lyapunov function for ϕ_t on U, then p is Liapunov stable.*
2. *If V is a Lyapunov function for ϕ_t on U, then p is asymptotically stable.*

Periodic orbits for flows

Definition (Periodic point)

Let $x' = f(x)$, and $\phi_t(x)$ be the associated flow. p is a *periodic point* with (*least*) *period* T, or T-periodic point, if $\phi_T(p) = p$ and $\phi_t(p) \neq p$ for $0 < t < T$.

Definition (Periodic orbit)

If p is a T-periodic point, then

$$\mathcal{O}(p) = \{\phi_t(p) : 0 \leq t \leq T\}$$

is the orbit of p, called a *periodic orbit* or a *closed orbit*.

Definition (Stable periodic orbit)

A periodic orbit γ is *stable* if for each $\varepsilon > 0$, there exists a neighborhood U of γ such that for all $x \in U$, $d(\gamma_x^+, \gamma) < \varepsilon$, i.e., if for all $x \in U$ and $t \geq 0$, $d(\phi_t(x), \gamma) < \varepsilon$.

Definition (Unstable periodic orbit)

A periodic orbit that is not stable is *unstable*.

Definition (Asymptotically stable periodic orbit)

A periodic orbit γ is *asymptotically stable* if it is stable and for all x in some neighborhood U of γ,

$$\lim_{t \to \infty} d(\phi_t(x), \gamma) = 0.$$

Hyperbolic periodic orbits

Definition (Characteristic multipliers)

If γ is a periodic orbit of period T, with $p \in \gamma$, then the eigenvalues of $D\phi_T(p)$ are $1, \lambda_1, \ldots, \lambda_{n-1}$. The eigenvalues $\lambda_1, \ldots, \lambda_{n-1}$ are called the *characteristic multipliers* of the periodic orbit.

Definition (Hyperbolic periodic orbit)

A periodic orbit is *hyperbolic* if none of the characteristic multipliers has modulus 1.

Definition (Periodic sink)

A periodic orbit which has all characteristic multipliers λ such that $|\lambda| < 1$.

Definition (Periodic source)

A periodic orbit which has all characteristic multipliers λ such that $|\lambda| > 1$.

Theorem

1. *If $\phi_t(x)$ is a solution of $x' = f(x)$, γ is a periodic orbit of period T, and $p \in \gamma$, then $D\phi_T(p)$ has 1 as an eigenvalue with eigenvector $f(p)$.*
2. *If p and q belong to the same T-periodic orbit γ, then $D\phi_T(p)$ and $D\phi_T(q)$ are linearly conjugate and thus have the same eigenvalues.*

Example

$$
\begin{aligned}
x' &= -y + x(1 - x^2 - y^2) \\
y' &= x + y(1 - x^2 - y^2) \\
z' &= z
\end{aligned}
$$

First, look at

$$
\begin{aligned}
x' &= -y + x(1 - x^2 - y^2) \\
y' &= x + y(1 - x^2 - y^2)
\end{aligned}
$$

In polar coordinates, if $x = r\cos\theta$ and $y = r\sin\theta$, then $r = \pm\sqrt{x^2+y^2}$ and $\theta = \arctan(y/x)$. So

$$\frac{d}{dt}r(t) = \frac{xx' + yy'}{\sqrt{x^2+y^2}}$$

and

$$\frac{d}{dt}\theta(t) = \frac{\frac{y'}{x} - \frac{x'y}{x^2}}{1 + \frac{y^2}{x^2}} = \frac{xy' - x'y}{x^2+y^2}$$

System is

$$
\begin{aligned}
x' &= -y + x(1 - x^2 - y^2) \\
y' &= x + y(1 - x^2 - y^2),
\end{aligned}
$$

so in polar coordinates,

$$
\begin{aligned}
r' &= \frac{xx' + yy'}{\sqrt{x^2 + y^2}} \\
&= \frac{x(-y + x(1 - x^2 - y^2)) + y(x + y(1 - x^2 - y^2))}{\sqrt{x^2 + y^2}} \\
&= \frac{(1 - x^2 - y^2)(x^2 + y^2)}{\sqrt{x^2 + y^2}} \\
&= (1 - x^2 - y^2)\sqrt{x^2 + y^2} \\
&= (1 - r^2)r
\end{aligned}
$$

$$
\begin{aligned}
\theta' &= \frac{xy' - x'y}{x^2 + y^2} \\
&= \frac{x(x + y(1 - x^2 - y^2)) - (-y + x(1 - x^2 - y^2))y}{x^2 + y^2} \\
&= \frac{x^2 + y^2 + (xy - xy)(1 - x^2 - y^2)}{x^2 + y^2} \\
&= 1,
\end{aligned}
$$

so, in polar coordinates, the system is

$$
\begin{aligned}
r' &= r(1 - r^2) \\
\theta' &= 1
\end{aligned}
$$