Sets

Definition
A *set* is a collection of elements.

- Two sets are equal if they contain exactly the same elements.
- A is a subset of B ($A \subset B$) if all the elements of A also belong to B.
- If A, B are two sets, A is a *proper subset* of B if $A \subset B$ and $A \neq B$ (sometimes, the notation $A \subsetneq B$ is used).

Definition
Let X and Y be two sets. Then the set

$$X \cup Y = \{x : x \in X \text{ or } x \in Y\}$$

is called the *union* of X and Y, and the set

$$X \cap Y = \{x : x \in X \text{ and } x \in Y\}$$

is the *intersection* of X and Y. If $X \subset Y$, then the complement of X in Y is given by

$$Y \setminus X = X^c = \{x : x \in Y, x \notin X\}.$$

De Morgan's laws

Theorem
Let $\{X_\alpha\}$ be a collection of subsets of X. Then

$$\left(\bigcup_{\alpha\in\Lambda} X_\alpha\right)^c = \bigcap_{\alpha\in\Lambda} (X_\alpha)^c$$

and

$$\left(\bigcap_{\alpha\in\Lambda} X_\alpha\right)^c = \bigcup_{\alpha\in\Lambda} (X_\alpha)^c.$$

Mappings

Definition
Let M and N be two arbitrary sets. A rule associating a unique element $b = f(a) \in N$ with each element $a \in M$ defines a *function* f on M (usually called a *mapping* of M into N).
If $a \in M$, $b = f(a) \in N$ is the *image* of a (under f). Every element of M with a given element $b \in N$ as its image is called a *preimage* of b.

- b may have several preimages;
- N may contain elements with no preimage;
- if b has a unique preimage, it is denoted $f^{-1}(b)$.

For sets: Let $A \subset X$, $B \subset Y$. Then the *image* of A and *inverse image* (or preimage) of B are, respectively,

$$f(A) = \{f(x) : x \in A\}, \quad \text{and} \quad f^{-1}(B) = \{x : f(x) \in B\}.$$

Definition

Let $f : X \to Y$.

- If $f(X) = Y$, f maps X *onto* Y (or, f is *surjective*).
- If for each $y \in Y$, $f^{-1}(y)$ consists of at most one element, then f is one-to-one (or, f is *injective*). If f is injective, then f^{-1} is a function with domain $f(X)$ and range X.
- If f is both injective and surjective, it is *bijective*.

The following properties hold, and can be extended to unions and intersections of collections of sets:

- $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$.
- $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$.
- $f(A \cup B) = f(A) \cup f(B)$.
- But $f(A \cap B) \subset (f(A) \cap f(B))$.

Distances, metric spaces

Definition

A *metric space* is a pair (X, d) consisting of a set X and a *distance* (or *metric*) d, i.e., a scalar valued real function $d(x, y)$ defined for all $x, y \in X$, satisfying the following properties:

1. $d(x, y) \geq 0$ for all $x, y \in X$,
2. $d(x, y) = 0$ if and only if $x = y$,
3. $d(x, y) = d(y, x)$ (symmetry),
4. $d(x, z) \leq d(x, y) + d(y, z)$ (triangle inequality).

Some useful inequalities

Let x_k, y_k for $k = 1, \dots, n$.

Theorem (Hölder's inequality)

Suppose $p, q > 1$ such that $1/p + 1/q = 1$. Then

$$\sum_{k=1}^{n} |x_k y_k| \leq \left(\sum_{k=1}^{n} |x_k|^p \right)^{1/p} \left(\sum_{k=1}^{n} |y_k|^q \right)^{1/q}.$$

Theorem (Cauchy-Schwarz inequality)

Case $p = q = 2$ of Hölder is the Cauchy-Schwarz inequality, also written

$$\left(\sum_{k=1}^{n} x_k y_k \right)^2 \leq \sum_{k=1}^{n} x_k^2 \sum_{k=1}^{n} y_k^2.$$

Theorem (Minkowski's inequality)

Suppose $p \geq 1$. Then

$$\left(\sum_{k=1}^{n} |x_k + y_k|^p \right)^{1/p} \leq \left(\sum_{k=1}^{n} |x_k|^p \right)^{1/p} + \left(\sum_{k=1}^{n} |y_k|^p \right)^{1/p}.$$

Works in the case of infinite sums ($n = \infty$), giving convergence of the sum on the LHS, provided $\sum_{k=1}^{\infty} x_k^p$ and $\sum_{k=1}^{\infty} y_k^p$ converge.

Examples of metric spaces

- The set $\mathbb{R}$ of real numbers with distance $d(x, y) = |x - y|$;
- The set of all ordered n-tuples $x = (x_1, \dots, x_n)$ with distance

$$d(x, y) = \sqrt{\sum_{k=1}^{n} (x_k - y_k)^2}$$

  is a metric space, denoted $\mathbb{R}^n$ and called the Euclidean n-space.
- Replacing the previous distance with $d(x, y) = \sum_{k=1}^{n} |x_k - y_k|$ gives the metric space $\mathbb{R}^n_1$.
- The set $C([a, b])$ of all continuous functions defined on the closed interval $[a, b]$ with distance

$$d(f, g) = \max_{a \le t \le b} |f(t) - g(t)|$$

  is a metric space.

Continuity

Definition

Let $f : (X, d_X) \to (Y, d_Y)$ be a mapping; f is continuous at $x_0 \in X$ if, for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$(d_X(x, x_0) < \delta \text{ and } x \in X) \Rightarrow d_Y(f(x), f(x_0)) < \varepsilon.$$

f is continuous on X if it is continuous at every point $x_0 \in X$.

Open balls, closed balls

Definition

An *open ball* with center x_0 and radius r in a metric space X is the set of points $x \in X$ that satisfy $d(x, x_0) < r$. It is also called an open (or ε-) neighborhood of x_0, and denoted $O_\varepsilon(x_0)$. A *closed ball* with center x_0 and radius r in a metric space X is the set of points $x \in X$ that satisfy $d(x, x_0) \le r$.

Definition

A point $x \in X$ is a *contact point* of the set $M \subset X$ if every neighborhood of x contains at least one point of M. The set of all contact points of a set M is denoted $\bar{M}$ or $[M]$ and is called the *closure* of M.

Properties of the closure and of closed sets

Denote [ ] the closure operator, and let M, N be subsets of a metric space (X, d).

- $M \subset [M]$.
- If $M \subset N$, then $[M] \subset [N]$.
- $[[M]] = [M]$.
- $[M \cup N] = [M] \cup [N]$.
- $[\emptyset] = \emptyset$.
- A set M is closed if $[M] = M$.

Theorem

The intersection of an arbitrary number of closed sets is closed. The union of a **finite** *number of closed sets is closed.*

Examples of closed sets

- Let (X, d) be a metric space. Then X and $\emptyset$ are closed sets.
- Every closed interval $[a, b]$ on the real line is a closed set.
- Every closed ball in a metric space is closed. For example, the set of all functions $f \in C([a, b])$ such that $|f(t)| \leq K$ (K a constant) is closed.
- Any set consisting of a finite number of points is closed.

Interior points, open sets

Definition

A point x is an *interior point* in a set $M \subset X$ if x has a neighborhood $O_\varepsilon(x) \subset M$, i.e., a neighborhood consisting entirely of points of M. The set of all interior points of $M \subset X$ is called the *interior* of M, and is denoted $Int(M)$ or M°. A set M is *open* if $M = M^\circ$.

Theorem

A set $M \subset X$ is open if and only if its complement $X \setminus M$ is closed.

Theorem

The union of an arbitrary number of open sets is open. The intersection of a **finite** *number of open sets is open.*

Examples of open sets

- Let (X, d) be a metric space. Then X and $\emptyset$ are open sets.
- Every open interval (a, b) on the real line is an open set.
- Every open ball in a metric space is open.

Cauchy sequences

Definition (Cauchy criterion)

A sequence $\{x_n\}$ of points in a metric space (X, d) satisfies the *Cauchy criterion* if, for any $\varepsilon > 0$, there exists N_ε such that

$$d(x_p, x_q) < \varepsilon \text{ for all } p, q > N_\varepsilon.$$

Definition (Cauchy sequence)

A sequence $\{x_n\}$ of points in a metric space (X, d) is a *Cauchy sequence* if it satisfies the Cauchy criterion.

Theorem

Every convergent sequence is a Cauchy sequence.

Complete metric spaces

Definition (Complete metric space)

A metric space (X, d) is *complete* if every Cauchy sequence in X converges to an element of X. Otherwise, X is *incomplete*.

Examples of complete metric spaces

- $(\mathbb{R}, d)$ with $d(x, y) = |x - y|$.
- $(\mathbb{C}, d)$ with $d(z, w) = |z - w|$.
- $(\mathbb{R}^n, d_p)$, with $d_p(x, y) = \left(\sum_{k=1}^{n}(x_k - y_k)^p\right)^{1/p}$, $p \geq 1$.
- $(\mathbb{R}^n, d_\infty)$, with $d_\infty(x, y) = \max\limits_{1 \leq k \leq n}\{|x_k - y_k|\}$.
- The space (X, d) of all bounded sequences with $d(x, y) = \sup\limits_{k \geq 1}\{|x_k - y_k|\}$.
- The space $\mathcal{B}(S)$ of all real-valued bounded functions f on S, with uniform metric $d(f, g) = \sup\limits_{x \in S}\{|f(x) - g(x)|\}$.
- The space $(C([a, b]), d)$ with $d(f, g) = \sup\limits_{a \leq x \leq b}\{|f(x) - g(x)|\}$.

Connectedness

Definition

A metric space (X, d) is *disconnected* if there exists two empty subsets of X, A and B, such that

1. $X = A \cup B$.
2. $A \cap [B] = \emptyset$ and $[A] \cap B = \emptyset$.

If no such subsets exist, then X is *connected*.

Theorem

Let Y be a subset of the metric space (X, d). If Y is a compact subset of X, then Y is closed and bounded.

Theorem

Let (X, d) be a compact metric space. Then (X, d) is complete.

Proposition

Let (X, d) be a metric space. The following statements are equivalent:

1. *every infinite set in (X, d) has at least one limit point in X;*
2. *every infinite sequence in (X, d) contains a convergent subsequence.*

Theorem

The metric space (X, d) is compact if and only if every sequence of points in X has a subsequence that converges to a point in X.