Summary of topics relevant for the final

Outline

Outline of this part

First-order difference equation

A difference equation takes the form

$$x(n+1) = f(x(n)),$$

which is also denoted

$$x_{n+1} = f(x_n).$$

Starting from an initial point x_0, we have

$$\begin{aligned}
x_1 &= f(x_0) \\
x_2 &= f(x_1) = f(f(x_0)) = f^2(x_0) \\
x_3 &= f(x_2) = f(f(f(x_0))) = f^3(x_0) \\
&\dots
\end{aligned}$$

Definition 1 (Iterates)

$f(x_0)$ is the *first iterate* of x_0 under f; $f^2(x_0)$ is the *second iterate* of x_0 under f. More generally, $f^n(x_0)$ is the nth iterate of x_0 under f. By convention, $f^0(x_0) = x_0$.

Definition 2 (Orbits)

The set

$$\{f^n(x_0) : n \geq 0\}$$

is called the *forward orbit* of x_0 and is denoted $O^+(x_0)$. The *backward* orbit $O^-(x_0)$ is defined, if f is invertible, by the negative iterates of f. Lastly, the (*whole*) orbit of x_0 is

$$\{f^k(x_0) : -\infty < k < \infty\}.$$

The forward orbit is also called the *positive* orbit. The function f is always assumed to be continuous. If its derivative or second derivative is used in a result, then the assumption is made that $f \in C^1$ or $f \in C^2$..

Periodic points

Definition 3 (Periodic point)

A point p is a *periodic point* of (*least*) *period* n if

$$f^n(p) = p \quad \text{and} \quad f^j(p) \neq p \text{ for } 0 < j < n.$$

Definition 4 (Fixed point)

A periodic point with period $n = 1$ is called a *fixed point*.

Definition 5 (Eventually periodic point)

A point p is an eventually periodic point of period n if there exists $m > 0$ such that

$$f^{m+n}(p) = f^m(p),$$

so that $f^{j+n}(p) = f^j(p)$ for all $j \geq m$ and $f^m(p)$ is a periodic point.

Finding fixed points and periodic points

- A fixed point is such that $f(x) = x$, so it lies at the intersection of the first bisectrix $y = x$ with the graph of $f(x)$.
- A periodic point is such that $f^n(x) = x$, it is thus a fixed point of the nth iterate of f, and so lies at the intersection of the first bisectrix $y = x$ with the graph of $f^n(x)$.

Stable set

Definition 6 (Forward asymptotic point)

q is *forward asymptotic* to p if

$$|f^j(q) - f^j(p)| \to 0 \text{ as } j \to \infty.$$

If p is n-periodic, then q is asymptotic to p if

$$|f^{jn}(q) - p| \to 0 \text{ as } j \to \infty.$$

Definition 7 (Stable set)

The *stable set* of p is

$$W^s(p) = \{q : q \text{ forward asymptotic to } p\}.$$

Unstable set

Definition 8 (Backward asymptotic point)

If f is invertible, then q is *backward asymptotic* to p if

$$|f^j(q) - f^j(p)| \to 0 \text{ as } j \to -\infty.$$

Definition 9 (Unstable set)

The *unstable set* of p is

$$W^u(p) = \{q : q \text{ backward asymptotic to } p\}.$$

Stability

Definition 10 (Stable fixed point)

A fixed point p is *stable* (or Lyapunov stable) if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that $|x_0 - p| < \delta$ implies $|f^n(x_0) - p| < \varepsilon$ for all $n > 0$. If a fixed point p is not stable, then it is *unstable*.

Definition 11 (Attracting fixed point)

A fixed point p is *attracting* if there exists $\eta > 0$ such that

$$|x(0) - p| < \eta \quad \text{implies} \quad \lim_{n\to\infty} x(n) = p.$$

If $\eta = \infty$, then p is a *global attractor* (or is *globally attracting*).

Definition 12 (Asymptotically stable point)

A fixed point p is *asymptotically stable* if it is stable and attracting. It is *globally asymptotically stable* if $\eta = \infty$.

The point does not have to be a fixed point to be stable.

Definition 13

A point p is stable if for every $\varepsilon > 0$, there exists $\delta > 0$ such that if $|x - p| < \delta$, then $|f^k(x) - f^k(p)| < \varepsilon$ for all $k \geq 0$.

Another characterization of asymptotic stability:

Definition 14

A point p is asymptotically stable if it is stable and $W^s(p)$ contains a neighborhood of p.

Can be used with periodic point, in which case we talk of *attracting periodic point* (or *periodic sink).* A periodic point p for which $W^u(p)$ is a neighborhood of p is a *repelling periodic point* (or *periodic source*).

Condition for stability/instability

Theorem 15

Let $f : \mathbb{R} \to \mathbb{R}$ be C^1.

1. *If p is a n-periodic point of f such that $|(f^n)'(p)| < 1$, then p is an attracting periodic point.*
2. *If p is a n-periodic point of f such that $|(f^n)'(p)| > 1$, then p is repelling.*

ω-limit points and sets

Definition 16
A point y is an ω-limit point of x for f is there exists a sequence $\{n_k\}$ going to infinity as $k \to \infty$ such that

$$\lim_{k\to\infty} d(f^{n_k}(x), y) = 0.$$

The set of all ω-limit points of x is the ω-limit set of x and is denoted $\omega(x)$.

α-limit points and sets

Definition 17
Suppose that f is invertible. A point y is an α-limit point of x for f is there exists a sequence $\{n_k\}$ going to minus infinity as $k \to \infty$ such that

$$\lim_{k\to\infty} d(f^{n_k}(x), y) = 0.$$

The set of all α-limit points of x is the α-limit set of x and is denoted $\alpha(x)$.

Invariant sets

Definition 18
Let $S \subset X$ be a set. S is *positively invariant* (under the flow of f) if $f(x) \in S$ for all $x \in S$, i.e., $f(S) \subset S$. S is *negatively invariant* if $f^{-1}(S) \subset S$. S is *invariant* if $f(S) = S$.

Theorem 19
Let $f : X \to X$ be continuous on a complete metric space X. Then

1. *If $f^j(x) = y$ for some j, then $\omega(x) = \omega(y)$.*
2. *For any x, $\omega(x)$ is closed and positively invariant.*
3. *If $O^+(x)$ is contained in some compact subset of X, then $\omega(x)$ is nonempty and compact and $d(f^n(x), \omega(x)) \to 0$ as $n \to \infty$.*
4. *If $D \subset X$ is closed and positively invariant, and $x \in D$, then $\omega(x) \subset D$.*
5. *If $y \in \omega(x)$, then $\omega(y) \subset \omega(x)$.*

Scalar difference equations

First-order difference equations

Periodic points

Stability

Limit sets

The cascade of bifurcations to chaos

Chaos – Devaney's definition

Parametrized families of functions

Consider the logistic map

$$x_{t+1} = \mu x_t(1 - x_t), \qquad (1)$$

where μ is a parameter in $\mathbb{R}_+$, and x will typically be taken in $[0, 1]$. Let

$$f_\mu(x) = \mu x(1 - x). \qquad (2)$$

The function f_μ is called a *parametrized family* of functions.

Bifurcations

Definition 20 (Bifurcation)

Let f_μ be a parametrized family of functions. Then there is a *bifurcation* at $\mu = \mu_0$ (or μ_0 is a bifurcation point) if there exists $\varepsilon > 0$ such that, if $\mu_0 - \varepsilon < a < \mu_0$ and $\mu_0 < b < \mu_0 + \varepsilon$, then the dynamics of $f_a(x)$ are "different" from the dynamics of $f_b(x)$.

An example of "different" would be that f_a has a fixed point (that is, a 1-periodic point) and f_b has a 2-periodic point.

Formally, f_a and f_b are *topologically conjugate* to two different functions.

Topological conjugacy

Definition 21 (Topological conjugacy)

Let $f : D \to D$ and $g : E \to E$ be functions. Then f *topologically conjugate* to g if there exists a homeomorphism $\tau : D \to E$, called a *topological conjugacy*, such that $\tau \circ f = g \circ \tau$.

Theorem 22
Let D and E be subsets of $\mathbb{R}$, $f : D \to D$, $g : E \to E$, and $\tau : D \to E$ be a topological conjugacy of f and g. Then

1. *$\tau^{-1} : E \to D$ is a topological conjugacy.*
2. *$\tau \circ f^n = g^n \circ \tau$ for all $n \in \mathbb{N}$.*
3. *p is a periodic point of f with least period n iff $\tau(p)$ is a periodic point of g with least period n.*
4. *If p is a periodic point of f with stable set $W^s(p)$, then the stable set of $\tau(p)$ is $\tau(W^s(p))$.*
5. *The periodic points of f are dense in D iff the periodic points of g are dense in E.*
6. *f is topologically transitive on D iff g is topologically transitive on E.*
7. *f is chaotic on D iff g is chaotic on E.*

Topologically transitive function

Definition 23
The function $f : D \to D$ is *topologically transitive* on D if for any open sets U and V that interset D, there exists $z \in U \cap D$ and $n \in \mathbb{N}$ such that $f^n(z) \in D$.
Equivalently, f is topologically transitive on D if for any two points $x, y \in D$ and any $\varepsilon > 0$, there exists $z \in D$ such that $|z - x| < \varepsilon$ and $|f^n(x) - y| < \varepsilon$ for some n.

Sensitive dependence on initial conditions

Definition 24
The function $f : D \to D$ exhibits *sensitive dependence on initial conditions* if there exists $\delta > 0$ such that for any $x \in D$ and any $\varepsilon > 0$, there exists $y \in D$ and $n \in \mathbb{N}$ such that $|x - y| < \varepsilon$ and $|f^n(x) - f^n(y)| > \delta$.

Chaos

The following in due to Devaney. There are other definitions.

Definition 25

The function $f : D \to D$ is *chaotic* if

1. the periodic points of f are dense in D,
2. f is topologically transitive,
3. and f exhibits sensitive dependence on initial conditions.

Outline of this part

General theory of ODEs

- ODEs
- Existence of solutions to IVPs

General theory of ODEs

- ODEs
- Existence of solutions to IVPs

Ordinary differential equations

Definition 26 (ODE)

An *ordinary differential equation* (ODE) is an equation involving one independent variable (often called time), t, and a dependent variable, $x(t)$, with $x \in \mathbb{R}^n$, $n \geq 1$, and taking the form

$$\frac{d}{dt}x = f(t, x),$$

where $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is a function, called the *vector field*.

Definition 27 (IVP)

An *initial value problem* (IVP) consists in an ODE and an initial condition,

$$\begin{aligned} \frac{d}{dt}x &= f(t, x) \\ x(t_0) &= x_0, \end{aligned} \tag{3}$$

where $t_0 \in \mathbb{R}$ and $x_0 \in \mathbb{R}^n$ is the initial condition.

General theory of ODEs

ODEs

Existence of solutions to IVPs

Flow

Consider an autonomous IVP,

$$\begin{aligned} \frac{d}{dt}x &= f(t,x) \\ x(0) &= x_0, \end{aligned} \tag{4}$$

that is, where f does not depend explicitly on t.
Let $\phi^t(x_0)$ (the notations $\phi_t(x_0)$ and $\phi(t,x_0)$ are also used) be the solution of (4) with given initial condition. We have

$$\phi^0(x_0) = x_0$$

and

$$\frac{d}{dt}\phi^t(x_0) = f(\phi^t(x_0))$$

for all t for which it is defined.

$\phi^t(x_0)$ is the *flow* of the ODE.

Lipschitz function

Definition 28 (Lipschitz function)

Let $f : U \subset \mathbb{R}^n \to \mathbb{R}^n$. If there exists $K > 0$ such that

$$|f(x) - f(y)| \leq K|x-y|$$

for all $x, y \in U$, then f is called a *Lipschitz* function with Lipschitz constant K. The smallest K for which the property holds is denoted $\mathrm{Lip}(f)$.

Remark: $f \in C^1 \Rightarrow f$ is Lipschitz.

Existence and uniqueness

Theorem 29 (Existence and Uniqueness)

Let $U \subset \mathbb{R}^n$ be an open set, and $f : U \to \mathbb{R}^n$ be a Lipschitz function. Let $x_0 \in U$ and $t_0 \in \mathbb{R}$. Then there exists

- $\alpha > 0$*, and*
- *a unique solution $x(t)$ to the differential equation $x' = f(x)$ defined on $t_0 - \alpha \leq t \leq t_0 + \alpha$,*

such that $x(t_0) = x_0$.

Continuous dependence on IC

Theorem 30 (Continuous dependence on initial conditions)

Let $U \subset \mathbb{R}^n$ be an open set, and $f : U \to \mathbb{R}^n$ be a Lipschitz function. Then the solution $\phi^t(x_0)$ depends continuously on the initial condition x_0.

Interval of existence of solutions

Theorem 31

Let U be an open subset of $\mathbb{R}^n$, anf $f : U \to \mathbb{R}^n$ be C^1.

- *Given $x \in U$, let (t_-, t_+) be the maximal interval of definition for $\phi^t(x)$. If $t_+ < \infty$, then given any compact subset $C \subset U$, there exists t_C with $0 \leq t_C < t_+$ such that $\phi^{t_C}(x) \notin C$.*
- *Similarly, if $t_- > -\infty$, then there exists t_{C-} with $t_- < t_{C-} \leq 0$ such that $\phi^{t_{C-}}(x) \notin C$.*
- *In particular, if $f : \mathbb{R}^n \to \mathbb{R}^n$ is defined on all of $\mathbb{R}^n$ and $|f(x)|$ is bounded, then the solutions exist for all t.*

Outline of this part

Linear ODEs

Definition 32 (Linear ODE)

A *linear* ODE is a differential equation taking the form

$$\frac{d}{dt}x = A(t)x + B(t), \tag{LNH}$$

where $A(t) \in \mathcal{M}_n(\mathbb{R})$ with continuous entries, $B(t) \in \mathbb{R}^n$ with real valued, continuous coefficients, and $x \in \mathbb{R}^n$. The associated IVP takes the form

$$\begin{aligned} \frac{d}{dt}x &= A(t)x + B(t) \\ x(t_0) &= x_0. \end{aligned} \tag{5}$$

Types of systems

- $x' = A(t)x + B(t)$ is linear nonautonomous ($A(t)$ depends on t) nonhomogeneous (also called *affine* system).
- $x' = A(t)x$ is linear nonautonomous homogeneous.
- $x' = Ax + B$, that is, $A(t) \equiv A$ and $B(t) \equiv B$, is linear autonomous nonhomogeneous (or affine autonomous).
- $x' = Ax$ is linear autonomous homogeneous.

- If $A(t + T) = A(t)$ for some $T > 0$ and all t, then linear periodic.

Existence and uniqueness of solutions

Theorem 33 (Existence and Uniqueness)
Solutions to (5) *exist and are unique on the whole interval over which A and B are continuous.*
In particular, if A, B are constant, then solutions exist on $\mathbb{R}$.

The vector space of solutions

Theorem 34
Consider the homogeneous system

$$\frac{d}{dt}x = A(t)x, \tag{LH}$$

with $A(t)$ defined and continuous on an interval J. The set of solutions of (LH) *forms an n-dimensional vector space.*

Fundamental matrix

Definition 35

A set of n linearly independent solutions of (LH) on J, $\{\phi_1, \dots, \phi_n\}$, is called a fundamental set of solutions of (LH) and the matrix

$$\Phi = [\phi_1 \; \phi_2 \; \dots \; \phi_n]$$

is called a fundamental matrix of (LH).

Fundamental matrix solution

Let $X \in \mathcal{M}_n(\mathbb{R})$ with entries $[x_{ij}]$. Define the derivative of X, X' (or $\frac{d}{dt}X$) as

$$\frac{d}{dt}X(t) = [\frac{d}{dt}x_{ij}(t)].$$

The system of n^2 equations

$$\frac{d}{dt}X = A(t)X$$

is called a *matrix differential equation.*

Theorem 36

A fundamental matrix Φ of (LH) satisfies the matrix equation $X' = A(t)X$ on the interval J.-

Abel's formula

Theorem 37

If Φ is a solution of the matrix equation $X' = A(t)X$ on an interval J and $\tau \in J$, then

$$det\Phi(t) = det\Phi(\tau)\exp\left(\int_\tau^t trA(s)ds\right)$$

for all $t \in J$.

The resolvent matrix

Definition 38 (Resolvent matrix)

Let $t_0 \in J$ and $\Phi(t)$ be a fundamental matrix solution of (LH) on J. Since the columns of Φ are linearly independent, it follows that $\Phi(t_0)$ is invertible. The *resolvent* (or *state transition matrix*, or *principal fundamental matrix*) of (LH) is then defined as

$$\mathcal{R}(t,t_0) = \Phi(t)\Phi(t_0)^{-1}.$$

Proposition 1

The resolvent matrix satisfies the identities

1. $\mathcal{R}(t,t) = I$,
2. $\mathcal{R}(t,s)\mathcal{R}(s,u) = \mathcal{R}(t,u)$,
3. $\mathcal{R}(t,s)^{-1} = \mathcal{R}(s,t)$,
4. $\frac{\partial}{\partial s}\mathcal{R}(t,s) = -\mathcal{R}(t,s)A(s)$,
5. $\frac{\partial}{\partial t}\mathcal{R}(t,s) = A(t)\mathcal{R}(t,s)$.

Proposition 2

$\mathcal{R}(t,t_0)$ is the only solution in $\mathcal{M}_n(\mathbb{K})$ of the initial value problem

$$\frac{d}{dt}M(t) = A(t)M(t)$$
$$M(t_0) = \mathbb{I},$$

with $M(t) \in \mathcal{M}_n(\mathbb{K})$.

Theorem 39

The solution to the IVP consisting of the linear homogeneous nonautonomous system (LH) *with initial condition $x(t_0) = x_0$ is given by*

$$\phi(t) = \mathcal{R}(t,t_0)x_0.$$

A variation of constants formula

Theorem 40 (Variation of constants formula)

Consider the IVP

$$x' = A(t)x + g(t,x) \tag{6a}$$
$$x(t_0) = x_0, \tag{6b}$$

where $g : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ a smooth function, and let $\mathcal{R}(t,t_0)$ be the resolvent associated to the homogeneous system $x' = A(t)x$, with $\mathcal{R}$ defined on some interval $J \ni t_0$. Then the solution ϕ of (6) *is given by*

$$\phi(t) = \mathcal{R}(t,t_0)x_0 + \int_{t_0}^{t} \mathcal{R}(t,s)g(\phi(s),s)ds, \tag{7}$$

on some subinterval of J.

Linear ODEs

Existence of solutions to linear IVPs

Resolvent matrix

Autonomous linear systems

Nonautonomous nonhomogeneous linear equations

Autonomous linear systems

Consider the autonomous affine system

$$\frac{d}{dt}x = Ax + B, \tag{A}$$

and the associated homogeneous autonomous system

$$\frac{d}{dt}x = Ax. \tag{L}$$

Exponential of a matrix

Definition 41 (Matrix exponential)

Let $A \in \mathcal{M}_n(\mathbb{K})$ with $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. The *exponential* of A, denoted e^{At}, is a matrix in $\mathcal{M}_n(\mathbb{K})$, defined by

$$e^{At} = \mathbb{I} + \sum_{k=1}^{\infty} \frac{t^k}{k!} A^k,$$

where $\mathbb{I}$ is the identity matrix in $\mathcal{M}_n(\mathbb{K})$.

Properties of the matrix exponential

- $\Phi(t) = e^{At}$ is a fundamental matrix for (L) for $t \in \mathbb{R}$.
- The resolvent for (L) is given for $t \in J$ by

$$\mathcal{R}(t, t_0) = e^{A(t-t_0)} = \Phi(t - t_0).$$

- $e^{At_1} e^{At_2} = e^{A(t_1+t_2)}$ for all $t_1, t_2 \in \mathbb{R}$. 1
- $Ae^{At} = e^{At}A$ for all $t \in \mathbb{R}$.
- $(e^{At})^{-1} = e^{-At}$ for all $t \in \mathbb{R}$.
- The unique solution ϕ of (L) with $\phi(t_0) = x_0$ is given by

$$\phi(t) = e^{A(t-t_0)} x_0.$$

Computing the matrix exponential

Let P be a nonsingular matrix in $\mathcal{M}_n(\mathbb{R})$. We transform the IVP

$$\begin{aligned}\frac{d}{dt}x &= Ax \\ x(t_0) &= x_0\end{aligned} \tag{L\_IVP}$$

using the transformation $x = Py$ or $y = P^{-1}x$.

The dynamics of y is

$$\begin{aligned}y' &= (P^{-1}x)' \\ &= P^{-1}x' \\ &= P^{-1}Ax \\ &= P^{-1}APy\end{aligned}$$

The initial condition is $y_0 = P^{-1}x_0$.

We have thus transformed IVP (L_IVP) into

$$\begin{aligned}\frac{d}{dt}y &= P^{-1}APy \\ y(t_0) &= P^{-1}x_0\end{aligned} \tag{L\_IVP\_y}$$

From the earlier result, we then know that the solution of (L_IVP_y) is given by

$$\psi(t) = e^{P^{-1}AP(t-t_0)}P^{-1}x_0,$$

and since $x = Py$, the solution to (L_IVP) is given by

$$\phi(t) = Pe^{P^{-1}AP(t-t_0)}P^{-1}x_0.$$

So everything depends on $P^{-1}AP$.

Diagonalizable case

Assume P nonsingular in $\mathcal{M}_n(\mathbb{R})$ such that

$$P^{-1}AP = \begin{pmatrix}\lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n\end{pmatrix}$$

with all eigenvalues $\lambda_1, \ldots, \lambda_n$ different. We have

$$e^{P^{-1}AP} = \mathbb{I} + \sum_{k=1}^{\infty} \frac{t^k}{k!} \begin{pmatrix}\lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n\end{pmatrix}^k$$

For a (block) diagonal matrix M of the form

$$M = \begin{pmatrix}m_{11} & & 0 \\ & \ddots & \\ 0 & & m_{nn}\end{pmatrix}$$

there holds

$$M^k = \begin{pmatrix}m_{11}^k & & 0 \\ & \ddots & \\ 0 & & m_{nn}^k\end{pmatrix}$$

Therefore,

$$e^{P^{-1}AP} = \begin{pmatrix} e^{\lambda_1 t} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n t} \end{pmatrix}$$

And so the solution to (L_IVP) is given by

$$\phi(t) = P \begin{pmatrix} e^{\lambda_1 t} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n t} \end{pmatrix} P^{-1} x_0.$$

Nondiagonalizable case

The Jordan canonical form is

$$P^{-1}AP = \begin{pmatrix} J_0 & & 0 \\ & \ddots & \\ 0 & & J_s \end{pmatrix}$$

so we use the same property as before (but with block matrices now), and

$$e^{P^{-1}APt} = \begin{pmatrix} e^{J_0 t} & & 0 \\ & \ddots & \\ 0 & & e^{J_s t} \end{pmatrix}$$

The first block in the Jordan canonical form takes the form

$$J_0 = \begin{pmatrix} \lambda_0 & & 0 \\ & \ddots & \\ 0 & & \lambda_k \end{pmatrix}$$

and thus, as before,

$$e^{J_0 t} = \begin{pmatrix} e^{\lambda_0 t} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_k t} \end{pmatrix}$$

Other blocks J_i are written as

$$J_i = \lambda_{k+i}\mathbb{I} + N_i$$

with $\mathbb{I}$ the $n_i \times n_i$ identity and N_i the $n_i \times n_i$ nilpotent matrix

$$N_i = \begin{pmatrix} 0 & 1 & 0 & & 0 \\ & & \ddots & & \\ & & & & 1 \\ 0 & & & & 0 \end{pmatrix}$$

$\lambda_{k+i}\mathbb{I}$ and N_i commute, and thus

$$e^{J_i t} = e^{\lambda_{k+i} t} e^{N_i t}$$

Since N_i is nilpotent, $N_i^k = 0$ for all $k \geq n_i$, and the series $e^{N_i t}$ terminates, and

$$e^{J_i t} = e^{\lambda_{k+i} t} \begin{pmatrix} 1 & t & \cdots & \frac{t^{n_i-1}}{(n_i-1)!} \\ 0 & 1 & \cdots & \frac{t^{n_i-2}}{(n_i-2)!} \\ & & & \\ 0 & & & 1 \end{pmatrix}$$

Fixed points (equilibria)

Definition 42

A *fixed point* (or *equilibrium point*, or *critical point*) of an autonomous differential equation

$$x' = f(x)$$

is a point p such that $f(p) = 0$. For a nonautonomous differential equation

$$x' = f(t, x),$$

a fixed point satisfies $f(t, p) = 0$ for all t.

A fixed point is a solution.

Orbits, limit sets

Orbits and limit sets are defined as for maps.

For the equation $x' = f(x)$, the subset $\{x(t), t \in I\}$, where I is the maximal interval of existence of the solution, is an *orbit*.

If the maximal solution $x(t, x_0)$ of $x' = f(x)$ is defined for all $t \geq 0$, where f is Lipschitz on an open subset V of $\mathbb{R}^n$, then the omega limit set of x_0 is the subset of V defined by

$$\omega(x_0) = \bigcap_{\tau=0}^{\infty} \left(\overline{\{x(t, x_0) : t \geq \tau\}} \cap V \right).$$

Proposition 3

A point q is in $\omega(x_0)$ iff there exists a sequence $\{t_k\}$ such that $\lim_{k\to\infty} t_k = \infty$ and $\lim_{k\to\infty} x(t_k, x_0) = q \in V$.

Definition 43 (Liapunov stable orbit)

The orbit of a point p is *Liapunov stable* for a flow ϕ_t if, given $\varepsilon > 0$, there exists $\delta > 0$ such that $d(x, p) < \delta$ implies that $d(\phi_t(x), \phi_t(p)) < \varepsilon$ for all $t \geq 0$. If p is a fixed point, then this is written $d(\phi_t(x), p) < \varepsilon$.

Definition 44 (Asymptotically stable orbit)

The orbit of a point p is *asymptotically stable* (or *attracting*) for a flow ϕ_t if it is Liapunov stable, and there exists $\delta_1 > 0$ such that $d(x, p) < \delta_1$ implies that $\lim_{t\to\infty} d(\phi_t(x), \phi_t(p)) = 0$. If p is a fixed point, then it is asymptotically stable if it is Liapunov stable and there exists $\delta_1 > 0$ such that $d(x, p) < \delta_1$ implies that $\omega(x) = \{p\}$.

Contracting linear equation

Theorem 45
Let $A \in \mathcal{M}_n(\mathbb{R})$, and consider the equation (L). *Then the following conditions are equivalent.*

1. *There is a norm $\| \|_A$ on $\mathbb{R}^n$ and a constant $a > 0$ such that for any $x_0 \in \mathbb{R}^n$ and all $t \geq 0$,*
$$\|e^{At}x_0\|_A \leq e^{-at}\|x_0\|_A.$$
2. *There is a norm $\| \|_B$ on $\mathbb{R}^n$ and constants $a > 0$ and $C \geq 1$ such that for any $x_0 \in \mathbb{R}^n$ and all $t \geq 0$,*
$$\|e^{At}x_0\|_B \leq Ce^{-at}\|x_0\|_B.$$
3. *All eigenvalues of A have negative real parts.*

In that case, the origin is a *sink* or *attracting*, the flow is a *contraction* (antonyms *source*, *repelling* and *expansion*).

Hyperbolic linear equation

Definition 46
The linear differential equation (L) is *hyperbolic* if A has no eigenvalue with zero real part.

Definition 47 (Stable eigenspace)
The *stable eigenspace* of $A \in \mathcal{M}_n(\mathbb{R})$ is

$$E^s = \text{span}\{v : v \text{ generalized eigenvector for eigenvalue } \lambda, \\ \text{with } \Re(\lambda) < 0\}$$

Definition 48 (Center eigenspace)
The *center eigenspace* of $A \in \mathcal{M}_n(\mathbb{R})$ is

$$E^c = \text{span}\{v : v \text{ generalized eigenvector for eigenvalue } \lambda, \\ \text{with } \Re(\lambda) = 0\}$$

Definition 49 (Unstable eigenspace)
The *unstable eigenspace* of $A \in \mathcal{M}_n(\mathbb{R})$ is

$$E^u = \text{span}\{v : v \text{ generalized eigenvector for eigenvalue } \lambda, \\ \text{with } \Re(\lambda) > 0\}$$

We can write
$$\mathbb{R}^n = E^s \oplus E^u \oplus +E^c,$$
and in the case that $E^c =$, then $\mathbb{R}^n = E^s \oplus E^u$ is called a *hyperbolic splitting*.

The symbol $\oplus$ stands for *direct sum*.

Definition 50 (Direct sum)
Let U, V be two subspaces of a vector space X. Then the span of U and V is defined by $u + v$ for $u \in U$ and $v \in V$. If U and V are disjoint except for 0, then the span of U and V is called the *direct sum* of U and V, and is denoted $U \oplus V$.

Trichotomy

Define

$$
\begin{aligned}
V^s &= \{v : \text{there exists } a > 0 \text{ and } C \geq 1 \text{ such that} \\
&\qquad \|e^{At}v\| \leq Ce^{-at}\|v\| \text{ for } t \geq 0\}. \\
V^u &= \{v : \text{there exists } a > 0 \text{ and } C \geq 1 \text{ such that} \\
&\qquad \|e^{At}v\| \leq Ce^{-a|t|}\|v\| \text{ for } t \leq 0\}. \\
V^c &= \{v : \text{for all } a > 0, \|e^{At}v\|e^{-a|t|} \to 0 \text{ as } t \to \pm\infty\}.
\end{aligned}
$$

Theorem 51

The following are true.

1. *The subspaces E^s, E^u and E^c are invariant under the flow e^{At}.*
2. *There holds that $E^s = V^s$, $E^u = V^u$ and $E^c = V^c$, and thus $e^{At}|_{E^u}$ is an exponential expansion, $e^{At}|_{E^s}$ is an exponential contraction, and $e^{At}|_{E^c}$ grows subexponentially as $t \to \pm\infty$.*

Topologically conjugate linear ODEs

Definition 52 (Topologically conjugate flows)

Let ϕ_t and ψ_t be two flows on a space M. ϕ_t and ψ_t are *topologically conjugate* if there exists an homeomorphism $h : M \to M$ such that

$$h \circ \phi_t(x) = \psi_t \circ h(x),$$

for all $x \in M$ and all $t \in \mathbb{R}$.

Definition 53 (Topologically equivalent flows)

Let ϕ_t and ψ_t be two flows on a space M. ϕ_t and ψ_t are *topologically equivalent* if there exists an homeomorphism $h : M \to M$ and a function $\alpha : \mathbb{R} \times M \to \mathbb{R}$ such that

$$h \circ \phi_{\alpha(t+s,x)}(x) = \psi_t \circ h(x),$$

for all $x \in M$ and all $t \in \mathbb{R}$, and where $\alpha(t,x)$ is monotonically increasing in t for each x and onto all of $\mathbb{R}$.

Theorem 54

Let $A, B \in \mathcal{M}_n(\mathbb{R})$.

1. *If all eigenvalues of A and B have negative real parts, then the linear flows e^{At} and e^{Bt} are topologically conjugate.*
2. *Assume that the system is hyperbolic, and that the dimension of the stable eigenspace of A is equal to the dimension of the eigenspace of B. Then the linear flows e^{At} and e^{Bt} are topologically conjugate.*

Theorem 55

Let $A, B \in \mathcal{M}_n(\mathbb{R})$. Assume that e^{At} and e^{Bt} are linearly conjugate, i.e., there exists M with $e^{Bt} = Me^{At}M^{-1}$. Then A and B have the same eigenvalues.

Theorem 56

Consider

$$x' = A(t)x + g(t) \tag{LNH}$$

and

$$x' = A(t)x \tag{LH}$$

1. *If* x_1 *and* x_2 *are two solutions of* (LNH)*, then* $x_1 - x_2$ *is a solution to* (LH)*.*
2. *If* x_n *is a solution to* (LNH) *and* x_h *is a solution to* (LH)*, then* $x_n + x_h$ *is a solution to* (LNH)*.*
3. *If* x_n *is a solution to* (LNH) *and* M *is a fundamental matrix solution of* (LH)*, then any solution of* (LNH) *can be written as* $x_n + M(t)v$*.*

Outline of this part

Linear maps

Similarities between ODEs and maps

Let $A \in \mathcal{M}_n(\mathbb{R})$. Let v be an eigenvector associated to the eigenvalue λ.

Then

$$\begin{aligned} A^2 v &= A(Av) \\ &= A(\lambda v) \\ &= \lambda A v \\ &= \lambda^2 v \end{aligned}$$

By induction,

$$A^n v = \lambda^n v,$$

i.e., v is an eigenvector of the matrix A^n, associated to the eigenvalue λ^n. Thus, if $|\lambda| < 1$, then $\|A^n v\| = |\lambda|^n \|v\|$ goes to zero as $n \to \infty$.

Linear map corresponding to a matrix with all eigenvalues of modulus less than 1 is a *linear contraction*, with the origin a *linear sink* or *attracting fixed point*. If all eigenvalues have modulus larger than 1, then the map induced by A is a *linear expansion*, and the origin is a *linear source* or *repelling fixed point*.

The map Ax is a *hyperbolic linear map* if all eigenvalues of A have modulus different of 1.

Definition 57 (Stable eigenspace)

The *stable eigenspace* of $A \in \mathcal{M}_n(\mathbb{R})$ is

$$E^s = \text{span}\{v : v \text{ generalized eigenvector for eigenvalue } \lambda,\\ \text{with } |\lambda| < 1\}$$

Definition 58 (Center eigenspace)

The *center eigenspace* of $A \in \mathcal{M}_n(\mathbb{R})$ is

$$E^c = \text{span}\{v : v \text{ generalized eigenvector for eigenvalue } \lambda,\\ \text{with } |\lambda| = 1\}$$

Definition 59 (Unstable eigenspace)

The *unstable eigenspace* of $A \in \mathcal{M}_n(\mathbb{R})$ is

$$E^u = \text{span}\{v : v \text{ generalized eigenvector for eigenvalue } \lambda,\\ \text{with } |\lambda| > 1\}$$

Outline of this part

Objective

Consider the autonomous nonlinear system in $\mathbb{R}^n$

$$x' = f(x) \tag{8}$$

The object here is to show two results which link the behavior of (8) near a hyperbolic equilibrium point x^* to the behavior of the linearized system

$$x' = Df(x^*)(x - x^*) \tag{9}$$

about that same equilibrium.

Stable manifold theorem

Theorem 60 (Stable manifold theorem)

Let $f \in C^1(E)$, E be an open subset of $\mathbb{R}^n$ containing a point x^* such that $f(x^*) = 0$, and let ϕ_t be the flow of the nonlinear system (8). Suppose that $Df(x^*)$ has k eigenvalues with negative real part and $n-k$ eigenvalues with positive real part. Then there exists a k-dimensional differentiable manifold S tangent to the stable subspace E^s of the linear system (9) at x^* such that for all $t \geq 0$, $\phi_t(S) \subset S$ and for all $x_0 \in S$,

$$\lim_{t\to\infty} \phi_t(x_0) = x^*$$

and there exists an $(n-k)$-dimensional differentiable manifold U tangent to the unstable subspace E^u of (9) at x^* such that for all $t \leq 0$, $\phi_t(U) \subset U$ and for all $x_0 \in U$,

$$\lim_{t\to-\infty} \phi_t(x_0) = x^*$$

HG theorem – Formulation 1

Theorem 61 (Hartman-Grobman)

Suppose that x^* is an equilibrium point of the nonlinear system (8). Let φ_t be the flow of (8), and ψ_t be the flow of the linearized system $x' = Df(x^*)(x - x^*)$. If x^* is a hyperbolic equilibrium, then there exists an open subset $\mathcal{D}$ of $\mathbb{R}^n$ containing x^*, and a homeomorphism G with domain in $\mathcal{D}$ such that $G(\varphi_t(x)) = \psi_t(G(x))$ whenever $x \in \mathcal{D}$ and both sides of the equation are defined.

HG theorem – Formulation 2

Theorem 62 (Hartman-Grobman)

Let $f \in C^1(E)$, E an open subset of $\mathbb{R}^n$ containing x^* where $f(x^*) = 0$, and let ϕ_t be the flow of the nonlinear system (8). Suppose that the matrix $A = Df(x^*)$ has no eigenvalue with zero real part.
Then there exists a homeomorphism H of an open set U containing x^* onto an open set V containing the origin such that for each $x_0 \in U$, there is an open interval $\mathcal{I}_0 \subset \mathbb{R}$ containing x^* such that for all $x_0 \in U$ and $t \in \mathcal{I}_0$,

$$H \circ \phi_t(x_0) = e^{At} H(x_0);$$

i.e., H maps trajectories of (8) near the origin onto trajectories of $x' = Df(x^*)(x - x^*)$ near the origin and preserves the parametrization by time.

Analysis near fixed points (linearization)

Stable manifold theorem

Hartman-Grobman theorem

Lyapunov functions

Periodic orbits

Lyapunov function

We consider $x' = f(x)$, $x \in \mathbb{R}^n$, with flow $\phi_t(x)$. Let p be a fixed point.

Definition 63 (Weak Lyapunov function)

The function $V \in C^1(U, \mathbb{R})$ is a *weak Lyapunov function* for ϕ_t on the open neighborhood $U \ni p$ if $V(x) > V(p)$ and $\frac{d}{dt}V(\phi_t(x)) \leq 0$ for all $x \in U \setminus \{p\}$.

Definition 64 (Lyapunov function)

The function $V \in C^1(U, \mathbb{R})$ is a (*strong*) *Lyapunov function* for ϕ_t on the open neighborhood $U \ni p$ if $V(x) > V(p)$ and $\frac{d}{dt}V(\phi_t(x)) < 0$ for all $x \in U \setminus \{p\}$.

Theorem 65

Suppose that p is a fixed point of $x' = f(x)$, U is a neighborhood of p, and $V : U \to \mathbb{R}$.

1. *If V is a weak Lyapunov function for ϕ_t on U, then p is Liapunov stable.*
2. *If V is a Lyapunov function for ϕ_t on U, then p is asymptotically stable.*

Analysis near fixed points (linearization)

Stable manifold theorem

Hartman-Grobman theorem

Lyapunov functions

Periodic orbits

Periodic orbits for flows

Definition 66 (Periodic point)

Let $x' = f(x)$, and $\phi_t(x)$ be the associated flow. p is a *periodic point* with (*least*) *period* T, or T-periodic point, if $\phi_T(p) = p$ and $\phi_t(p) \neq p$ for $0 < t < T$.

Definition 67 (Periodic orbit)

If p is a T-periodic point, then

$$\mathcal{O}(p) = \{\phi_t(p) : 0 \leq t \leq T\}$$

is the orbit of p, called a *periodic orbit* or a *closed orbit*.

Definition 68 (Stable periodic orbit)

A periodic orbit γ is *stable* if for each $\varepsilon > 0$, there exists a neighborhood U of γ such that for all $x \in U$, $d(\gamma_x^+, \gamma) < \varepsilon$, i.e., if for all $x \in U$ and $t \geq 0$, $d(\phi_t(x), \gamma) < \varepsilon$.

Definition 69 (Unstable periodic orbit)

A periodic orbit that is not stable is *unstable*.

Definition 70 (Asymptotically stable periodic orbit)

A periodic orbit γ is *asymptotically stable* if it is stable and for all x in some neighborhood U of γ,

$$\lim_{t\to\infty} d(\phi_t(x), \gamma) = 0.$$

Hyperbolic periodic orbits

Definition 71 (Characteristic multipliers)

If γ is a periodic orbit of period T, with $p \in \gamma$, then the eigenvalues of the Poincaré map $D\phi_T(p)$ are $1, \lambda_1, \ldots, \lambda_{n-1}$. The eigenvalues $\lambda_1, \ldots, \lambda_{n-1}$ are called the *characteristic multipliers* of the periodic orbit.

Definition 72 (Hyperbolic periodic orbit)

A periodic orbit is *hyperbolic* if none of the characteristic multipliers has modulus 1.

Definition 73 (Periodic sink)

A periodic orbit which has all characteristic multipliers λ such that $|\lambda| < 1$.

Definition 74 (Periodic source)

A periodic orbit which has all characteristic multipliers λ such that $|\lambda| > 1$.

Theorem 75

1. *If $\phi_t(x)$ is a solution of $x' = f(x)$, γ is a periodic orbit of period T, and $p \in \gamma$, then $D\phi_T(p)$ has 1 as an eigenvalue with eigenvector $f(p)$.*
2. *If p and q belong to the same T-periodic orbit γ, then $D\phi_T(p)$ and $D\phi_T(q)$ are linearly conjugate and thus have the same eigenvalues.*

Outline of this part

The general context of bifurcations

Consider the discrete time system

$$x_{t+1} = f(x_t, \mu) = f_\mu(x_t) \tag{10}$$

or the continuous time system

$$x' = f(x, \mu) = f_\mu(x) \tag{11}$$

for $\mu \in \mathbb{R}$. We start with a function $f : \mathbb{R}^2 \to \mathbb{R}$, C^r when a map is considered, C^1 when continuous time is considered.

In both cases, the function f can depend on some parameters. We are interested in the differences of qualitative behavior, as one of these parameters, which we call μ, varies.

Types of bifurcations (discrete time)

Saddle-node (or tangent):

$$x_{t+1} = \mu + x_t + x_t^2$$

Transcritical:

$$x_{t+1} = (\mu + 1)x_t + x_t^2$$

Pitchfork:

$$x_{t+1} = (\mu + 1)x_t - x_t^3$$

Types of bifurcations (continuous time)

- Saddle-node
$$x' = \mu - x^2$$
- Transcritical
$$x' = \mu x - x^2$$
- Pitchfork
  - supercritical
$$x' = \mu x - x^3$$
  - subcritical
$$x' = \mu x + x^3$$

Saddle-node for maps

Theorem 76

Assume $f \in C^r$ with $r \geq 2$, for both x and μ. Suppose that

1. $f(x_0, \mu_0) = x_0$,
2. $f'_{\mu_0}(x_0) = 1$,
3. $f''_{\mu_0}(x_0) \neq 0$ *and*
4. $\dfrac{\partial f}{\partial \mu}(x_0, \mu_0) \neq 0$.

Then $\exists I \ni x_0$ and $N \ni \mu_0$, and $m \in C^r(I, N)$, such that

1. $f_{m(x)}(x) = x$,
2. $m(x_0) = \mu_0$,
3. *the graph of m gives all the fixed points in $I \times N$.*

Theorem 77 (cont.)

Moreover, $m'(x_0) = 0$ and

$$m''(x_0) = \frac{-\dfrac{\partial^2 f}{\partial x^2}(x_0,\mu_0)}{\dfrac{\partial f}{\partial \mu}(x_0,\mu_0)} \neq 0.$$

These fixed points are attracting on one side of x_0 and repelling on the other.

Saddle-node for continuous equations

Consider the system $x' = f(x,\mu)$, $x \in \mathbb{R}$. Suppose that $f(x_0,\mu_0) = 0$. Further, assume that the following nondegeneracy conditions hold:

1. $a_0 = \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(x_0,\mu_0) \neq 0$,
2. $\frac{\partial f}{\partial \mu}(x_0,\mu_0) \neq 0$.

Then, in a neighborhood of (x_0,μ_0), the equation $x' = f(x,\mu)$ is topologically equivalent to the normal form

$$x' = \gamma + \mathrm{sign}(a_0)x^2$$

Saddle-node for continuous systems

Theorem 78

Consider the system $x' = f(x,\mu)$, $x \in \mathbb{R}^n$. Suppose that $f(x,0) = x_0 = 0$. Further, assume that

1. *The Jacobian matrix $A_0 = Df(0,0)$ has a simple zero eigenvalue,*
2. *$a_0 \neq 0$, where*

$$a_0 = \frac{1}{2}\langle p, B(q,q)\rangle = \frac{1}{2}\frac{d^2}{d\tau^2}\langle p, f(\tau q, 0)\rangle\Big|_{\tau=0}$$

3. *$f_\mu(0,0) \neq 0$.*

B is the bilinear function with components

$$B_j(x,y) = \sum_{k,\ell=1}^{n} \frac{\partial^2 f_j(\xi,0)}{\partial \xi_k \partial \xi_\ell}\Big|_{\xi=0} x_k y_\ell, \quad j = 1,\ldots,n$$

and $\langle p,q\rangle = p^T q$ the standard inner product.

Theorem 79 (cont.)

Then, in a neighborhood of the origin, the system $x' = f(x,\mu)$ is topologically equivalent to the suspension of the normal form by the standard saddle,

$$\begin{aligned} y' &= \gamma + sign(a_0)y^2 \\ y_S' &= -y_S \\ y_U' &= y_U \end{aligned}$$

with $y \in \mathbb{R}$, $y_S \in \mathbb{R}^{n_S}$ and $y_U \in \mathbb{R}^{n_U}$, where $n_S + n_U + 1 = n$ and n_S is number of eigenvalues of A_0 with negative real parts.

Bifurcations

General context

A few types of bifurcations

Saddle-node

Pitchfork

Hopf

Pitchfork bifurcation

The ODE $x' = f(x, \mu)$, with the function $f(x, \mu)$ satisfying

$$-f(x, \mu) = f(-x, \mu)$$

(f is odd),

$$\frac{\partial f}{\partial x}(0, \mu_0) = 0, \frac{\partial^2 f}{\partial x^2}(0, \mu_0) = 0, \frac{\partial^3 f}{\partial x^3}(0, \mu_0) \neq 0,$$
$$\frac{\partial f}{\partial r}(0, \mu_0) = 0, \frac{\partial^2 f}{\partial r \partial x}(0, \mu_0) \neq 0.$$

has a pitchfork bifurcation at $(x, \mu) = (0, \mu_0)$. The form of the pitchfork is determined by the sign of the third derivative:

$$\frac{\partial^3 f}{\partial x^3}(0, \mu_0) \begin{cases} < 0, & \text{supercritical} \\ > 0, & \text{subcritical} \end{cases}$$

Bifurcations

General context

A few types of bifurcations

Saddle-node

Pitchfork

Hopf

Canonical example

Consider the system

$$\begin{aligned} x' &= -y + x(\mu - x^2 - y^2) \\ y' &= x + y(\mu - x^2 - y^2) \end{aligned}$$

Transform to polar coordinates:

$$\begin{aligned} r' &= r(\mu - r^2) \\ \theta' &= 1 \end{aligned}$$

Hopf bifurcation

Theorem 80 (Hopf bifurcation theorem)

Let $x' = A(\mu)x + F(\mu, x)$ be a C^k planar vector field, with $k \geq 0$, depending on the scalar parameter μ such that $F(\mu, 0) = 0$ and $D_x F(\mu, 0) = 0$ for all μ sufficiently close enough to the origin. Assume that the linear part $A(\mu)$ at the origin has the eigenvalue $\alpha(\mu) \pm i\beta(\mu)$, with $\alpha(0) = 0$ and $\beta(0) \neq 0$. Furthermore, assume the eigenvalues cross the imaginary axis with nonzero speed, i.e.,

$$\frac{d}{d\mu}\alpha(\mu)\Big|_{\mu=0} \neq 0.$$

Then, in any neighborhood $\mathcal{U} \ni (0,0)$ in $\mathbb{R}^2$ and any given $\mu_0 > 0$, there exists a $\bar{\mu}$ with $|\bar{\mu}| < \mu_0$ such that the differential equation $x' = A(\bar{\mu})x + F(\bar{\mu}, x)$ has a nontrivial periodic orbit in $\mathcal{U}$.

Supercritical or subcritical Hopf?

Transform the system into

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \alpha(\mu) & \beta(\mu) \\ -\beta(\mu) & \alpha(\mu) \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} f_1(x,y,\mu) \\ g_1(x,y,\mu) \end{pmatrix} = \begin{pmatrix} f(x,y,\mu) \\ g(x,y,\mu) \end{pmatrix}$$

The Jacobian at the origin is

$$J(\mu) = \begin{pmatrix} \alpha(\mu) & \beta(\mu) \\ -\beta(\mu) & \alpha(\mu) \end{pmatrix}$$

and thus eigenvalues are $\alpha(\mu) \pm i\beta(\mu)$, and $\alpha(0) = 0$ and $\beta(0) > 0$.

Supercritical or subcritical Hopf? (cont.)

Define

$$\begin{aligned} C &= f_{xxx} + f_{xyy} + g_{xxy} + g_{yyy} \\ &\quad + \frac{1}{\beta(0)}\left(-f_{xy}\left(f_{xx} + f_{yy}\right) + g_{xy}\left(g_{xx} + g_{yy}\right) + f_{xx}g_{xx} - f_{yy}g_{yy}\right), \end{aligned}$$

evaluated at $(0,0)$ and for $\mu = 0$. Then, if $d\alpha(0)/d\mu > 0$,

1. If $C < 0$, then for $\mu < 0$, the origin is a stable spiral, and for $\mu > 0$, there exists a stable periodic solution and the origin is unstable (**supercritical Hopf**).
2. If $C > 0$, then for $\mu < 0$, there exists an unstable periodic solution and the origin is unstable, and for $\mu > 0$, the origin is unstable (**subcritical Hopf**).
3. If $C = 0$, the test is inconclusive.

Outline of this part

How to analyze a system

Position of the problem

You are given an autonomous system, whether in discrete time

$$x_{t+1} = f(x_t) \tag{12}$$

or in continuous time

$$x' = f(x) \tag{13}$$

with $x \in \mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}^n$ a C^k function ($k \geq 2$ for (12) or $k \geq 1$ for (13)).

What do you do now?

Battle plan

1. If you can solve explicitly (12) or (13), solve it explicitly.
2. If not (99% of the time, in real life), plan B:
   2.1 Determine invariants.
   2.2 Determine equilibria.
   2.3 Study (local) stability of the equilibria.
   2.4 Seek Lyapunov functions for global stability.
   2.5 Study bifurcations that occur equilibria lose stability.
   2.6 Use numerical techniques (not relevant for the final, though).

Explicit solutions

It happens.. so infrequently with nonlinear systems that most of the times, you will overlook the possibility.

If a nonlinear system is integrable explicitly, it is often linked to the presence of invariants, that allow to reduce the dimension (typically, 2d to 1d).

In case of linear systems, solutions can be found explicitly (they can be complicated, or can be in an implicit form).

Look for invariants

If the system lives on a hyperplane, which is characterized by

$$\sum_i x_i(t) \equiv C \in \mathbb{R}$$

or

$$\sum_i x_i' = 0$$

then its dimension can be reduced, since one of the variables, say x_i, can be expressed as $C - \sum_{j \neq i} x_i$.

The same can be true with subparts of the system, if for example some variables always appear as sums in the remaining equations.

Study local stability

Compute the Jacobian matrix, and evaluate it at the equilibria (fixed points).

If DTE, the fixed point is locally asymptotically stable if all eigenvalues have modulus less than 1, repelling (unstable) otherwise.

If ODE, the fixed point is locally asymptotically stable if all eigenvalues have negative real parts, unstable otherwise.

Seek Lyapunov functions

In your case, if you need to use a Lyapunov function, it will be provided..

Be sure to know how to differentiate the function, it is not always simple..

Study bifurcations

It can be a good way to figure out what is happening..

Also, sometimes checking for a bifurcation can give you information about the stability of the equilibrium, without having to do the stability analysis.

Outline of this part

Some matrix properties

Perron-Frobenius theorem

Routh-Hurwitz criterion

Nonnegative matrices

Definition 81 (Nonnegative matrix)

Let $A = (a_{ij}) \in \mathcal{M}_n(\mathbb{R})$. A is *nonnegative* iff $\forall i, j$, $a_{ij} \geq 0$.

Definition 82 (Positive matrix)

A is *positive* iff $a_{ij} > 0$ for all $i, j = 1, \ldots, n$.

Definition 83 (Irreducible matrix)

A is *irreducible* iff for all i, j, there exists $q \in \mathbb{N}$ such that $a_{ij}^q > 0$. If A is not irreducible, it is *reducible*, and there is a permutation matrix P such that A is written in block triangular form,

$$P^{-1}AP = \begin{pmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{pmatrix}$$

Definition 84 (Primitive matrix)

A is *primitive* iff there exists $q \in \mathbb{N}$ such that $\forall i, j$, $a_{ij}^q > 0$.

Perron-Frobenius theorem

Theorem 85

Let $A \in \mathcal{M}_n(\mathbb{R})$ be primitive.

1. *There exists an eigenvalue λ_1, real and positive, that is a simple, and such that any other eigenvalue λ verifies $|\lambda| < \lambda_1$. To this eigenvalue, there corresponds a strongly positive eigenvector, i.e., with all entries positive, and all other (left and right) eigenvectors of A have components of both signs.*

Some matrix properties

Perron-Frobenius theorem

Routh-Hurwitz criterion

Properties of 2×2 matrices

Consider the matrix

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

The characteristic polynomial of M is

$$\begin{aligned} P(\lambda) &= (a-\lambda)(d-\lambda) - bc \\ &= \lambda^2 - (a+d)\lambda + (ad - bc) \\ &= \lambda^2 - \mathrm{tr}(M)\lambda + \det(M) \end{aligned}$$

Theorem 86

The matrix M has eigenvalues with negative real parts if, and only if, $det(M) > 0$ and $tr(M) < 0$.