Delay differential equations

ODEs

In an ODE, the evolution at time t depends on the current state at time t and possibly on t:

$$\frac{d}{dt}x(t) = f(t, x(t))$$

This is analogous to Markov property, where

> *a stochastic process has the Markov property if the conditional probability distribution of future states of the process depends only upon the present state, not on the sequence of events that preceded it*

No dependence on the past?

So an ODE does cannot directly account for such things as

- gestation
- incubation
- life history
- duration of events
- anything that happened in the past but is important for the current evolution of the system

DDE

One way to overcome this limitation: delay differential equations (DDE)

Simplest DDE: at time t, evolution of the system depends on t, current state of the system and state of the system some time $\tau > 0$ in the past

$$\frac{d}{dt}x(t) = f(t, x(t), x(t-\tau))$$

More complicated: at time t, evolution depends on t, current state of the system and state the system was in for the last τ units of time

$$\frac{d}{dt}x(t) = f\left(t, x(t), \int_{t-\tau}^{t} \phi(x(s))ds\right)$$

where ϕ is some function

Initial data for DDE

For an ODE

$$\frac{d}{dt}x(t) = f(t, x(t))$$

an IVP is obtained by providing an **initial conditional**

$$x(t_0) = x_0$$

For a simple DDE

$$\frac{d}{dt}x(t) = f(t, x(t), x(t-\tau))$$

we must provide **initial data** on $[t_0 - \tau, t_0]$

$$\forall t \in [t_0 - \tau, t_0], \quad x(t) = \phi(t)$$

where ϕ is some function

A very simple example

Let us use techniques from ODE to consider some basic properties of the equation

$$\frac{d}{dt}x(t) = x(t-1) \tag{1a}$$

considered with initial data

$$x(t) = 1, \quad \forall t \in [-1, 0] \tag{1b}$$

The method of steps

Equation (1a) can be written on the interval $[0, 1]$ as the nonautonomous ODE

$$\frac{d}{dt}x(t) = f(t, x(t)) \tag{2}$$

with

$$f(t, x) = \phi_0(t - 1)$$

where $\phi_0(t) = 1$ for $t \in [-1, 0]$

Using the integral form of the solution, we have, for all $t \in [0, 1]$,

$$\begin{aligned} x(t) &= x(0) + \int_0^t f(s, x(s))ds \\ &= x(0) + \int_0^t \phi_0(s-1)ds \\ &= x(0) + \int_0^t ds \\ &= 1 + t \end{aligned}$$

(initial data is $x(t) = 1, \forall t \in [-1, 0]$ so $x(0) = 1$)

Proceeding as before, we can write (1a) on the interval $[1, 2]$ as a nonautonomous ODE

$$\frac{d}{dt}x(t) = f(t, x(t)),$$

with

$$f(t, x(t)) = \phi_1(t-1)$$

where $\phi_1(t)$ is defined on $[0, 1]$ as $x(t) = 1 + t$. Therefore, (1a) can be written

$$\frac{d}{dt}x(t) = \phi_1(t-1), \quad t \in [1, 2]$$

where $\phi_1(t) = 1 + t$ for $t \in [0, 1]$

Using the integral form of the solution, for $t \in [1, 2]$,

$$\begin{aligned} x(t) &= x(1) + \int_1^t \phi_1(s-1)ds \\ &= x(1) + \int_0^t \phi_1(s)ds \end{aligned}$$

From the solution $x(t) = 1 + t$ found earlier, $x(1) = 2$. Therefore, for $t \in [1, 2]$,

$$x(t) = 2 + \left[s + \frac{s^2}{2}\right]_0^t = 2 + t + \frac{t^2}{2}$$

Continuing as previously, for $t \in [2, 3]$,

$$x(t) = x(2) + \int_2^t \phi_2(s-1)ds,$$

where $\phi_2(t) = 2 + t + t^2/2$ for $t \in [1, 2]$ and $x(2) = 6$. So, for $t \in [2, 3]$,

$$x(t) = 6 + \left[2s + \frac{s^2}{2} + \frac{s^3}{6}\right]_1^t = \frac{10}{3} + 2t + \frac{t^2}{2} + \frac{t^3}{6}$$

We could continue.. In the case of (1a) with initial data (1b), we can find an expression for the general form of the solution

Now suppose that instead of the constant initial data (1b), we consider the initial data

$$x(t) = \xi(t), \quad \forall t \in [-1, 0], \tag{3}$$

with $\xi(t)$ a function of class C^p on $[-1, 0]$.

Then the solution to (1a) with initial data (3) is of class C^{p+n+1} on the interval $(n, n+1)$, for $n \in \mathbb{N}$

To compute the solution on the interval $[0, 1]$, we integrated the initial data. So, if the initial data on $[-1, 0]$ is $\xi(t) \in C^p$, then the solution on $(0, 1)$ is of class C^{p+1}. (Regularity at the endpoints of the interval is undetermined.) In turn, we use this to construct the solution on $(1, 2)$, giving a C^{p+2} solution on the interval $(1, 2)$. An easy induction shows the result

Linear stability analysis

Consider the linear DDE

$$x' + px(t-\tau) = 0$$

As in the ODE case, we use the **ansatz** $x(t) = Ce^{\lambda t}$. Using the ansatz, we have

$$x' = \lambda Ce^{\lambda t} \quad \text{and} \quad x(t-\tau) = Ce^{\lambda(t-\tau)} = Ce^{\lambda t}e^{-\lambda\tau}$$

so the ansatz is solution to the DDE iff

$$\lambda Ce^{\lambda t} + pCe^{\lambda t}e^{-\lambda\tau} = 0$$

Divide both sides by $Ce^{\lambda t}$, giving the **characteristic equation**

$$\lambda + pe^{-\lambda\tau} = 0$$

So same as ODE, but the characteristic equation is a little more complicated: it is a **transcendental equation**

Concerning

$$x' + px(t - \tau) = 0$$

and its associated characteristic equation $\lambda + pe^{-\lambda\tau} = 0$, we have the following result: every solution to the equation oscillates iff the characteristic equation has no real roots

Stability analysis of DDE

Consider the delayed logistic equation

$$N' = rN\left(1 - \frac{N(t-\tau)}{K}\right)$$

Equilibria are such that the solution does not change over time

As a consequence, in the limit, $N(t) = N(t-\tau)$ at an equilibrium, and we seek equilibria by solving

$$rN^*\left(1 - \frac{N^*}{K}\right) = 0$$

for N^*

So we find the same EP as in the ODE case, $N^* = 0$ and $N^* = K$

How to linearize

Consider

$$x' = F(x(t), x(t - \tau))$$

If $F(x_0, x_0) = 0$, then x_0 is an equilibrium and the linearized system about $x = x_0$ is

$$x' = F_x(x_0, x_0)x + F_y(x_0, x_0)x(t - \tau)$$