Discrete-time population growth models

Discrete-time systems

In continuous-time models $t \in \mathbb{R}$. Another way to model natural phenomena is to consider equations of the form

$$x_{t+1} = f(x_t),$$

where $t \in \mathbb{N}$ or $\mathbb{Z}$, that is, t takes values in a discrete valued (countable) set

Time could for example be days, years, etc.

Suppose we have a system in the form

$$x_{t+1} = f(x_t),$$

with initial condition given for $t = 0$ by x_0. Then,

$$\begin{aligned} x_1 &= f(x_0) \\ x_2 &= f(x_1) = f(f(x_0)) \stackrel{\Delta}{=} f^2(x_0) \\ &\vdots \\ x_k &= f^k(x_0). \end{aligned}$$

The $f^k = \underbrace{f \circ f \circ \cdots \circ f}_{k \text{ times}}$ are called the **iterates** of f.

Fixed points

Definition (Fixed point)

Let f be a function. A point p such that $f(p) = p$ is called a **fixed point** of f.

Indeed, if $f(p) = p$, then

$$\begin{aligned} f(p) &= p \\ f(f(p)) &= f(p) = p \\ f(f(f(p))) &= f(p) = p \\ &\vdots \\ f^k(p) &= p \quad \forall k \in \mathbb{N} \end{aligned}$$

so the system is *fixed* (stuck) at p..

Theorem

Consider the closed interval $I = [a, b]$. If $f : I \to I$ is continuous, then f has a fixed point in I.

Theorem

Let I be a closed interval and $f : I \to \mathbb{R}$ be a continuous function. If $f(I) \supset I$, then f has a fixed point in I.

Periodic points

Definition (Periodic point)

Let f be a function. If there exists a point p and an integer n such that

$$f^n(p) = p, \quad \text{but} \quad f^k(p) \neq p \text{ for } k < n,$$

then p is a periodic point of f with (least) period n (or a n-periodic point of f).

Thus, p is a n-periodic point of f iff p is a 1-periodic point of f^n.

Stability of fixed points, of periodic points

Theorem

Let $f : \mathbb{R} \to \mathbb{R}$ be a continuously differentiable function (that is, differentiable with continuous derivative, or C^1), and p be a fixed point of f.

1. *If $|f'(p)| < 1$, then there is an open interval $\mathcal{I} \ni p$ such that $\lim_{k\to\infty} f^k(x) = p$ for all $x \in \mathcal{I}$.*
2. *If $|f'(p)| > 1$, then there is an open interval $\mathcal{I} \ni p$ such that if $x \in \mathcal{I}$, $x \neq p$, then there exists k such that $f^k(x) \notin \mathcal{I}$.*

Definition

Suppose that p is a n-periodic point of f, with $f : \mathbb{R} \to \mathbb{R} \in C^1$.

- If $|\left(f^n\right)'(p)| < 1$, then p is an **attracting** periodic point of f.
- If $|\left(f^n\right)'(p)| > 1$, then p is an **repelling** periodic point of f.

Robert May

- Born 1938 in Sydney, Australia
- 1962, Professor of Theoretical Physics, University of Sydney
- 1973, Professor of Zoology, Princeton University
- 1988, Professor, Imperial College and University of Oxford

Known for

Simple mathematical models with very complicated dynamics (Nature, 1976)

The logistic map

The logistic **map** is, for $t \geq 0$,

$$N_{t+1} = rN_t \left(1 - \frac{N_t}{K}\right). \tag{DT1}$$

To transform this into an initial value problem, we need to provide an initial condition $N_0 \geq 0$ for $t = 0$.

Parametrized families of functions

Consider the equation (DT1), which for convenience we rewrite as

$$N_{t+1} = rN_t(1 - N_t), \tag{DT2}$$

where r is a parameter in $\mathbb{R}_+$, and N will typically be taken in $[0, 1]$. Let

$$f_r(x) = rx(1 - x).$$

The function f_r is called a **parametrized family** of functions.

Bifurcations

Definition (Bifurcation)

Let f_μ be a parametrized family of functions. Then there is a **bifurcation** at $\mu = \mu_0$ (or μ_0 is a bifurcation point) if there exists $\varepsilon > 0$ such that, if $\mu_0 - \varepsilon < a < \mu_0$ and $\mu_0 < b < \mu_0 + \varepsilon$, then the dynamics of $f_a(x)$ are "different" from the dynamics of $f_b(x)$.

An example of "different" would be that f_a has a fixed point (that is, a 1-periodic point) and f_b has a 2-periodic point.

Back to the logistic map

Consider the simplified version (DT2),

$$N_{t+1} = rN_t(1 - N_t) \stackrel{\Delta}{=} f_r(N_t).$$

Are solutions well defined? Suppose $N_0 \in [0, 1]$, do we stay in $[0, 1]$? f_r is continuous on $[0, 1]$, so it has a extrema on $[0, 1]$. We have

$$f_r'(x) = r - 2rx = r(1 - 2x),$$

which implies that f_r increases for $x < 1/2$ and decreases for $x > 1/2$, reaching a maximum at $x = 1/2$.

$f_r(0) = f_r(1) = 0$ are the minimum values, and $f(1/2) = r/4$ is the maximum. Thus, if we want $N_{t+1} \in [0, 1]$ for $N_t \in [0, 1]$, we need to consider $r \leq 4$.

- Note that if $N_0 = 0$, then $N_t = 0$ for all $t \geq 1$.
- Similarly, if $N_0 = 1$, then $N_1 = 0$, and thus $N_t = 0$ for all $t \geq 1$.
- This is true for all t: if there exists t_k such that $N_{t_k} = 1$, then $N_t = 0$ for all $t \geq t_k$.
- This last case might occur if $r = 4$, as we have seen.
- Also, if $r = 0$ then $N_t = 0$ for all t.

For these reasons, we generally consider

$$N \in (0, 1)$$

and

$$r \in (0, 4).$$

Fixed points: existence

Fixed points of (DT2) satisfy $N = rN(1-N)$, giving:

- $N = 0$;
- $1 = r(1-N)$, that is, $p \triangleq \dfrac{r-1}{r}$.

Note that $\lim_{r\to 0^+} p = 1 - \lim_{r\to 0^+} 1/r = -\infty$, $\frac{\partial}{\partial r} p = 1/r^2 > 0$ (so p is an increasing function of r), $p = 0 \Leftrightarrow r = 1$ and $\lim_{r\to\infty} p = 1$. So we come to this first conclusion:

- 0 always is a fixed point of f_r.
- If $0 < r < 1$, then p takes negative values so is not relevant.
- If $1 < r < 4$, then p exists.

Stability of the fixed points

Stability of the fixed points is determined by the (absolute) value f_r' at these fixed points. We have

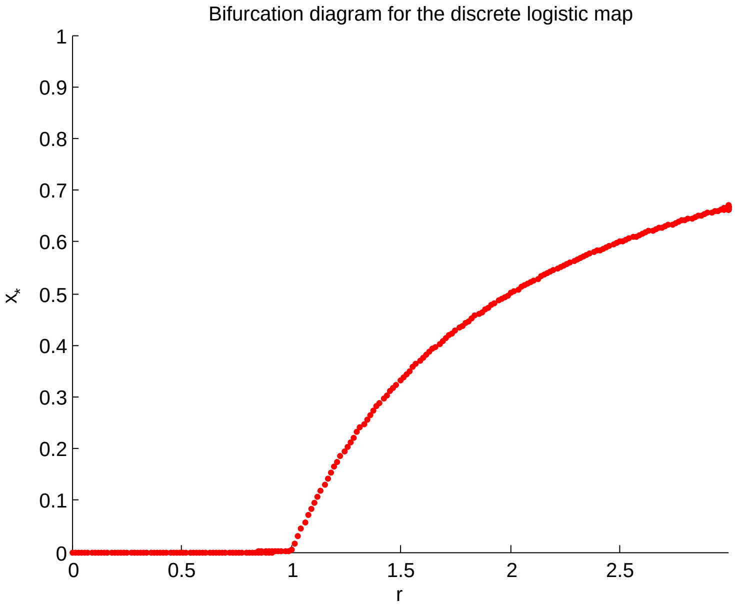
$$|f_r'(0)| = r,$$

and

$$\begin{aligned} |f_r'(p)| &= \left| r - 2r\frac{r-1}{r} \right| \\ &= |r - 2(r-1)| \\ &= |2 - r| \end{aligned}$$

Therefore, we have

- if $0 < r < 1$, then the fixed point $N = p$ does not exist and $N = 0$ is attracting,
- if $1 < r < 3$, then $N = 0$ is repelling, and $N = p$ is attracting,
- if $r > 3$, then $N = 0$ and $N = p$ are repelling.


Bifurcation diagram for the discrete logistic map
1
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Another bifurcation

Thus the points $r = 1$ and $r = 3$ are bifurcation points. To see what happens when $r > 3$, we need to look for period 2 points.

$$\begin{aligned} f_r^2(x) &= f_r(f_r(x)) \\ &= rf_r(x)(1 - f_r(x)) \\ &= r^2 x(1 - x)(1 - rx(1 - x)). \end{aligned} \tag{1}$$

0 and p are points of period 2, since a fixed point x^* of f satisfies $f(x^*) = x^*$, and so, $f^2(x^*) = f(f(x^*)) = f(x^*) = x^*$.
This helps localizing the other periodic points. Writing the fixed point equation as

$$Q(x) \stackrel{\Delta}{=} f_r^2(x) - x = 0,$$

we see that, since 0 and p are fixed points of f_μ^2, they are roots of $Q(x)$. Therefore, Q can be factorized as

$$Q(x) = x(x - p)(-r^3 x^2 + Bx + C),$$

Substitute the value $(r-1)/r$ for p in Q, develop Q and (1) and equate coefficients of like powers gives

$$Q(x) = x\left(x - \frac{r-1}{r}\right)\left(-r^3x^2 + r^2(r+1)x - r(r+1)\right). \quad (2)$$

We already know that $x = 0$ and $x = p$ are roots of (2). So we search for roots of

$$R(x) := -r^3x^2 + r^2(r+1)x - r(r+1).$$

Discriminant is

$$\begin{aligned}\Delta &= r^4(r+1)^2 - 4r^4(r+1) \\ &= r^4(r+1)(r+1-4) \\ &= r^4(r+1)(r-3).\end{aligned}$$

Therefore, R has distinct real roots if $r > 3$. Remark that for $r = 3$, the (double) root is $p = 2/3$. For $r > 3$ but very close to 3, it follows from the continuity of R that the roots are close to $2/3$.

Descartes' rule of signs

Theorem (Descartes' rule of signs)

Let $p(x) = \sum_{i=0}^{m} a_i x^i$ be a polynomial with real coefficients such that $a_m \neq 0$. Define v to be the number of variations in sign of the sequence of coefficients $a_m, \ldots, a_0$. By 'variations in sign' we mean the number of values of n such that the sign of a_n differs from the sign of a_{n-1}, as n ranges from m down to 1. Then

- *the number of positive real roots of $p(x)$ is $v - 2N$ for some integer N satisfying $0 \leq N \leq \frac{v}{2}$,*
- *the number of negative roots of $p(x)$ may be obtained by the same method by applying the rule of signs to $p(-x)$.*

Example of use of Descartes' rule

Example

Let

$$p(x) = x^3 + 3x^2 - x - 3.$$

Coefficients have signs $+ + - -$, i.e., 1 sign change. Thus $v = 1$. Since $0 \le N \le 1/2$, we must have $N = 0$. Thus $v - 2N = 1$ and there is exactly one positive real root of $p(x)$.

To find the negative roots, we examine $p(-x) = -x^3 + 3x^2 + x - 3$. Coefficients have signs $- + + -$, i.e., 2 sign changes. Thus $v = 2$ and $0 \le N \le 2/2 = 1$. Thus, there are two possible solutions, $N = 0$ and $N = 1$, and two possible values of $v - 2N$. Therefore, there are either two or no negative real roots. Furthermore, note that $p(-1) = (-1)^3 + 3 \cdot (-1)^2 - (-1) - 3 = 0$, hence there is at least one negative root. Therefore there must be exactly two.

Back to the logistic map and the polynomial R..

We use Descartes' rule of signs.

- R has signed coefficients $-+-$, so 2 sign changes imlying 0 or 2 positive real roots.
- $R(-x)$ has signed coefficients $---$, so no negative real roots.
- Since $\Delta > 0$, the roots are real, and thus it follows that both roots are positive.

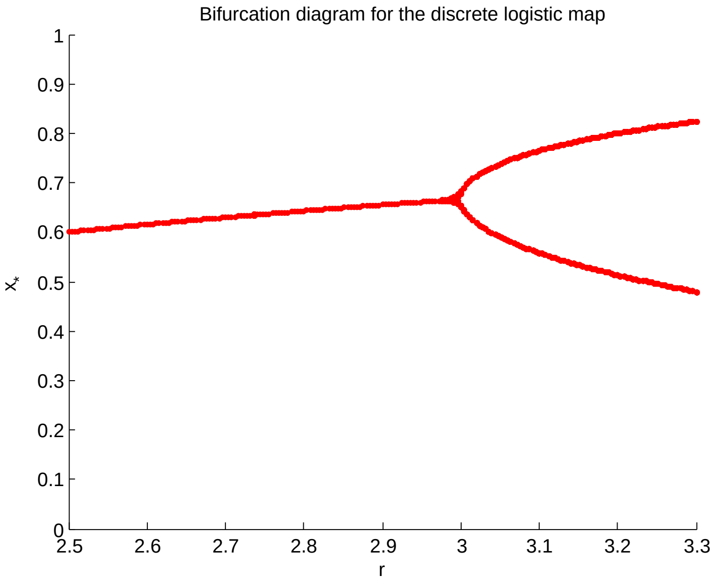
To show that the roots are also smaller than 1, consider the change of variables $z = x - 1$. The polynomial R is transformed into

$$\begin{aligned} R_2(z) &= -r^3(z+1)^2 + r^2(r+1)(z+1) - r(r+1) \\ &= -r^3 z^2 + r^2(1-r)z - r. \end{aligned}$$

For $r > 1$, the signed coefficients are $---$, so R_2 has no root $z > 0$, implying in turn that R has no root $x > 1$.

Summing up

- If $0 < r < 1$, then $N = 0$ is attracting, p does not exist and there are no period 2 points.
- At $r = 1$, there is a bifurcation (called a **transcritical** bifurcation).
- If $1 < r < 3$, then $N = 0$ is repelling, $N = p$ is attracting, and there are no period 2 points.
- At $r = 3$, there is another bifurcation (called a **period-doubling** bifurcation).
- For $r > 3$, both $N = 0$ and $N = p$ are repelling, and there is a period 2 point.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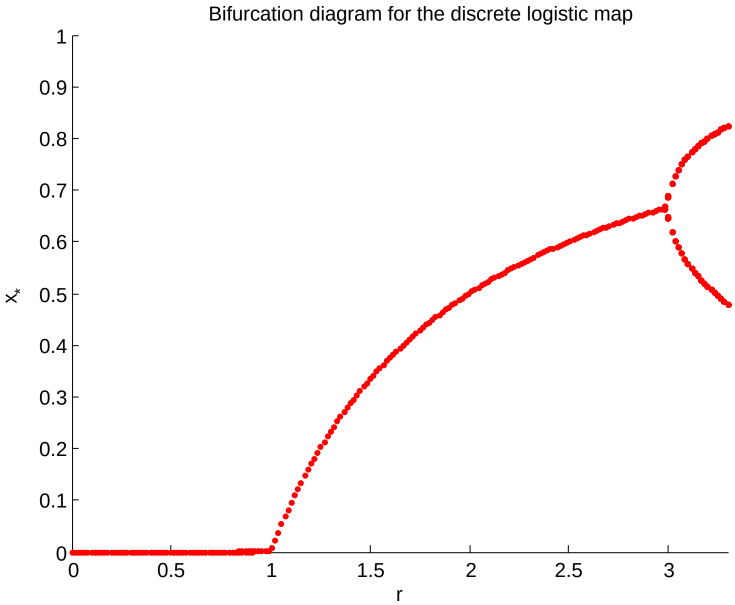
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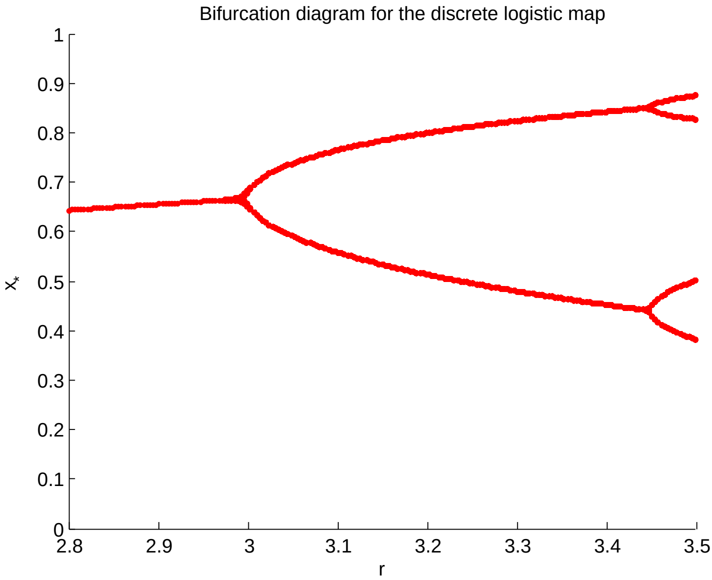
This process continues



The period-doubling cascade to chaos

The logistic map undergoes a sequence of period doubling bifurcations, called the **period-doubling cascade**, as r increases from 3 to 4.

- Every successive bifurcation leads to a doubling of the period.
- The bifurcation points form a sequence, $\{r_n\}$, that has the property that

  $$\lim n \to \infty \frac{r_n - r_{n-1}}{r_{n+1} - r_n}$$

  exists and is a constant, called the Feigenbaum constant, equal to $4.669202\ldots$
- This constant has been shown to exist in many of the maps that undergo the same type of cascade of period doubling bifurcations.

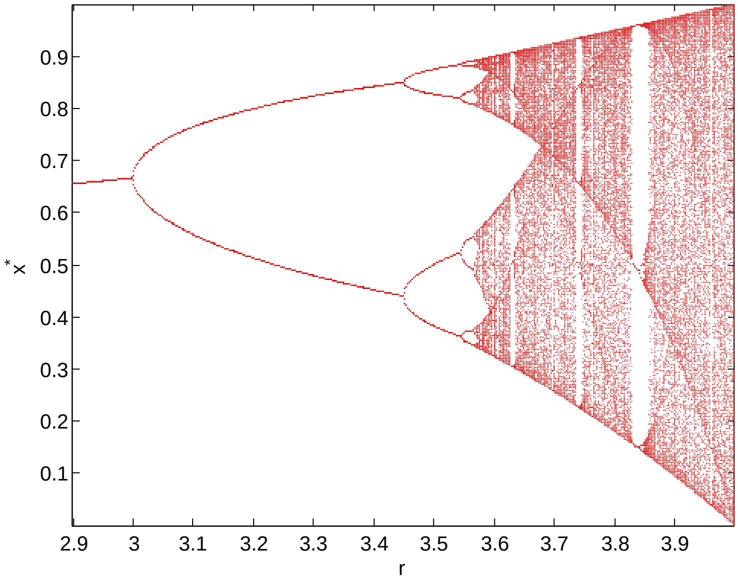
Chaos

After a certain value of r, there are periodic points with all periods. In particular, there are periodic points of period 3.

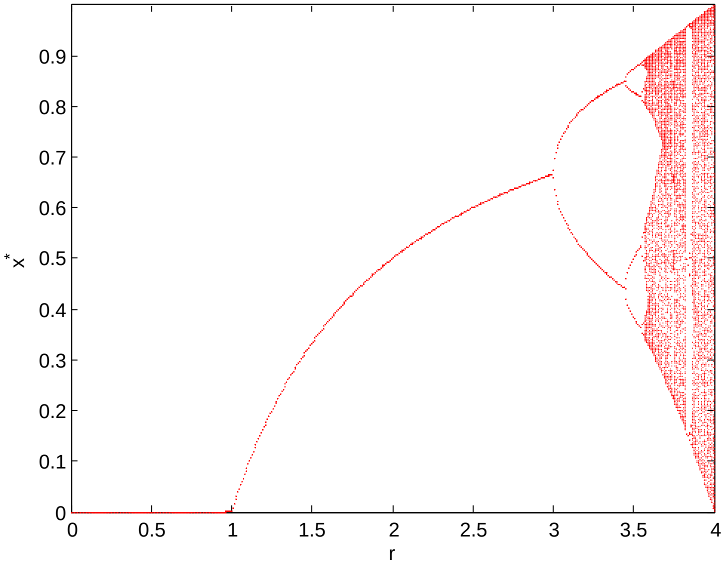
By a theorem (called **Sarkovskii's theorem**), the presence of period 3 points implies the presence of points of all periods.

At this point, the system is said to be in a **chaotic regime**, or **chaotic**.

Bifurcation cascade for $2.9 \leq r \leq 4$



The complete bifurcation cascade



The tent map

[May's 1976 paper]

$$X_{t+1} = \begin{cases} aX_t & \text{if } X_t < 1/2 \\ a(1 - X_t) & \text{if } X_t > 1/2 \end{cases}$$

defined for $0 < X < 1$.

For $0 < a < 1$, all trajectories are attracted to $X = 0$; for $1 < a < 2$, there are infinitely many periodic orbits, along with an uncountable number of aperiodic trajectories, none of which are locally stable. The first odd period cycle appears at $a = \sqrt{2}$ and all integer periods are represented beyond $a = (1 + \sqrt{5})/2$.

Yet another chaotic map

[May's 1976 paper]

$$X_{t+1} = \begin{cases} \lambda X_t & \text{if } X_t < 1 \\ \lambda X_t^{1-b} & \text{if } X_t > 1 \end{cases}$$

If $\lambda > 1$, GAS point for $b < 2$. For $b > 2$, chaotic regime with all integer periods present after $b = 3$.

The Ricker model

$$N(t+1) = N(t)\exp\left\{r\left(1-\frac{N(t)}{K}\right)\right\} = f(N(t)),$$

r intrinsic growth rate, K carrying capacity. Growth rate $f(N(t))$ increasing in $N(t)$ and per capita growth $\frac{f(N)}{N}$ decreasing in $N(t)$. Increase in population not sufficient to compensate for decrease in per capita growth, so $\lim_{N(t)\to+\infty} f(N(t)) = 0$ (Ricker model is overcompensatory).

- $r < 2$ Globally asymptotically stable equilibrium $\bar{x} = K$
- r $=$ 2 Bifurcation into a stable 2-cycle
- r $=$ 2.5 Bifurcation into a stable 4-cycle
- Series of cycle duplication: 8-cycle, 16-cycle, etc.
- r $=$ 2.692 chaos
- For $r > 2.7$ there are some regions where dynamics returns to a cycle, e.g., r=3.15.

Perron-Frobenius theorem

Theorem

If M is a nonnegative primitive matrix, then:

- *M has a positive eigenvalue λ_1 of maximum modulus.*
- *λ_1 is a simple root of the characteristic polynomial.*
- *for every other eigenvalue λ_i, $\lambda_1 > \lambda_i$ (it is strictly dominant)*
- $$\min_i \sum_j m_{ij} \leq \lambda_1 \leq \max_i \sum_j m_{ij}$$
  $$\min_j \sum_i m_{ij} \leq \lambda_1 \leq \max_j \sum_i m_{ij}$$
- *row and column eigenvectors associated with λ_1 are $\gg 0$.*
- *the sequence M^t is asymptotically one-dimensional, its columns converge to the column eigenvector associated with λ_1; and its rows converges to the row eigenvector associated with λ_1.*