Some stochastic models

CTMC

Continuous-time Markov chain

$\{X(t)\}$, $t \in [0, \infty)$ a collection of discrete random variables with values in a finite or infinite set

Definition

Stochastic process $\{X(t)\}$, $t \in [0, \infty)$ a **continuous time Markov chain** if for any sequence of real numbers $0 \leq t_0 < t_1 < \cdots < t_n < t_{n+1}$,

$$\begin{aligned} \mathbb{P}\left(X(t_{n+1}) = i_{n+1} \mid X(t_1) = i_i, \ldots, X(t_n) = i_n\right) \\ = \mathbb{P}\left(X(t_{n+1}) = i_{n+1} \mid X(t_n) = i_n\right) \end{aligned}$$

Each r.v. $X(t)$ has probability distribution $\{p_i(t)\}_{i=0}^{\infty}$ with

$$p_i(t) = \mathbb{P}(X(t) = i)$$

and let $p(t) = (p_0(t), p_1(t), \ldots)^T$. To link r.v., for $s < t$,

$$p_{ji}(t, s) = \mathbb{P}\left(X(t) = j \mid X(s) = i\right)$$

is **transition probability**. Transition probability is **stationary** (or **homogeneous**) if $p_{ji}(t, s)$ depends on $t - s$ (length of time interval) but not explicitly on t or s, i.e., for $s < t$,

$$p_{ji}(t - s) = \mathbb{P}(X(t) = j \mid X(s) = i) = \mathbb{P}(X(t - s) = j \mid X(0) = i)$$

Transition matrix

Transition matrix is

$$P(t) = [p_{ji}(t)]$$

$p_{ji}(t) \geq 0$ and in general,

$$\sum_{j=0}^{\infty} p_{ji}(t) = 1, \quad t \geq 0$$

(proba of transition from state i to some other state is 1)

$P(t)$ is a **stochastic matrix** for $t \geq 0$

Transition probas satisfy **Chapman-Kolmogorov** equations

$$\sum_{k=0}^{\infty} p_{jk}(s) p_{ki}(t) = p_{ji}(t+s)$$

or $P(s)P(t) = P(s+t)$, $\forall s, t \in [0, \infty)$

CTMC as jump processes

CTMC starting in state $X(0)$

- stays in state $X(0)$ for random amount of time W_1 then jumps to new state $X(W_1)$
- stays in state $X(W_1)$ for random amount of time W_2 then jumps to new state $X(W_2)$
- ...

W_i r.v. for time of i^{th} jump. Define $W_0 = 0$. Then collection of r.v. $\{W_i\}$ is **jump times** (or **waiting times**) and r.v.

$$T_i = W_{i+1} - W_i$$

are **interevent times** (or **holding times** or **sojourn times**)

Poisson process

Initial condition $X(0) = 0$, then for Δt sufficiently small

- $p_{i+1,i}(\Delta t) = \mathbb{P}(X(t+\Delta t) = i+1 \mid X(t) = i) = \lambda \Delta t + o(\Delta t)$
- $p_{ii}(\Delta t) = \mathbb{P}(X(t+\Delta t) = i \mid X(t) = i) = 1 - \lambda \Delta t + o(\Delta t)$
- $p_{ji}(\Delta t) = \mathbb{P}(X(t+\Delta t) = j \mid X(t) = i) = o(\Delta t)$, $j \geq i+2$
- $p_{ji}(\Delta t) = 0$, $j < i$