MATH 8430

Fundamental Theory of Ordinary Differential Equations

Lecture Notes

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Contents

Introduction

This course deals with the *elementary* theory of ordinary differential equations. The word elementary should not be understood as *simple*. The underlying assumption here is that, to understand the more advanced topics in the analysis of nonlinear systems, it is important to have a good understanding of how solutions to differential equations are constructed.

If you are taking this course, you most likely know how to analyze systems of nonlinear ordinary differential equations. You know, for example, that in order for solutions to a system to exist and be unique, the system must have a C^1 vector field. What you do not necessarily know is why that is. This is the object of Chapter 1, where we consider the general theory of existence and uniqueness of solutions. We also consider the continuation of solutions as well as continuous dependence on initial data and on parameters.

In Chapter 2, we explore linear systems. We first consider homogeneous linear systems, then linear systems in full generality. Homogeneous linear systems are linked to the theory for nonlinear systems by means of linearization, which we study in Chapter 4, in which we show that the behavior of nonlinear systems can be approximated, in the vicinity of a hyperbolic equilibrium point, by a homogeneous linear system. As for autonomous systems, nonautonomous nonlinear systems are linked to a linearized form, this time through exponential dichotomy, which is explained in Chapter 5.

Chapter 1

General theory of ODEs

We begin with the general theory of ordinary differential equations (ODEs). First, we define ODEs, initial value problems (IVPs) and solutions to ODEs and IVPs in Section 1.1. In Section 1.2, we discuss existence and uniqueness of solutions to IVPs.

1.1 ODEs, IVPs, solutions

1.1.1 Ordinary differential equation, initial value problem

Definition 1.1.1 (ODE)**.** *An nth order* ordinary differential equation *(ODE) is a functional relationship taking the form*

$$F\left(t, x(t), \frac{d}{dt}x(t), \frac{d^2}{dt^2}x(t), \ldots, \frac{d^n}{dt^n}x(t)\right) = 0,$$

that involves an independent variable $t \in \mathcal{I} \subset \mathbb{R}$, an unknown function $x(t) \in \mathcal{D} \subset \mathbb{R}^n$ of the independent variable, its derivative and derivatives of order up to n. For simplicity, the time dependence of x is often omitted, and we in general write equations as

$$F\left(t, x, x', x'', \ldots, x^{(n)}\right) = 0, \tag{1.1}$$

where $x^{(n)}$ denotes the nth order derivative of x. An equation such as (1.1) is said to be in general *(or* implicit*) form.*

An equation is said to be in *normal* (or *explicit*) form when it is written as

$$x^{(n)} = f\left(t, x, x', x'', \ldots, x^{(n-1)}\right).$$

Note that it is not always possible to write a differential equation in normal form, as it can be impossible to solve $F(t, x, \ldots, x^{(n)}) = 0$ in terms of $x^{(n)}$.

Definition 1.1.2 (First-order ODE)**.** *In the following, we consider for simplicity the more restrictive case of a first-order ordinary differential equation in normal form*

$$x' = f(t, x). \tag{1.2}$$

Note that the theory developed here holds usually for nth order equations; see Section 1.5. The function f is assumed continuous and real valued on a set $\mathcal{U} \subset \mathbb{R} \times \mathbb{R}^n$.

Definition 1.1.3 (Initial value problem)**.** *An initial value problem (IVP) for equation (1.2) is given by*

$$\begin{aligned} x' &= f(t,x) \\ x(t_0) &= x_0, \end{aligned} \tag{1.3}$$

where f is continuous and real valued on a set $\mathcal{U} \subset \mathbb{R} \times \mathbb{R}^n$, with $(t_0, x_0) \in \mathcal{U}$.

Remark – The assumption that f be continuous can be relaxed, piecewise continuity only is needed. However, this leads in general to much more complicated problems and is beyond the scope of this course. Hence, unless otherwise stated, we assume that f is at least continuous. The function f could also be complex valued, but this too is beyond the scope of this course. ◦

Remark – An IVP for an nth order differential equation takes the form

$$\begin{gathered} x^{(n)} = f(t, x, x', \ldots, x^{(n-1)}) \\ x(t_0) = x_0, x'(t_0) = x'_0, \ldots, x^{(n-1)}(t_0) = x_0^{(n-1)}, \end{gathered}$$

i.e., initial conditions have to be given for derivatives up to order $n-1$. ◦

We have already seen that the order of an ODE is the order of the highest derivative involved in the equation. An equation is then classified as a function of its *linearity*. A linear equation is one in which the vector field f takes the form

$$f(t,x) = a(t)x(t) + b(t).$$

If $b(t) = 0$ for all t, the equation is linear *homogeneous*; otherwise it is linear *nonhomogeneous*. If the vector field f depends only on x, *i.e.*, $f(t,x) = f(x)$ for all t, then the equation is *autonomous*; otherwise, it is *nonautonomous*. Thus, a linear equation is autonomous if $a(t) = a$ and $b(t) = b$ for all t. Nonlinear equations are those that are not linear. They too, can be autonomous or nonautonomous.

Other types of classifications exist for ODEs, which we shall not deal with here, the previous ones being the only one we will need.

1.1.2 Solutions to an ODE

Definition 1.1.4 (Solution)**.** *A function $\phi(t)$ (or ϕ, for short) is a* solution *to the ODE* (1.2) *if it satisfies this equation, that is, if*

$$\phi'(t) = f(t, \phi(t)),$$

for all $t \in \mathcal{I} \subset \mathbb{R}$, an open interval such that $(t, \phi(t)) \in \mathcal{U}$ for all $t \in \mathcal{I}$.

The notations ϕ and x are used indifferently for the solution. However, in this chapter, to emphasize the difference between the equation and its solution, we will try as much as possible to use the notation x for the unknown and ϕ for the solution.

Definition 1.1.5 (Integral form of the solution)**.** *The function*

$$\phi(t) = x_0 + \int_{t_0}^{t} f(s, \phi(s))ds \tag{1.4}$$

is called the integral form *of the solution to the IVP (1.3).*

Let $\mathcal{R} = \mathcal{R}((t_0, x_0), a, b)$ be the domain defined, for $a > 0$ and $b > 0$, by

$$\mathcal{R} = \{(t, x) : \; |t - t_0| \leq a, \|x - x_0\| \leq b\},$$

where $\| \; \|$ is any appropriate norm of $\mathbb{R}^n$. This domain is illustrated in Figures 1.1 and 1.2; it is sometimes called a *security system*, *i.e.*, the union of a security *interval* (for the independent variable) and a security *domain* (for the dependent variables) [19]. Suppose

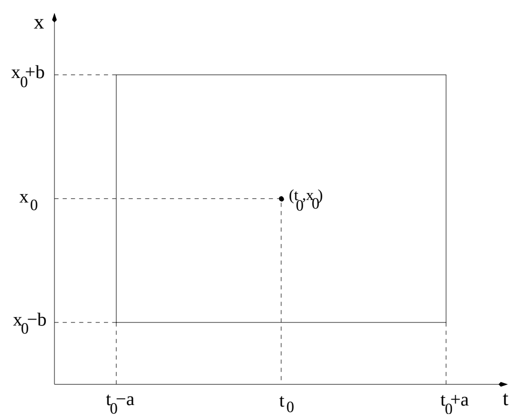

Figure 1.1: The domain $\mathcal{R}$ for $\mathcal{D} \subset \mathbb{R}$.

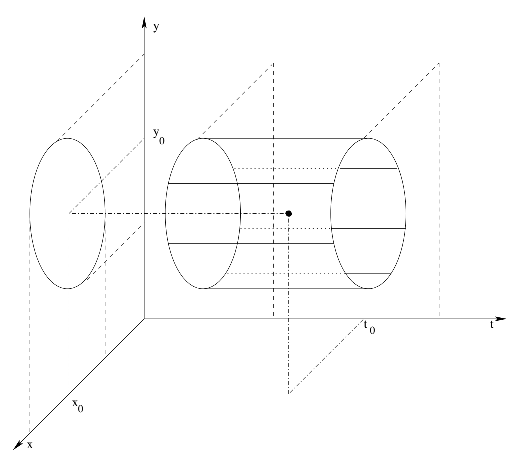

Figure 1.2: The domain $\mathcal{R}$ for $\mathcal{D} \subset \mathbb{R}^2$: "security tube".

that f is continuous on $\mathcal{R}$, and let $M = \max_{\mathcal{R}} \|f(t,x)\|$, which exists since f is continuous on the compact set $\mathcal{R}$.

In the following, existence of solutions will be obtained generally in relation to the domain $\mathcal{R}$ by considering a subset of the time interval $|t-t_0| \leq a$ defined by $|t-t_0| \leq \alpha$, with

$$\alpha = \begin{cases} a & \text{if } M = 0 \\ \min(a, \frac{b}{M}) & \text{if } M > 0. \end{cases}$$

This choice of $\alpha = \min(a, b/M)$ is natural. We endow f with specific properties (continuity, Lipschitz, etc.) on the domain $\mathcal{R}$. Thus, in order to be able to use the definition of $\phi(t)$ as the solution of $x' = f(t,x)$, we must be working in $\mathcal{R}$. So we require that $|t-t_0| \leq a$ **and** $\|x - x_0\| \leq b$. In order to satisfy the first of these conditions, choosing $\alpha \leq a$ and working on $|t-t_0| \leq \alpha$ implies of course that $|t-t_0| \leq a$. The requirement that $\alpha \leq b/M$ comes from the following argument. If we assume that $\phi(t)$ is a solution of (1.3) defined on $[t_0, t_0+\alpha]$, then we have, for $t \in [t_0, t_0+\alpha]$,

$$\begin{aligned} \|\phi(t) - x_0\| &= \left\| \int_{t_0}^t f(s,\phi(s))ds \right\| \\ &\leq \int_{t_0}^t \|f(s,\phi(s))\|\, ds \\ &\leq M \int_{t_0}^t ds \\ &= M(t-t_0), \end{aligned}$$

where the first inequality is a consequence of the definition of the integrals by Riemann sums (Lemma A.2.1 in Appendix A.2). Similarly, we have $\|\phi(t) - x_0\| \leq -M(t-t_0)$ for all $t \in [t_0 - \alpha, t_0]$. Thus, for $|t-t_0| \leq \alpha$, $\|\phi(t) - x_0\| \leq M|t-t_0|$. Suppose now that $\alpha \leq b/M$. It follows that $\|\phi - x_0\| \leq M|t-t_0| \leq Mb/M = b$. Taking $\alpha = \min(a, b/M)$ then ensures that both $|t-t_0| \leq a$ and $\|\phi - x_0\| \leq b$ hold simultaneously.

The following two theorems deal with the localization of the solutions to an IVP. They make more precise the previous discussion. Note that for the moment, the existence of a solution is only assumed. First, we establish that the security system described above performs properly, in the sense that a solution on a smaller time interval stays within the security domain.

Theorem 1.1.6. *If $\phi(t)$ is a solution of the IVP (1.3) in an interval $|t-t_0| < \tilde{\alpha} \leq \alpha$, then $\|\phi(t) - x_0\| < b$ in $|t-t_0| < \tilde{\alpha}$, i.e., $(t, \phi(t)) \in \mathcal{R}((t_0, x_0), \tilde{\alpha}, b)$ for $|t-t_0| < \tilde{\alpha}$.*

Proof. Assume that ϕ is a solution with $(t,\phi(t)) \notin \mathcal{R}((t_0,x_0),\tilde{\alpha},b)$. Since ϕ is continuous, it follows that there exists $0 < \beta < \tilde{\alpha}$ such that

$$\big(\|\phi(t) - x_0\| < b \text{ for } |t-t_0| < \beta\big) \text{ and } \big(\|\phi(t_0+\beta) - x_0\| = b \text{ or } \|\phi(t_0-\beta) - x_0\| = b\big), \quad (1.5)$$

i.e., the solution escapes the security domain at $t = t_0 \pm \beta$. Since $\tilde{\alpha} \leq \alpha \leq a$, $\beta < a$. Thus

$$(t, \phi(t)) \in \mathcal{R} \text{ for } |t-t_0| \leq \beta.$$

Thus $\|f(t,\phi(t))\| \leq M$ for $|t-t_0| \leq \beta$. Since ϕ is a solution, we have that $\phi'(t) = f(t,\phi(t))$ and $\phi(t_0) = x_0$. Thus

$$\phi(t) = x_0 + \int_{t_0}^{t} f(s,\phi(s))ds \text{ for } |t-t_0| \leq \beta.$$

Hence

$$\begin{aligned}\|\phi(t) - x_0\| &= \left\| \int_{t_0}^{t} f(s,\phi(s))ds \right\| \text{ for } |t-t_0| \leq \beta \\ &\leq M|t-t_0| \text{ for } |t-t_0| \leq \beta.\end{aligned}$$

As a consequence,

$$\|\phi(t) - x_0\| \leq M\beta < M\tilde{\alpha} \leq M\alpha \leq M\frac{b}{M} = b \text{ for } |t-t_0| \leq \beta.$$

In particular, $\|\phi(t_0 \pm \beta) - x_0\| < b$. Hence contradiction with (1.5). □

The following theorem is proved using the same sort of technique as in the proof of Theorem 1.1.6. It links the variation of the solution to the nature of the vector field.

Theorem 1.1.7. *If $\phi(t)$ is a solution of the IVP (1.3) in an interval $|t-t_0| < \tilde{\alpha} \leq \alpha$, then $\|\phi(t_1) - \phi(t_2)\| \leq M|t_1 - t_2|$ whenever t_1, t_2 are in the interval $|t-t_0| < \tilde{\alpha}$.*

Proof. Let us begin by considering $t \geq t_0$. On $t_0 \leq t \leq t_0 + \tilde{\alpha}$,

$$\begin{aligned}\phi(t_1) - \phi(t_2) &= x_0 + \int_{t_0}^{t_1} f(s,\phi(s))ds - x_0 - \int_{t_0}^{t_2} f(s,\phi(s))ds \\ &= -\int_{t_1}^{t_2} f(s,\phi(s))ds \text{ if } t_2 > t_1 \\ &\ \int_{t_1}^{t_2} f(s,\phi(s))ds \text{ if } t_1 > t_2\end{aligned}$$

□

Now we can see formally what is needed for a solution.

Theorem 1.1.8. *Suppose f is continuous on an open set $\mathcal{U} \subset \mathbb{R} \times \mathbb{R}^n$. Let $(t_0, x_0) \in \mathcal{U}$, and ϕ be a function defined on an open set $\mathcal{I}$ of $\mathbb{R}$ such that $t_0 \in \mathcal{I}$. Then ϕ is a solution of the IVP (1.3) if, and only if,*

i) $\forall t \in \mathcal{I}$, $(t,\phi(t)) \in \mathcal{U}$.

ii) ϕ *is continuous on* $\mathcal{I}$.

iii) $\forall t \in \mathcal{I}$, $\phi(t) = x_0 + \int_{t_0}^{t} f(s,\phi(s))ds$.

Proof. ($\Rightarrow$) Let us suppose that $\phi' = f(t, \phi)$ for all $t \in \mathcal{I}$ and that $\phi(t_0) = x_0$. Then for all $t \in \mathcal{I}$, $(t, \phi(t)) \in \mathcal{U}$ (i). Also, ϕ is differentiable and thus continuous on $\mathcal{I}$ (ii). Finally,

$$\phi'(s) = f(s, \phi(s))$$

so integrating both sides from t_0 to t,

$$\phi(t) - \phi(t_0) = \int_{t_0}^{t} f(s, \phi(s))ds$$

and thus

$$\phi(t) = x_0 + \int_{t_0}^{t} f(s, \phi(s))ds$$

hence (iii).
($\Leftarrow$) Assume $i)$, $ii)$ and $iii)$. Then ϕ is differentiable on $\mathcal{I}$ and $\phi'(t) = f(t, \phi(t))$ for all $t \in \mathcal{I}$. From (3), $\phi(t_0) = x_0 + \int_{t_0}^{t_0} f(s, \phi(s))ds = x_0$. □

Note that Theorem 1.1.8 states that ϕ should be continuous, whereas the solution should of course be C^1, for its derivative needs to be continuous. However, this is implied by point $iii)$. In fact, more generally, the following result holds about the regularity of solutions.

Theorem 1.1.9 (Regularity). *Let $f : \mathcal{U} \to \mathbb{R}^n$, with $\mathcal{U}$ an open set of $\mathbb{R} \times \mathbb{R}^n$. Suppose that $f \in C^k$. Then all solutions of* (1.2) *are of class C^{k+1}.*

Proof. The proof is obvious, since a solution ϕ is such that $\phi' \in C^k$. □

1.1.3 Geometric interpretation

The function f is the *vector field* of the equation. At every point in (t, x) space, a solution ϕ is tangent to the value of the vector field at that point. A particular consequence of this fact is the following theorem.

Theorem 1.1.10. *Let $x' = f(x)$ be a scalar autonomous differential equation. Then the solutions of this equation are monotone.*

Proof. The direction field of an autonomous scalar differential equation consists of vectors that are parallel for all t (since $f(t, x) = f(x)$ for all t). Suppose that a solution ϕ of $x' = f(x)$ is non monotone. Then this means that, given an initial point (t_0, x_0), one the following two occurs, as illustrated in Figure 1.3.

i) $f(x_0) \neq 0$ and there exists t_1 such that $\phi(t_1) = x_0$.

ii) $f(x_0) = 0$ and there exists t_1 such that $\phi(t_1) \neq x_0$.

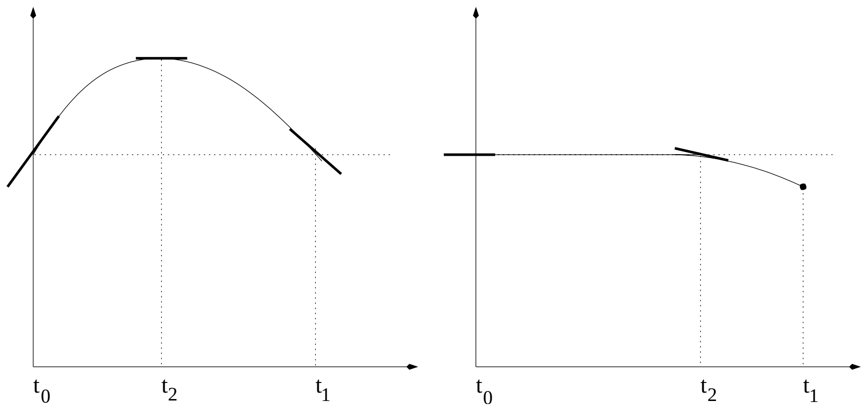

Figure 1.3: Situations that would lead to a scalar autonomous differential equation having nonmonotone solutions.

Suppose we are in case i), and assume we are in the case $f(x_0) > 0$. Thus, the solution curve ϕ is increasing at (t_0, x_0), *i.e.*, $\phi'(t_0) > 0$. As ϕ is continuous, i) implies that there exists $t_2 \in (t_0, t_1)$ such that $\phi(t_2)$ is a maximum, with ϕ increasing for $t \in [t_0, t_2)$ and ϕ decreasing for $t \in (t_2, t_1]$. It follows that $\phi'(t_1) < 0$, which is a contradiction with $\phi'(t_0) > 0$.

Now assume that we are in case ii). Then there exists $t_2 \in (t_0, t_1)$ with $\phi(t_2) = x_0$ but such that $\phi'(t_2) < 0$. This is a contradiction. $\square$

Remark – If we have uniqueness of solutions, it follows from this theorem that if ϕ_1 and ϕ_2 are two solutions of the scalar autonomous differential equation $x' = f(x)$, then $\phi_1(t_0) < \phi_2(t_0)$ implies that $\phi_1(t) < \phi_2(t)$ for all t. ◦

Remark – Be careful: Theorem 1.1.10 is only true for scalar equations. ◦

1.2 Existence and uniqueness theorems

Several approaches can be used to show existence and/or uniqueness of solutions. In Sections 1.2.2 and 1.2.3, we take a direct path: using either a fixed point method (Section 1.2.2) or an iterative approach (Section 1.2.3), we obtain existence and uniqueness of solutions under the assumption that the vector field is Lipschitz. In Section 1.2.4, the Lipschitz assumption is dropped and therefore a different approach must be used, namely that of *approximate solutions*, with which only existence can be established.

1.2.1 Successive approximations

Picard's successive approximation method consists in using the integral form (1.4) of the solution to the IVP (1.3) to construct a sequence of approximation of the solution, that converges to the solution. The steps followed in constructing this approximating sequence are the following.

Step 1. Start with an initial estimate of the solution, say, the constant function $\phi_0(t) = \phi_0 = x_0$, for $|t - t_0| \leq h$. Evidently, this function satisfies the IVP.
Step 2. Use ϕ_0 in (1.4) to define the second element in the sequence:

$$\phi_1(t) = x_0 + \int_{t_0}^{t} f(s, \phi_0(s))ds.$$

Step 3. Use ϕ_1 in (1.4) to define the third element in the sequence:

$$\phi_2(t) = x_0 + \int_{t_0}^{t} f(s, \phi_1(s))ds.$$

...
Step n. Use ϕ_{n-1} in (1.4) to define the nth element in the sequence:

$$\phi_n(t) = x_0 + \int_{t_0}^{t} f(s, \phi_{n-1}(s))ds.$$

At this stage, there are two major ways to tackle the problem, which use the same idea: if we can prove that the sequence $\{\phi_n\}$ converges, and that the limit happens to satisfy the differential equation, then we have the solution to the IVP (1.3). The first method (Section 1.2.2) uses a fixed point approach. The second method (Section 1.2.3) studies explicitly the limit.

1.2.2 Local existence and uniqueness – Proof by fixed point

Here are two slightly different formulations of the same theorem, which establishes that if the vector field is continuous and Lipschitz, then the solutions exist and are unique. We prove the result in the second case. For the definition of a Lipschitz function, see Section A.6 in the Appendix.

Theorem 1.2.1 (Picard local existence and uniqueness). *Assume $f : \mathcal{U} \subset \mathbb{R} \times \mathbb{R}^n \to \mathcal{D} \subset \mathbb{R}^n$ is continuous, and that $f(t, x)$ satisfies a Lipschitz condition in $\mathcal{U}$ with respect to x. Then, given any point $(t_0, x_0) \in \mathcal{U}$, there exists a unique solution of (1.3) on* some *interval containing t_0 in its interior.*

Theorem 1.2.2 (Picard local existence and uniqueness). *Consider the IVP (1.3), and assume f is (piecewise) continuous in t and satisfies the Lipschitz condition*

$$\|f(t, x_1) - f(t, x_2)\| \leq L\|x_1 - x_2\|$$

for all $x_1, x_2 \in \mathcal{D} = \{x : \|x - x_0\| \leq b\}$ and all t such that $|t - t_0| \leq a$. Then there exists $0 < \delta \leq \alpha = \min\left(a, \frac{b}{M}\right)$ such that (1.3) has a unique solution in $|t - t_0| \leq \delta$.

To set up the proof, we proceed as follows. Define the operator F by

$$F : x \mapsto x_0 + \int_{t_0}^{t} f(s, x(s))ds.$$

Note that the function $(F\phi)(t)$ is a continuous function of t. Then Picard's successives approximations take the form $\phi_1 = F\phi_0$, $\phi_2 = F\phi_1 = F^2\phi_0$, where F^2 represents $F \circ F$. Iterating, the general term is given for $k = 0, \ldots$ by

$$\phi_k = F^k \phi_0.$$

Therefore, finding the limit $\lim_{k\to\infty} \phi_k$ is equivalent to finding the function ϕ, solution of the fixed point problem

$$x = Fx,$$

with x a continuously differentiable function. Thus, a solution of (1.3) is a fixed point of F, and we aim to use the contraction mapping principle to verify the existence (and uniqueness) of such a fixed point. We follow the proof of [14, p. 56-58].

Proof. We show the result on the interval $t - t_0 \leq \delta$. The proof for the interval $t_0 - t \leq \delta$ is similar. Let X be the space of continuous functions defined on the interval $[t_0, t_0 + \delta]$, $X = C([t_0, t_0 + \delta])$, that we endow with the sup norm, *i.e.*, for $x \in X$,

$$\|x\|_c = \max_{t\in[t_0,t_0+\delta]} \|x(t)\|$$

Recall that this norm is the norm of uniform convergence. Let then

$$S = \{x \in X : \|x - x_0\|_c \leq b\}$$

Of course, $S \subset X$. Furthermore, S is closed, and X with the sup norm is a complete metric space. Note that we have transformed the problem into a problem involving the space of continuous functions; hence we are now in an infinite dimensional case. The proof proceeds in 3 steps.
Step 1. We begin by showing that $F : S \to S$. From (1.4),

$$\begin{aligned}(F\phi)(t) - x_0 &= \int_{t_0}^{t} f(s, \phi(s))ds \\ &= \int_{t_0}^{t} f(s, \phi(s)) - f(s, x_0) + f(s, x_0)ds\end{aligned}$$

Therefore, by the triangle inequality,

$$\|F\phi - x_0\| \leq \int_{t_0}^{t} \|f(s, \phi(s)) - f(t, x_0)\| + \|f(t, x_0)\|ds$$

As f is (piecewise) continuous, it is bounded on $[t_0, t_1]$ and there exists $M = \max_{t\in[t_0,t_1]} \|f(t, x_0)\|$. Thus

$$\begin{aligned}\|F\phi - x_0\| &\leq \int_{t_0}^{t} \|f(s, \phi(s)) - f(t, x_0)\| + Mds \\ &\leq \int_{t_0}^{t} L\|\phi(s) - x_0\| + Mds,\end{aligned}$$

since f is Lipschitz. Since $\phi \in S$ for all $\|\phi - x_0\| \leq b$, we have that for all $\phi \in S$,

$$\begin{aligned}\|F\phi - x_0\| &\leq \int_{t_0}^{t} Lb + M ds \\ &\leq (t - t_0)(Lb + M)\end{aligned}$$

As $t \in [t_0, t_0 + \delta]$, $(t - t_0) \leq \delta$, and thus

$$\|F\phi - x_0\|_c = \max_{[t_0, t_0+\delta]} \|F\phi - x_0\| \leq (Lb + M)\delta$$

Choose then δ such that $\delta \leq b/(Lb + M)$, *i.e.*, t sufficiently close to t_0. Then we have

$$\|F\phi - x_0\|_c \leq b$$

This implies that for $\phi \in S$, $F\phi \in S$, *i.e.*, $F : S \to S$.
Step 2. We now show that F is a contraction. Let $\phi_1, \phi_2 \in S$,

$$\begin{aligned}\|(F\phi_1)(t) - (F\phi_2)(t)\| &= \left\| \int_{t_0}^{t} f(s, \phi_1(s)) - f(s, \phi_2(s)) ds \right\| \\ &\leq \int_{t_0}^{t} \|f(s, \phi_1(s)) - f(s, \phi_2(s))\| ds \\ &\leq \int_{t_0}^{t} L\|\phi_1(s) - \phi_2(s)\| ds \\ &\leq L\|\phi_1 - \phi_2\|_c \int_{t_0}^{t} ds\end{aligned}$$

and thus

$$\|F\phi_1 - F\phi_2\|_c \leq L\delta\|\phi_1 - \phi_2\|_c \leq \rho\|\phi_1 - \phi_2\|_c \text{ for } \delta \leq \frac{\rho}{L}$$

Thus, choosing $\rho < 1$ and $\delta \leq \rho/L$, F is a contraction. Since, by Step 1, $F : S \to S$, the contraction mapping principle (Theorem A.11) implies that F has a unique fixed point in S, and (1.3) has a unique solution in S.
Step 3. It remains to be shown that any solution in X is in fact in S (since it is on X that we want to show the result). Considering a solution starting at x_0 at time t_0, the solution leaves S if there exists a $t > t_0$ such that $\|\phi(t) - x_0\| = b$, *i.e.*, the solution crosses the border of $\mathcal{D}$. Let $\tau > t_0$ be the first of such t's. For all $t_0 \leq t \leq \tau$,

$$\begin{aligned}\|\phi(t) - x_0\| &\leq \int_{t_0}^{t} \|f(s, \phi(s)) - f(s, x_0)\| + \|f(s, x_0)\| ds \\ &\leq \int_{t_0}^{t} L\|\phi(s) - x_0\| + M ds \\ &\leq \int_{t_0}^{t} Lb + M ds\end{aligned}$$

As a consequence,
$$b = \|\phi(\tau) - x_0\| \leq (Lb + M)(\tau - t_0)$$
As $\tau = t_0 + \mu$, for some $\mu > 0$, it follows that if
$$\mu > \frac{b}{Lb + M}$$
then the solution ϕ is confined to $\mathcal{D}$. □

Note that the condition $x_1, x_2 \in \mathcal{D} = \{x : \|x - x_0\| \leq b\}$ in the statement of the theorem refers to a local Lipschitz condition. If the function f is Lipschitz, then the following theorem holds.

Theorem 1.2.3 (Global existence)**.** *Suppose that f is piecewise continuous in t and is Lipschitz on $\mathcal{U} = \mathcal{I} \times \mathcal{D}$. Then* (1.3) *admits a unique solution on $\mathcal{I}$.*

1.2.3 Local existence and uniqueness – Proof by successive approximations

Using the method of successive approximations, we can prove the following theorem.

Theorem 1.2.4. *Suppose that f is continuous on a domain $\mathcal{R}$ of the (t, x)-plane defined, for $a, b > 0$, by $\mathcal{R} = \{(t, x) : |t - t_0| \leq a, \|x - x_0\| \leq b\}$, and that f is locally Lipschitz in x on $\mathcal{R}$. Let then, as previously defined,*
$$M = \sup_{(t,x)\in\mathcal{R}} \|f(t, x)\| < \infty$$
and
$$\alpha = \min(a, \frac{b}{M})$$
Then the sequence defined by
$$\begin{aligned}
\phi_0 &= x_0, \quad |t - t_0| \leq \alpha \\
\phi_i(t) &= x_0 + \int_{t_0}^{t} f(s, \phi_{i-1}(s))ds, \quad i \geq 1, \quad |t - t_0| \leq \alpha
\end{aligned}$$
converges uniformly on the interval $|t - t_0| \leq \alpha$ to ϕ, unique solution of (1.3).

Proof. We follow [20, p. 3-6].

Existence. Suppose that $|t - t_0| \leq \alpha$. Then
$$\begin{aligned}
\|\phi_1 - \phi_0\| &= \left\| \int_{t_0}^{t} f(s, \phi_0(s))ds \right\| \\
&\leq M|t - t_0| \\
&\leq \alpha M \leq b
\end{aligned}$$

from the definitions of M and α, and thus $\|\phi_1 - \phi_0\| \leq b$. So $\int_{t_0}^t f(s, \phi_1(s))ds$ is defined for $|t - t_0| \leq \alpha$, and, for $|t - t_0| \leq \alpha$,

$$\|\phi_2(t) - \phi_0\| = \left\| \int_{t_0}^t f(s, \phi_1(s))ds \right\| \leq \| \int_{t_0}^t \|f(s, \phi_1(s))\|ds \leq \alpha M \leq b.$$

All subsequent terms in the sequence can be similarly defined, and, by induction, for $|t - t_0| \leq \alpha$,

$$\|\phi_k(t) - \phi_0\| \leq \alpha M \leq b, \quad k = 1, \ldots, n.$$

Now, for $|t - t_0| \leq \alpha$,

$$\begin{aligned}
\|\phi_{k+1}(t) - \phi_k(t)\| &= \left\| x_0 + \int_{t_0}^t f(s, \phi_k(s))ds - x_0 - \int_{t_0}^t f(s, \phi_{k-1}(s))ds \right\| \\
&= \left\| \int_{t_0}^t f(s, \phi_k(s)) - f(s, \phi_{k-1}(s))\; ds \right\| \\
&\leq L \int_{t_0}^t \|\phi_k(s) - \phi_{k-1}(s)\|ds,
\end{aligned}$$

where the inequality results of the fact that f is locally Lipschitz in x on $\mathcal{R}$.

We now prove that, for all k,

$$\|\phi_{k+1} - \phi_k\| \leq b \frac{(L|t - t_0|)^k}{k!} \text{ for } |t - t_0| \leq \alpha \tag{1.6}$$

Indeed, (1.6) holds for $k = 1$, as previously established. Assume that (1.6) holds for $k = n$. Then

$$\begin{aligned}
\|\phi_{n+2} - \phi_{n+1}\| &= \left\| \int_{t_0}^t f(s, \phi_{n+1}(s)) - f(s, \phi_n(s))ds \right\| \\
&\leq \int_{t_0}^t L\|\phi_{n+1}(s) - \phi_n(s)\|ds \\
&\leq \int_{t_0}^t Lb \frac{(L|s - t_0|)^n}{n!} ds \text{ for } |t - t_0| \leq \alpha \\
&\leq b \frac{L^{n+1}}{n!} \frac{|t - t_0|^{n+1}}{n+1} \Big|_{s=t_0}^{s=t} \\
&\leq b \frac{(L|t - t_0|)^{n+1}}{(n+1)!}
\end{aligned}$$

and thus (1.6) holds for $k = 1, \ldots$.

Thus, for $N > n$ we have

$$\|\phi_N(t) - \phi_n(t)\| \leq \sum_{k=n}^{N-1} \|\phi_{k+1}(t) - \phi_k(t)\| \leq \sum_{k=n}^{N-1} b \frac{(L|t - t_0|)^k}{k!} \leq b \sum_{k=n}^{N-1} \frac{(L\alpha)^k}{k!}$$

The rightmost term in this expression tends to zero as $n \to \infty$. Therefore, $\{\phi_k(t)\}$ converges uniformly to a function $\phi(t)$ on the interval $|t - t_0| \leq \alpha$. As the convergence is uniform, the limit function is continuous. Moreover $\phi(t_0) = x_0$. Indeed, $\phi_N(t) = \phi_0(t) + \sum_{k=1}^{N}(\phi_k(t) - \phi_{k-1}(t))$, so $\phi(t) = \phi_0(t) + \sum_{k=1}^{\infty}(\phi_k(t) - \phi_{k-1}(t))$.

The fact that ϕ is a solution of (1.3) follows from the following result. If a sequence of functions $\{\phi_k(t)\}$ converges uniformly and that the $\phi_k(t)$ are continuous on the interval $|t - t_0| \leq \alpha$, then

$$\lim_{n\to\infty} \int_{t_0}^{t} \phi_n(s)ds = \int_{t_0}^{t} \lim_{n\to\infty} \phi_n(s)ds$$

Hence,

$$\begin{aligned}
\phi(t) &= \lim_{n\to\infty} \phi_n(t) \\
&= x_0 + \lim_{n\to\infty} \int_{t_0}^{t} f(s, \phi_{n-1}(s))ds \\
&= x_0 + \int_{t_0}^{t} \lim_{n\to\infty} f(s, \phi_{n-1}(s))ds \\
&= x_0 + \int_{t_0}^{t} f(s, \phi(s))ds,
\end{aligned}$$

which is to say that

$$\phi(t) = x_0 + \int_{t_0}^{t} f(s, \phi(s))ds \quad \text{for } |t - t_0| \leq \alpha.$$

As the integrand $f(t, \phi)$ is a continuous function, ϕ is differentiable (with respect to t), and $\phi'(t) = f(t, \phi(t))$, so ϕ is a solution to the IVP (1.3).

Uniqueness. Let ϕ and ψ be two solutions of (1.3), *i.e.*, for $|t - t_0| \leq \alpha$,

$$\begin{aligned}
\phi(t) &= x_0 + \int_{t_0}^{t} f(s, \phi(s))ds \\
\psi(t) &= x_0 + \int_{t_0}^{t} f(s, \psi(s))ds.
\end{aligned}$$

Then, for $|t - t_0| \leq \alpha$,

$$\begin{aligned}
\|\phi(t) - \psi(t)\| &= \left\| \int_{t_0}^{t} f(s, \phi(s)) - f(s, \psi(s))ds \right\| \\
&\leq L \int_{t_0}^{t} \|\phi(s) - \psi(s)\| ds. \qquad (1.7)
\end{aligned}$$

We now apply Gronwall's Lemma A.7) to this inequality, using $K = 0$ and $g(t) = \|\phi(t) - \psi(t)\|$. First, applying the lemma for $t_0 \leq t \leq t_0 + \alpha$, we get $0 \leq \|\phi(t) - \psi(t)\| \leq 0$, that is,

$$\|\phi(t) - \psi(t)\| = 0,$$

and thus $\phi(t) = \psi(t)$ for $t_0 \leq t \leq t_0 + \alpha$. Similarly, for $t_0 - \alpha \leq t \leq t_0$, $\|\phi(t) - \psi(t)\| = 0$. Therefore, $\phi(t) = \psi(t)$ on $|t - t_0| \leq \alpha$. □

Example – Let us consider the IVP $x' = -x$, $x(0) = x_0 = c$, $c \in \mathbb{R}$. For initial solution, we choose $\phi_0(t) = c$. Then

$$\begin{aligned}\phi_1(t) &= x_0 + \int_0^t f(s, \phi_0(s))ds \\ &= c + \int_0^t -\phi_0(s)ds \\ &= c - c\int_0^t ds \\ &= c - ct.\end{aligned}$$

To find ϕ_2, we use ϕ_1 in (1.4).

$$\begin{aligned}\phi_2(t) &= x_0 + \int_0^t f(s, \phi_1(s))ds \\ &= c - \int_0^t (c - cs)ds \\ &= c - ct + c\frac{t^2}{2}.\end{aligned}$$

Continuing this method, we find a general term of the form

$$\phi_n(t) = \sum_{i=0}^{n} c\frac{(-1)^i t^i}{i!}.$$

This is the power series expansion of ce^{-t}, so $\phi_n \to \phi = ce^{-t}$ (and the approximation is valid on $\mathbb{R}$), which is the solution of the initial value problem. ◇

Note that the method of successive approximations is a very general method that can be used in a much more general context; see [8, p. 264-269].

1.2.4 Local existence (non Lipschitz case)

The following theorem is often called Peano's existence theorem. Because the vector field is not assumed to be Lipschitz, something is lost, namely, uniqueness.

Theorem 1.2.5 (Peano)**.** *Suppose that f is continuous on some region*

$$\mathcal{R} = \{(t, x) : \ |t - t_0| \leq a, \ \|x - x_0\| \leq b\},$$

with $a, b > 0$, and let $M = \max_{\mathcal{R}} \|f(t, x)\|$. Then there exists a continuous function $\phi(t)$, differentiable on $\mathcal{R}$, such that

i) $\phi(t_0) = x_0$,

ii) $\phi'(t) = f(t, \phi)$ *on* $|t - t_0| \leq \alpha$*, where*

$$\alpha = \begin{cases} a & \text{if } M = 0 \\ \min\left(a, \frac{b}{M}\right) & \text{if } M > 0. \end{cases}$$

Before we can prove this result, we need a certain number of preliminary notations and results. The definition of *equicontinuity* and a statement of the Ascoli lemma are given in Section A.5. To construct a solution without the Lipschitz condition, we approximate the differential equation by another one that does satisfy the Lipschitz condition. The unique solution of such an approximate problem is an ε*-approximate solution.* It is formally defined as follows [8, p. 285].

Definition 1.2.6 ($\boldsymbol{\varepsilon}$-approximate solution)**.** *A differentiable mapping* u *of an open ball* $J \in \mathcal{I}$ *into* $\mathcal{U}$ *is an approximate solution of* $x' = f(t, x)$ *with approximation* ε *(or an* ε*-approximate solution) if we have*

$$\|u'(t) - f(t, u(t))\| \leq \varepsilon,$$

for any $t \in J$*.*

Lemma 1.2.7. *Suppose that* $f(t, x)$ *is continuous on a region*

$$\mathcal{R} = \{(t, x) : \ |t - t_0| \leq a, \ \|x - x_0\| \leq b\}.$$

Then, for every positive number ε*, there exists a function* $F_\varepsilon(t, x)$ *such that*

i) F_ε *is continuous for* $|t - t_0| \leq a$ *and all* x*,*

ii) F_ε *has continuous partial derivatives of all orders with respect to* $x_1, \ldots, x_n$ *for* $|t - t_0| \leq a$ *and all* x*,*

iii) $\|F_\varepsilon(t, x)\| \leq \max_{\mathcal{R}} \|f(t, x)\| = M$ *for* $|t - t_0| \leq a$ *and all* x*,*

iv) $\|F_\varepsilon(t, x) - f(t, x)\| \leq \varepsilon$ *on* $\mathcal{R}$*.*

See a proof in [12, p. 10-12]; note that in this proof, the property that f defines a differential equation is not used. Hence Lemma 1.2.7 can be used in a more general context than that of differential equations. We now prove Theorem 1.2.5.

Proof of Theorem 1.2.5. The proof takes four steps.

1. We construct, for every positive number ε, a function $F_\varepsilon(t, x)$ that satisfies the requirements given in Lemma 1.2.7. Using an existence-uniqueness result in the Lipschitz case (such as Theorem 1.2.2), we construct a function $\phi_\varepsilon(t)$ such that

(P1) $\phi_\varepsilon(t_0) = x_0$,

(P2) $\phi'_\varepsilon(t) = F_\varepsilon(t, \phi_\varepsilon(t))$ on $|t - t_0| < \alpha$.

(P3) $(t, \phi_\varepsilon(t)) \in \mathcal{R}$ on $|t - t_0| \leq \alpha$.

2. The set $\mathcal{F} = \{\phi_\varepsilon : \varepsilon > 0\}$ is bounded and equicontinuous on $|t - t_0| \leq \alpha$. Indeed, property (P3) of ϕ_ε implies that $\mathcal{F}$ is bounded on $|t - t_0| \leq \alpha$ and that $\|F_\varepsilon(t, \phi_\varepsilon(t))\| \leq M$ on $|t - t_0| \leq \alpha$. Hence property (P2) of ϕ_ε implies that

$$\|\phi_\varepsilon(t_1) - \phi_\varepsilon(t_2)\| \leq M|t_1 - t_2|,$$

if $|t_1 - t_0| \leq \alpha$ and $|t_2 - t_0| \leq \alpha$ (this is a consequence of Theorem 1.1.7). Therefore, for a given positive number μ, we have $\|\phi_\varepsilon(t_1) - \phi_\varepsilon(t_2)\| \leq \mu$ whenever $|t_1 - t_0| \leq \alpha, |t_2 - t_0| \leq \alpha$, and $|t_1 - t_2| \leq \mu/M$.

3. Using Lemma A.5, choose a sequence $\{\varepsilon_k : k = 1, 2, \ldots\}$ of positive numbers such that $\lim_{k\to\infty} \varepsilon_k = 0$ and that the sequence $\{\phi_{\varepsilon_k} : k = 1, 2, \ldots\}$ converges uniformly on $|t - t_0| \leq \alpha$ as $k \to \infty$. Then set

$$\phi(t) = \lim_{k\to\infty} \phi_{\varepsilon_k}(t) \text{ on } |t - t_0| \leq \alpha.$$

4. Observe that

$$\begin{aligned}\phi_\varepsilon(t) &= x_0 + \int_{t_0}^{t} F_\varepsilon(s, \phi_\varepsilon(s))ds \\ &= x_0 + \int_{t_0}^{t} f(s, \phi_\varepsilon(s))ds + \int_{t_0}^{t} F_\varepsilon(s, \phi_\varepsilon(s)) - f(s, \phi_\varepsilon(s))ds,\end{aligned}$$

and that it follows from iv) in Lemma 1.2.7 that

$$\left\| \int_{t_0}^{t} F_\varepsilon(s, \phi_\varepsilon(s)) - f(s, \phi_\varepsilon(s))ds \right\| \leq \varepsilon|t - t_0| \text{ on } |t - t_0| \leq \alpha.$$

This is true for all $\varepsilon \geq 0$, so letting $\varepsilon \to 0$, we obtain

$$\phi(t) = x_0 + \int_{t_0}^{t} f(s, \phi(s))ds,$$

which completes the proof. □

See [12, p. 13-14] for the outline of two other proofs of this result. A proof, by Hartman [11, p. 10-11], now follows.

Proof. Let $\delta > 0$ and $\phi_\ell(t)$ be a C^1 n-dimensional vector-valued function on $[t_0 - \delta, t_0]$ satisfying $\phi_\ell(t_0) = x_0$, $\phi_\ell'(t_0) = f(t_0, x_0)$ and $\|\phi_\ell(t) - x_0\| \leq b$, $\|\phi_\ell'(t)\| \leq M$. For $0 < \varepsilon \leq \delta$, define a function $\phi_\varepsilon(t)$ on $[t_0 - \delta, t_0 + \alpha]$ by putting $\phi_\varepsilon(t) = \phi_\ell(t)$ on $[t_0 - \delta, t_0]$ and

$$\phi_\varepsilon(t) = x_0 + \int_{t_0}^{t} f(s, \phi_\varepsilon(s - \varepsilon))ds \text{ on } [t_0, t_0 + \alpha]. \tag{1.8}$$

The function ϕ_ε can indeed be thus defined on $[t_0 - \delta, t_0 + \alpha]$. To see this, remark first that this formula is meaningful and defines $\phi_\varepsilon(t)$ for $t_0 \leq t \leq t_0 + \alpha_1$, $\alpha_1 = \min(\alpha, \varepsilon)$, so that $\phi_\varepsilon(t)$ is C^1 on $[t_0 - \delta, t_0 + \alpha_1]$ and, on this interval,

$$\|\phi_\varepsilon(t) - x_0\| \leq b, \qquad \|\phi_\varepsilon(t) - \phi_\varepsilon(s)\| \leq M|t - s|. \tag{1.9}$$

It then follows that (1.8) can be used to extend $\phi_\varepsilon(t)$ as a C^1 function over $[t_0-\delta, t_0+\alpha_2]$, where $\alpha_2 = \min(\alpha, 2\varepsilon)$, satisfying relation (1.9). Continuing in this fashion, (1.8) serves to define $\phi_\varepsilon(t)$ over $[t_0, t_0+\alpha]$ so that $\phi_\varepsilon(t)$ is a C^0 function on $[t_0-\delta, t_0+\alpha]$, satisfying relation (1.9).

Since $\|\phi'_\varepsilon(t)\| \leq M$, M can be used as a Lipschitz constant for ϕ_ε, giving uniform continuity of ϕ_ε. It follows that the family of functions, $\phi_\varepsilon(t)$, $0 < \varepsilon \leq \delta$, is equicontinuous. Thus, using Ascoli's Lemma (Lemma A.5), there exists a sequence $\varepsilon(1) > \varepsilon(2) > \ldots$, such that $\varepsilon(n) \to 0$ as $n \to \infty$ and

$$\phi(t) = \lim_{n\to\infty} \phi_{\varepsilon(n)}(t) \text{ exists uniformly}$$

on $[t_0-\delta, t_0+\alpha]$. The continuity of f implies that $f(t, \phi_{\varepsilon(n)}(t-\varepsilon(n))$ tends uniformly to $f(t,\phi(t))$ as $n \to \infty$; thus term-by-term integration of (1.8) where $\varepsilon = \varepsilon(n)$ gives

$$\phi(t) = x_0 + \int_{t_0}^t f(s, \phi(s))ds$$

and thus $\phi(t)$ is a solution of (1.3). □

An important corollary follows.

Corollary 1.2.8. *Let $f(t,x)$ be continuous on an open set E and satisfy $\|f(t,x)\| \leq M$. Let E_0 be a compact subset of E. Then there exists an $\alpha = \alpha(E, E_0, M) > 0$ with the property that if $(t_0, x_0) \in E_0$, then the IVP (1.3) has a solution and every solution exists on $|t-t_0| \leq \alpha$.*

In fact, hypotheses can be relaxed a little. Coddington and Levinson [5] define an ε-approximate solution as

Definition 1.2.9. *An ε-approximate solution of the differential equation* (1.2)*, where f is continuous, on a t interval I is a function $\phi \in C$ on I such that*

i) $(t,\phi(t)) \in \mathcal{U}$ *for* $t \in \mathcal{I}$*;*

ii) $\phi \in C^1$ *on I, except possibly for a finite set of points S on I, where ϕ' may have simple discontinuities (g has finite discontinuities at c if the left and right limits of g at c exist but are not equal);*

iii) $\|\phi'(t) - f(t,\phi(t))\| \leq \varepsilon$ *for* $t \in I \setminus S$*.*

Hence it is assumed that ϕ has a piecewise continuous derivative on I, which is denoted by $\phi \in C^1_p(I)$.

Theorem 1.2.10. *Let $f \in C$ on the rectangle*

$$\mathcal{R} = \{(t,x) : |t-t_0| \leq a, \|x-x_0\| \leq b\}.$$

Given any $\varepsilon > 0$, there exists an ε-approximate solution ϕ of (1.3) *on $|t-t_0| \leq \alpha$ such that $\phi(t_0) = x_0$.*

Proof. Let $\varepsilon > 0$ be given. We construct an ε-approximate solution on the interval $[t_0, t_0+\varepsilon]$; the construction works in a similar way for $[t_0 - \alpha, t_0]$. The ε-approximate solution that we construct is a polygonal path starting at (t_0, x_0).

Since $f \in C$ on $\mathcal{R}$, it is uniformly continuous on $\mathcal{R}$, and therefore for the given value of ε, there exists $\delta_\varepsilon > 0$ such that

$$\|f(t,\phi) - f(\tilde{t},\tilde{\phi})\| \leq \varepsilon \tag{1.10}$$

if

$$(t,\phi) \in \mathcal{R}, (\tilde{t},\tilde{\phi}) \in \mathcal{R} \text{ and } |t-\tilde{t}| \leq \delta_\varepsilon \quad \|\phi - \tilde{\phi}\| \leq \delta_\varepsilon.$$

Now divide the interval $[t_0, t_0+\alpha]$ into n parts $t_0 < t_1 < \cdots < t_n = t_0 + \alpha$, in such a way that

$$\max |t_k - t_{k-1}| \leq \min\left(\delta_\varepsilon, \frac{\delta_\varepsilon}{M}\right). \tag{1.11}$$

From (t_0, x_0), construct a line segment with slope $f(t_0, x_0)$ intercepting the line $t = t_1$ at (t_1, x_1). From the definition of α and M, it is clear that this line segment lies inside the triangular region T bounded by the lines segments with slopes $\pm M$ from (t_0, x_0) to their intercept at $t = t_0 + \alpha$, and the line $t = t_0 + \alpha$. In particular, $(t_1, x_1) \in T$.

At the point (t_1, x_1), construct a line segment with slope $f(t_1, x_1)$ until the line $t = t_2$, obtaining the point (t_2, x_2). Continuing similarly, a polygonal path ϕ is constructed that meets the line $t = t_0 + \alpha$ in a finite number of steps, and lies entirely in T.

The function ϕ, which can be expressed as

$$\begin{aligned} \phi(t_0) &= x_0 \\ \phi(t) &= \phi(t_{k-1}) + f(t_{k-1}, \phi(t_{k-1}))(t - t_{k-1}), \quad t \in [t_{k-1}, t_k],\ k = 1, \ldots, n, \end{aligned} \tag{1.12}$$

is the ε-approximate solution that we seek. Clearly, $\phi \in C_p^1([t_0, t_0+\alpha])$ and

$$\|\phi(t) - \phi(\tilde{t})\| \leq M|t - \tilde{t}| \text{ for } t, \tilde{t} \in [t_0, t_0 + \alpha]. \tag{1.13}$$

If $t \in [t_{k-1}, t_k]$, then (1.13) together with (1.11) imply that $\|\phi(t) - \phi(t_{k-1})\| \leq \delta_\varepsilon$. But from (1.12) and (1.10),

$$\|\phi'(t) - f(t,\phi(t))\| = \|f(t_{k-1}, \phi(t_{k-1})) - f(t, \phi(t))\| \leq \varepsilon.$$

Therefore, ϕ is an ε-approximation. □

We can now turn to their proof of Theorem 1.2.5.

Proof. Let $\{\varepsilon_n\}$ be a monotone decreasing sequence of positive real numbers with $\varepsilon_n \to 0$ as $n \to \infty$. By Theorem 1.2.10, for each ε_n, there exists an ε_n-approximate solution ϕ_n of (1.3) on $|t - t_0| \leq \alpha$ such that $\phi_n(t_0) = x_0$. Choose one such solution ϕ_n for each ε_n. From (1.13), it follows that

$$\|\phi_n(t) - \phi_n(\tilde{t})\| \leq M|t - \tilde{t}|. \tag{1.14}$$

Applying (1.14) to $\tilde{t} = t_0$, it is clear that the sequence $\{\phi_n\}$ is uniformly bounded by $\|x_0\| + b$, since $|t - t_0| \leq b/M$. Moreover, (1.14) implies that $\{\phi_n\}$ is an equicontinuous set. By Ascoli's lemma (Lemma A.5), there exists a subsequence $\{\phi_{n_k}\}$, $k = 1, \ldots,$ of $\{\phi_n\}$, converging uniformly on $[t_0 - \alpha, t_0 + \alpha]$ to a limit function ϕ, which must be continuous since each ϕ_n is continuous.

This limit function ϕ is a solution to (1.3) which meets the required specifications. To see this, write

$$\phi_n(t) = x_0 + \int_{t_0}^{t} f(s, \phi_n(s)) + \Delta_n(s)ds, \tag{1.15}$$

where $\Delta_n(t) = \phi'(t) - f(t, \phi_n(t))$ at those points where ϕ_n' exists, and $\Delta_n(t) = 0$ otherwise. Because ϕ_n is an ε_n-approximate solution, $\|\Delta_n(t)\| \leq \varepsilon_n$. Since f is uniformly continuous on $\mathcal{R}$, and $\phi_{n_k} \to \phi$ uniformly on $[t_0 - \alpha, t_0 + \alpha]$ as $k \to \infty$, it follows that $f(t, \phi_{n_k}) \to f(t, \phi(t))$ uniformly on $[t_0 - \alpha, t_0 + \alpha]$ as $k \to \infty$.

Replacing n by n_k in (1.15) and letting $k \to \infty$ gives

$$\phi(t) = x_0 + \int_{t_0}^{t} f(s, \phi(s))ds. \tag{1.16}$$

Clearly, $\phi(t_0) = 0$, when evaluated using (1.16), and also $\phi'(t) = f(t, \phi(t))$ since f is continuous. Thus ϕ as defined by (1.16) is a solution to (1.3) on $|t - t_0| \leq \alpha$. $\square$

1.2.5 Some examples of existence and uniqueness

Example – Consider the IVP

$$\begin{aligned} x' &= 3|x|^{\frac{2}{3}} \\ x(t_0) &= x_0 \end{aligned} \tag{1.17}$$

Here, Theorem 1.2.5 applies, since $f(t, x) = 3x^{2/3}$ is continuous. However, Theorem 1.2.2 does not apply, since $f(t, x)$ is not locally Lipschitz in $x = 0$ (or, f is not Lipschitz on any interval containing 0). This means that we have existence of solutions to this IVP, but not uniqueness of the solution.

The fact that f is not Lipschitz on any interval containing 0 is established using the following argument. Suppose that f is Lipschitz on an interval $\mathcal{I} = (-\varepsilon, \varepsilon)$, with $\varepsilon > 0$. Then, there exists $L > 0$ such that for all $x_1, x_2 \in \mathcal{I}$,

$$\|f(t, x_1) - f(t, x_2)\| \leq L|x_1 - x_2|$$

that is,

$$3\left||x_1|^{\frac{2}{3}} - |x_2|^{\frac{2}{3}}\right| \leq L|x_1 - x_2|$$

Since this has to hold true for all $x_1, x_2 \in \mathcal{I}$, it must hold true in particular for $x_2 = 0$. Thus

$$3|x_1|^{\frac{2}{3}} \leq L|x_1|$$

Given an $\varepsilon > 0$, it is possible to find $N_\varepsilon > 0$ such that $\frac{1}{n} < \varepsilon$ for all $n \geq N_\varepsilon$. Let $x_1 = \frac{1}{n}$. Then for $n \geq N_\varepsilon$, if f is Lipschitz there must hold

$$3\left(\frac{1}{n}\right)^{\frac{2}{3}} \leq \frac{L}{n}$$

So, for all $n \geq N_\varepsilon$,

$$n^{\frac{1}{3}} \leq \frac{L}{3}$$

This is a contradiction, since $\lim_{n\to\infty} n^{1/3} = \infty$, and so f is not Lipschitz on $\mathcal{I}$.

Let us consider the set

$$E = \{t \in \mathbb{R} : \ x(t) = 0\}$$

The set E can be have several forms, depending on the situation.

1) $E = \emptyset$,
2) $E = [a, b]$, (closed since x is continuous and thus reaches its bounds),
3) $E = (-\infty, b)$,
4) $E = (a, +\infty)$,
5) $E = \mathbb{R}$.

Note that case 2) includes the case of a single intersection point, when $a = b$, giving $E = \{a\}$. Let us now consider the nature of x in these different situations. Recall that from Theorem 1.1.10, since (1.17) is defined by a scalar autonomous equation, its solutions are monotone. For simplicity, we consider here the case of monotone increasing solutions. The case of monotone decreasing solutions can be treated in a similar fashion.

1) Here, there is no intersection with the $x = 0$ axis. Thus if follows that

$$x(t) \text{ is } \begin{cases} > 0, & \text{if } x_0 > 0 \\ < 0, & \text{if } x_0 < 0 \end{cases}$$

2) In this case,

$$x(t) \text{ is } \begin{cases} < 0, & \text{if } t < a \\ = 0, & \text{if } t \in [a, b] \\ > 0, & \text{if } t > b \end{cases}$$

3) Here,

$$x(t) \text{ is } \begin{cases} = 0, & \text{if } t < b \\ > 0, & \text{if } t > b \end{cases}$$

4) In this case,

$$x(t) \text{ is } \begin{cases} < 0, & \text{if } t < a \\ = 0, & \text{if } t > a \end{cases}$$

5) In this last case, $x(t) = 0$ for all $t \in \mathbb{R}$.

Now, depending on the sign of x, we can integrate the equation. First, if $x > 0$, then $|x| = x$ and so

$$\begin{aligned} & x' = 3x^{2/3} \\ \Leftrightarrow\ & \frac{1}{3}x^{-2/3}x' = 1 \\ \Leftrightarrow\ & x^{1/3} = t + k_1 \\ \Leftrightarrow\ & x(t) = (t + k_1)^3 \end{aligned}$$

for $k_1 \in \mathbb{R}$. Then, if $x < 0$, then $|x| = -x$, and

$$\begin{aligned} x' &= 3\,(-x)^{2/3} \\ \Leftrightarrow \frac{1}{3}(-x)^{-2/3}(-x') &= -1 \\ \Leftrightarrow (-x)^{1/3} &= -t + k_2 \\ \Leftrightarrow x(t) &= -(-t+k_2)^3 \end{aligned}$$

for $k_2 \in \mathbb{R}$. We can now use these computations with the different cases that were discussed earlier, depending on the value of t_0 and x_0. We begin with the case of $t_0 > 0$ and $x_0 > 0$.

1) The case $E = \emptyset$ is impossible, for all initial conditions (t_0, x_0). Indeed, as $x_0 > 0$, we have $x(t) = (t + k_1)^3$. Using the initial condition, we find that $x(t_0) = x_0 = (t_0 + k_1)^3$, *i.e.*, $k_1 = x_0^{1/3} - t_0$, and $x(t) = (t + x_0^{1/3} - t_0)^3$.

2) If $E = [a, b]$, then

$$x(t) = \begin{cases} -(-t+k_2)^3 \text{ if } t < a \\ 0 \text{ if } t \in [a,b] \\ (t+k_1)^3 \text{ if } t > b \end{cases}$$

Since $x_0 > 0$, we have to be in the $t > b$ region, so $t_0 > b$, and $(t_0 + k_1)^3 = x_0$, which implies that $k_1 = x_0^{1/3} - t_0$. Thus

$$x(t) = \begin{cases} -(-t+k_2)^3 \text{ if } t < a \\ 0 \text{ if } t \in [a,b] \\ (t+x_0^{1/3}-t_0)^3 \text{ if } t > b \end{cases}$$

Since x is continuous,

$$\lim_{t\to b, t>b} (t + x_0^{1/3} - t_0)^3 = 0$$

and

$$\lim_{t\to a, t<a} -(-t+k_2)^3 = 0$$

This implies that $b = t_0 - x_0^{1/3}$ and $k_2 = a$. So finally,

$$x(t) = \begin{cases} -(-t+a)^3 \text{ if } t < a \\ 0 \text{ if } t \in [a, t_0 - x_0^{\frac{1}{3}}] \qquad (a \le t_0 - x_0^{\frac{1}{3}}) \\ (t + x_0^{1/3} - t_0)^3 \text{ if } t > t_0 - x_0^{\frac{1}{3}} \end{cases}$$

Thus, choosing $a \le t_0 - x_0^{1/3}$, we have solutions of the form shown in Figure 1.4. Indeed, any a_i satisfying this property yields a solution.

3) The case $[a, +\infty)$ is impossible. Indeed, there does not exist a solution through (t_0, x_0) such that $x(t) = 0$ for all $t \in [a, +\infty)$; since we are in the case of monotone increasing functions, if $x_0 > 0$ then $x(t) \ge x_0$ for all $t \ge t_0$.

4) $E = \mathbb{R}$ is also impossible, for the same reason.

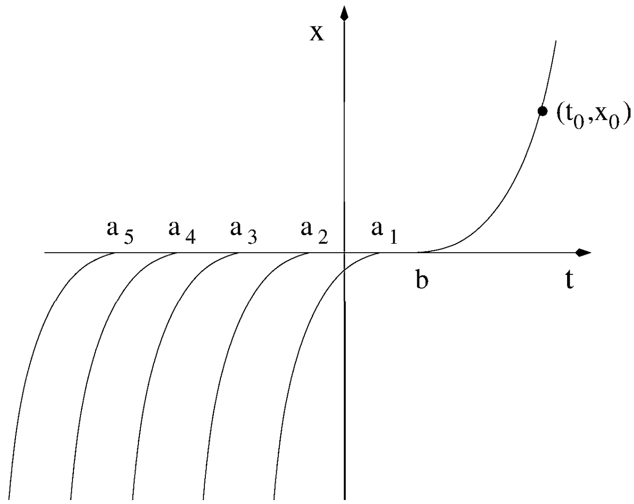

Figure 1.4: Case $t_0, x_0 > 0$, subcase 2, in the resolution of (1.17).

5) For the case $E = (-\infty, b]$, we have

$$x(t) = \begin{cases} 0 \text{ if } t \in (-\infty, b] \\ (t + k_1)^3 \text{ if } t > b \end{cases}$$

Since $x(t_0) = x_0$, $k_1 = x_0^{1/3} - t_0$, and since x is continuous, $b = -k_1 = t_0 - x_0^{1/3}$. So,

$$x(t) = \begin{cases} 0 \text{ if } t \in (-\infty, t_0 - x_0^{1/3}] \\ (t + x_0^{1/3} - t_0)^3 \text{ if } t > t_0 - x_0^{1/3} \end{cases}$$

The other cases are left as an exercise. ⋄

Example – Consider the IVP

$$\begin{aligned} x' &= 2tx^2 \\ x(0) &= 0 \end{aligned} \tag{1.18}$$

Here, we have existence and uniqueness of the solutions to (1.18). Indeed, $f(t, x) = 2tx^2$ is continuous and locally Lipschitz on $\mathbb{R}$. ⋄

1.3 Continuation of solutions

The results we have seen so far deal with the local existence (and uniqueness) of solutions to an IVP, in the sense that solutions are shown to exist in a neighborhood of the initial data. The *continuation* of solutions consists in studying criteria which allow to define solutions on possibly larger intervals.

Consider the IVP

$$\begin{aligned} x' &= f(t, x) \\ x(t_0) &= x_0, \end{aligned} \tag{1.19}$$

with f continuous on a domain $\mathcal{U}$ of the (t,x) space, and the initial point $(t_0,x_0)\in\mathcal{U}$.

Lemma 1.3.1. *Let the function $f(t,x)$ be continuous in an open set $\mathcal{U}$ in (t,x)-space, and assume that a function $\phi(t)$ satisfies the condition $\phi'(t)=f(t,\phi(t))$ and $(t,\phi(t))\in\mathcal{U}$, in an open interval $\mathcal{I}=\{t_1<t<t_2\}$. Under this assumption, if $\lim_{j\to\infty}(\tau_j,\phi(\tau_j))=(t_1,\eta)\in\mathcal{U}$ for some sequence $\{\tau_j:\ j=1,2,\ldots\}$ of points in the interval $\mathcal{I}$, then $\lim_{\tau\to t_1}(\tau,\phi(\tau))=(t_1,\eta)$. Similarly, if $\lim_{j\to\infty}(\tau_j,\phi(\tau_j))=(t_2,\eta)\in\mathcal{U}$ for some sequence $\{\tau_j:\ j=1,2,\ldots\}$ of points in the interval $\mathcal{I}$, then $\lim_{\tau\to t_2}(\tau,\phi(\tau))=(t_2,\eta)$.*

Proof. Let $\mathcal{W}$ be an open neighborhood of (t_1,η). Then $(t,\phi(t))\in\mathcal{W}$ in an interval $\tau_1<t<\tau(\mathcal{W})$ for some $\tau(\mathcal{W})$ determined by $\mathcal{W}$. Indeed, assume that the closure of $\mathcal{W}$, $\bar{\mathcal{W}}\subset\mathcal{U}$, and that $|f(t,x)|\leq M$ in $\mathcal{W}$ for some positive number M. For every positive integer j and every positive number ε, consider a rectangular region

$$\mathcal{R}_j(\varepsilon)=\{(t,x):\ |t-t_j|\leq\varepsilon,\ \|x-\phi(t_j)\|\leq M\varepsilon\}$$

Then there exists an $\varepsilon>0$ and a j such that $(\tau_1,\eta)\in R_j(\varepsilon)\subset\mathcal{W}$, with $\varepsilon=\min\left(\varepsilon,\frac{M\varepsilon}{M}\right)$ and $t_j-\varepsilon\leq\tau_1$.

From Theorem 1.1.6 applied to the solution ϕ of the IVP $x'=f(t,x)$, $x(\tau_j)=\phi(\tau_j)$, we obtain that $(\tau,\phi(\tau))\in\mathcal{R}_j(\varepsilon)\in\mathcal{U}$ on the interval $t_1<\tau\leq\tau_j$. Since $\mathcal{U}$ is an arbitrary open neighborhood of (t_1,η), we conclude that $\lim_{j\to\infty}(\tau_j,\phi(\tau_j))=(t_1,\eta)\in\mathcal{R}$. □

From the previous result, we can derive a result concerning the maximal interval over which a solution can be extended. To emphasize the fact that the solution ϕ of an ODE exists in some interval $\mathcal{I}$, we denote $(\phi,\mathcal{I})$. We need the notion of extension of a solution. It is defined in the classical manner (see Figure 1.5).

Definition 1.3.2 (Extension)**.** *Let $(\phi,\mathcal{I})$ and $(\tilde{\phi},\tilde{\mathcal{I}})$ be two solutions of the same ODE. We say that $(\tilde{\phi},\tilde{\mathcal{I}})$ is an* extension *of $(\phi,\mathcal{I})$ if, and only if,*

$$\mathcal{I}\subset\tilde{\mathcal{I}},\quad \tilde{\phi}_{|\mathcal{I}}=\phi$$

where $_{|\mathcal{I}}$ denotes the restriction to $\mathcal{I}$.

Theorem 1.3.3. *Let $f(t,x)$ be continuous in an open set $\mathcal{U}$ in (t,x)-space, and the function $\phi(t)$ be a function satisfying the condition $\phi'(t)=f(t,\phi(t))$ and $(t,\phi(t))\in\mathcal{U}$, in an open interval $\mathcal{I}=\{t_1<t<t_2\}$. If the following two conditions are satisfied:*

i) $\phi(t)$ cannot be extended to the left of t_1 (or, respectively, to the right of t_2),

ii) $\lim_{j\to\infty}(\tau_j,\phi(\tau_j))=(t_1,\eta)$ (or, respectively, (t_2,η)) exists for some sequence $\{\tau_j:\ j=1,2,\ldots\}$ of points in the interval $\mathcal{I}$,

then the limit point (t_1,η) (or, respectively, (t_2,η)) must be on the boundary of $\mathcal{U}$.

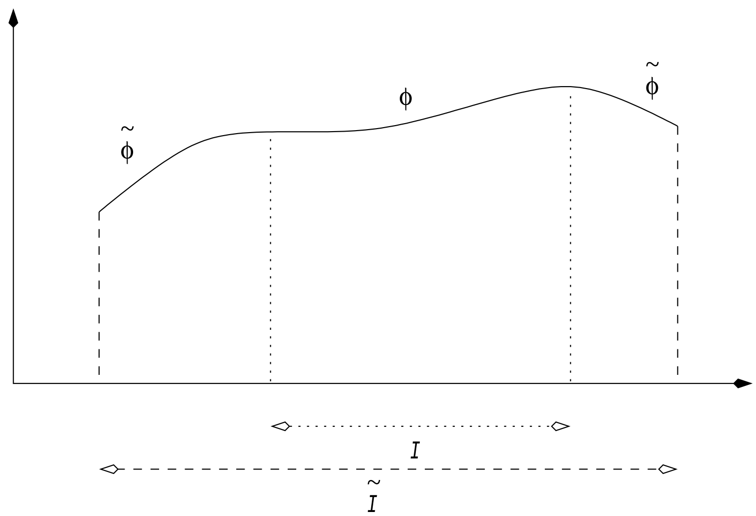

Figure 1.5: The extension $\tilde{\phi}$ on the interval $\tilde{\mathcal{I}}$ of the solution ϕ (defined on the interval $\mathcal{I}$).

Proof. Suppose that the hypotheses of the theorem are satisfied, and that $(t_1, \eta) \in \mathcal{U}$ (respectively, $(t_2, \eta) \in \mathcal{U}$). Then, from Lemma 1.3.1, it follows that

$$\lim_{\tau \to t_1} (\tau, \phi(\tau)) = (t_1, \eta)$$

(or, respectively, $\lim_{\tau \to t_2}(\tau, \phi(\tau)) = (t_2, \eta)$). Thus we can apply Theorem 1.2.5 (Peano's Theorem) to the IVP

$$\begin{aligned} x' &= f(t, x) \\ x(t_1) &= \eta, \end{aligned}$$

(or, respectively, $x' = f(t, x)$, $x(t_2) = \eta$). This implies that the solution ϕ can be extended to the left of t_1 (respectively, to the right of t_2), since Theorem 1.2.5 implies existence in a neighborhood of t_1. This is a contradiction. □

A particularly important consequence of the previous theorem is the following corollary.

Corollary 1.3.4. *Assume that $f(t, x)$ is continuous for $t_1 < t < t_2$ and all $x \in \mathbb{R}^n$. Also, assume that there exists a function $\phi(t)$ satisfying the following conditions:*

a) ϕ and ϕ' are continuous in a subinterval $\mathcal{I}$ of the interval $t_1 < t < t_2$,

b) $\phi'(t) = f(t, \phi(t))$ in $\mathcal{I}$.

Then, either

i) $\phi(t)$ can be extended to the entire interval $t_1 < t < t_2$ as a solution of the differential equation $x' = f(t, x)$, or

ii) $\lim_{t \to \tau} \|\phi(t)\| = \infty$ for some τ in the interval $t_1 < t < t_2$.

1.3.1 Maximal interval of existence

Another way of formulating these results is with the notion of maximal intervals of existence. Consider the differential equation

$$x' = f(t, x) \tag{1.20}$$

Let $x = x(t)$ be a solution of (1.20) on an interval $\mathcal{I}$.

Definition 1.3.5 (Right maximal interval of existence). *The interval $\mathcal{I}$ is a* right maximal interval *of existence for x if there does not exist an extension of $x(t)$ over an interval $\mathcal{I}_1$ so that x remains a solution of (1.20), with $\mathcal{I} \subset \mathcal{I}_1$ (and $\mathcal{I}$ and $\mathcal{I}_1$ having different right endpoints). A left maximal interval of existence is defined in a similar way.*

Definition 1.3.6 (Maximal interval of existence). *An interval which is both a left and a right maximal interval of existence is called a* maximal interval *of existence.*

Theorem 1.3.7. *Let $f(t, x)$ be continuous on an open set $\mathcal{U}$ and $\phi(t)$ be a solution of (1.20) on some interval. Then $\phi(t)$ can be extended (as a solution) over a maximal interval of existence (ω_-, ω_+). Also, if (ω_-, ω_+) is a maximal interval of existence, then $\phi(t)$ tends to the boundary $\partial\mathcal{U}$ of $\mathcal{U}$ as $t \to \omega_-$ and $t \to \omega_+$.*

Remark – The extension need not be unique, and $\omega_\pm$ depends on the extension. Also, to say, for example, that $\phi \to \partial\mathcal{U}$ as $t \to \omega_+$ is interpreted to mean that either $\omega_+ = \infty$ or that $\omega_+ < \infty$ and if $\mathcal{U}^0$ is any compact subset of $\mathcal{U}$, then $(t, \phi(t)) \notin \mathcal{U}^0$ when t is near ω_+. ◦

Two interesting corollaries, from [11].

Corollary 1.3.8. *Let $f(t, x)$ be continuous on a strip $t_0 \leq t \leq t_0 + a$ $(< \infty)$, $x \in \mathbb{R}^n$ arbitrary. Let ϕ be a solution of (1.3) on a right maximal interval J. Then either*

i) $J = [t_0, t_0 + a]$,

ii) or $J = [t_0, \delta)$, $\delta \leq t_0 + a$, *and* $\|\phi(t)\| \to \infty$ *as* $t \to \delta$.

Corollary 1.3.9. *Let $f(t, x)$ be continuous on the closure $\bar{\mathcal{U}}$ of an open (t, x)-set $\mathcal{U}$, and let (1.3) possess a solution ϕ on a maximal right interval J. Then either*

i) $J = [t_0, \infty)$,

ii) or $J = [t_0, \delta)$, *with* $\delta < \infty$ *and* $(\delta, \phi(\delta)) \in \partial\mathcal{U}$,

iii) or $J = [t_0, \delta)$ *with* $\delta < \infty$ *and* $\|\phi(t)\| \to \infty$ *as* $t \to \delta$.

1.3.2 Maximal and global solutions

Linked to the notion of maximal intervals of existence of solutions is the notion of maximal and global solutions.

Definition 1.3.10 (Maximal solution). *Let $\mathcal{I}_1 \subset \mathbb{R}$ and $\mathcal{I}_2 \subset \mathbb{R}$ be two intervals such that $\mathcal{I}_1 \subset \mathcal{I}_2$. A solution $(\phi, \mathcal{I}_1)$ is* maximal *in $\mathcal{I}_2$ if ϕ has no extension $(\tilde{\phi}, \tilde{\mathcal{I}})$ solution of the ODE such that $\mathcal{I}_1 \subsetneq \tilde{\mathcal{I}} \subset \mathcal{I}_2$.*

Definition 1.3.11 (Global solution). *A solution $(\phi, \mathcal{I}_1)$ is* global *on $\mathcal{I}_2$ if ϕ admits a extension $\tilde{\phi}$ defined on the whole interval $\mathcal{I}_2$.*

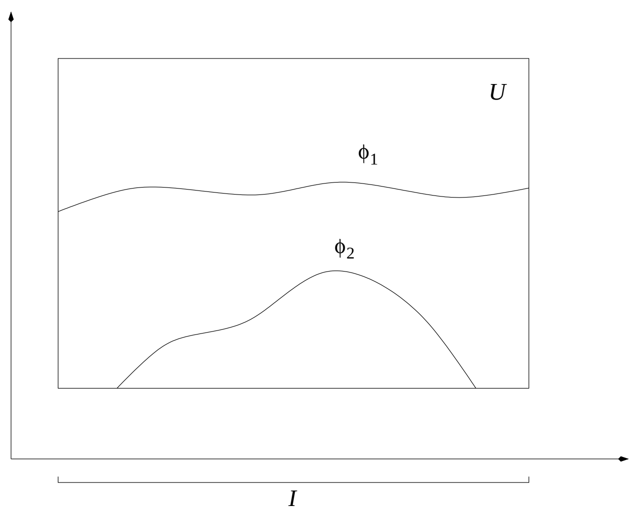

Figure 1.6: ϕ_1 is a global and maximal solution on $\mathcal{I}$; ϕ_2 is a maximal solution on $\mathcal{I}$, but it is not global on $\mathcal{I}$.

Every global solution on a given interval $\mathcal{I}$ is maximal on that same interval. The converse is false.

Example – Consider the equation $x' = -2tx^2$ on $\mathbb{R}$. If $x \neq 0$, $x'x^{-2} = -2t$, which implies that $x(t) = 1/(t^2 - c)$, with $c \in \mathbb{R}$. Depending on c, there are several cases.

- if $c < 0$, then $x(t) = 1/(t^2 - c)$ is a global solution on $\mathbb{R}$,
- if $c > 0$, the solutions are defined on $(-\infty, -\sqrt{c})$, $(-\sqrt{c}, \sqrt{c})$ and $(\sqrt{c}, \infty)$. The solutions are maximal solutions on $\mathbb{R}$, but are not global solutions.
- if $c = 0$, then the maximal non global solutions on $\mathbb{R}$ are defined on $(-\infty, 0)$ and $(0, \infty)$.

Another solution is $x \equiv 0$, which is a global solution on $\mathbb{R}$. ◇

Lemma 1.3.12. *Every solution ϕ of the differential equation $x' = f(t, x)$ is contained in a maximal solution $\tilde{\phi}$.*

The following theorem extends the uniqueness property to an interval of existence of the solution.

Theorem 1.3.13. *Let $\phi_1, \phi_2 : \mathcal{I} \to \mathbb{R}^n$ be two solutions of the equation $x' = f(t,x)$, with f locally Lipschitz in x on $\mathcal{U}$. If ϕ_1 and ϕ_2 coincide at a point $t_0 \in \mathcal{I}$, then $\phi_1 = \phi_2$ on $\mathcal{I}$.*

Proof. Under the assumptions of the theorem, $\phi_1(t_0) = \phi_2(t_0)$. Suppose that there exists a t_1, $t_1 \neq t_0$, such that $\phi_1(t_1) \neq \phi_2(t_1)$. For simplicity, let us assume that $t_1 > t_0$.

By the local uniqueness of the solution, it follows from $\phi_1(t_0) = \phi_2(t_0)$ that there exists a neighborhood $\mathcal{N}$ of t_0 such that $\phi_1(t) = \phi_2(t)$ for all $t \in \mathcal{N}$. Let

$$E = \{t \in [t_0, t_1] : \ \phi_1(t) \neq \phi_2(t)\}$$

Since $t_1 \in E$, $E \neq \emptyset$. Let $\alpha = \inf(E)$, we have $\alpha \in (t_0, t_1]$, and for all $t \in [t_0, \alpha)$, $\phi_1(t) = \phi_2(t)$.

By continuity of ϕ_1 and ϕ_2, we thus have $\phi_1(\alpha) = \phi_2(\alpha)$. This implies that there exists a neighborhood $\mathcal{W}$ of α on which $\phi_1 = \phi_2$. This is a contradiction, since $\phi_1(t) \neq \phi_2(t)$ for $t > \alpha$, hence there exists no such t_1, and $\phi_1 = \phi_2$ on $\mathcal{I}$. $\square$

Corollary 1.3.14 (Global uniqueness)**.** *Let $f(t,x)$ be locally Lipschitz in x on $\mathcal{U}$. Then by any point $(t_0, x_0) \in \mathcal{U}$, there passes a unique maximal solution $\phi : \mathcal{I} \to \mathbb{R}^n$. If there exists a global solution on $\mathcal{I}$, then it is unique.*

1.4 Continuous dependence on initial data, on parameters

Let ϕ be a solution of (1.3). To emphasize the fact the this solution depends on the initial condition (t_0, x_0), we denote it ϕ_{t_0,x_0}. Let η be a parameter of (1.3). When we study the dependence of ϕ_{t_0,x_0} on η, we denote the solution as $\phi_{t_0,x_0,\eta}$.

We suppose that $\|f(t,x)\| \leq M$ and $|\partial f(t,x)/\partial x_i| \leq K$ for $i = 1, \ldots, n$ for $(t,x) \in \mathcal{U}$, with $\mathcal{U} \in \mathbb{R} \times \mathbb{R}^n$. Note that these conditions are automatically satisfied on a closed bounded region of the form $\mathcal{R} = \{(t,x) : \ |t - t_0| \leq a, \|x - x_0\| \leq b\}$, where $a, b > 0$.

Our objective here is to characterize the nature of the dependence of the solution on the initial time t_0 and the initial data x_0.

Theorem 1.4.1. *Suppose that f and $\partial f/\partial x$ are continuous and bounded in a given region $\mathcal{U}$. Let ϕ_{t_0,x_0} be a solution of (1.3) passing through (t_0, x_0) and $\psi_{\hat{t}_0,\hat{x}_0}$ be a solution of (1.3) passing through $(\hat{t}_0, \hat{x}_0)$. Suppose that ϕ and ψ exist on some interval $\mathcal{I}$.*

Then, for each $\varepsilon > 0$, there exists $\delta > 0$ such that if $|t - \hat{t}| < \delta$ and $\|x_0 - \hat{x}_0\| < \delta$, then $\|\phi(t) - \psi(\hat{t})\| < \varepsilon$, for $t, \hat{t} \in \mathcal{I}$.

Proof. The prooof is from [2, p. 135-136]. Since ϕ is the solution of (1.3) through the point (t_0, x_0), we have, for all $t \in \mathcal{I}$,

$$\phi(t) = x_0 + \int_{t_0}^{t} f(s, \phi(s))ds \tag{1.21}$$

As ψ is the solution of (1.3) through the point $(\tilde{t}_0, \tilde{x}_0)$, we have, for all $t \in \mathcal{I}$,

$$\psi(t) = \tilde{x}_0 + \int_{\tilde{t}_0}^{t} f(s, \psi(s))ds \tag{1.22}$$

Since

$$\int_{t_0}^{t} f(s, \phi(s))ds = \int_{t_0}^{\tilde{t}_0} f(s, \phi(s))ds + \int_{\tilde{t}_0}^{t} f(s, \phi(s))ds,$$

substracting (1.22) from (1.21) gives

$$\phi(t) - \psi(t) = x_0 - \tilde{x}_0 + \int_{t_0}^{\tilde{t}_0} f(s, \phi(s))ds + \int_{\tilde{t}_0}^{t} f(s, \phi(s)) - f(s, \psi(s))ds$$

and therefore

$$\|\phi(t) - \psi(t)\| \leq \|x_0 - \tilde{x}_0\| + \left\|\int_{t_0}^{\tilde{t}_0} f(s, \phi(s))ds\right\| + \left\|\int_{\tilde{t}_0}^{t} f(s, \phi(s)) - f(s, \psi(s))ds\right\|$$

Using the boundedness assumptions on f and $\partial f/\partial x$ to evaluate the right hand side of the latter inequation, we obtain

$$\|\phi(t) - \psi(t)\| \leq \|x_0 - \tilde{x}_0\| + M|\tilde{t}_0 - t_0| + K\left\|\int_{\tilde{t}_0}^{t} \phi(s) - \psi(s)ds\right\|$$

If $|t_0 - \tilde{t}_0| < \delta$, $\|x_0 - \tilde{x}_0\| < \delta$, then we have

$$\|\phi(t) - \psi(t)\| \leq \delta + M\delta + K\left\|\int_{\tilde{t}_0}^{t} \phi(s) - \psi(s)ds\right\| \tag{1.23}$$

Applying Gronwall's inequality (Appendix A.7) to (1.23) gives

$$\|\phi(t) - \psi(t)\| \leq \delta(1 + M)e^{K|t-\tilde{t}_0|} \leq \delta(1 + M)e^{K(\tau_2-\tau_1)}$$

using the fact that $|t - \tilde{t}_0| < \tau_2 - \tau_1$, if we denote $\mathcal{I} = (\tau_1, \tau_2)$. Since

$$\|\psi(t) - \psi(\tilde{t})\| < \left\|\int_{\tilde{t}}^{t} f(s, \psi(s))ds\right\| \leq M|t - \tilde{t}| \leq M\delta$$

if $|t - \tilde{t}| < \delta$, we have

$$\|\phi(t) - \psi(\tilde{t})\| \leq \|\phi(t) - \psi(t)\| + \|\psi(t) - \psi(\tilde{t})\| \leq \delta(1 + M)e^{K(\tau_2-\tau_1)} + \delta M$$

Now, given $\varepsilon > 0$, we need only choose $\delta < \varepsilon/[M + (1 + M)^{K(\tau_2-\tau_1)}]$ to obtain the desired inequality, completing the proof. □

What we have shown is that the solution passing through the point (t_0, x_0) is a continuous function of the triple (t, t_0, x_0). We now consider the case where the parameters also vary, comparing solutions to two different but "close" equations.

Theorem 1.4.2. *Let f, g be defined in a domain $\mathcal{U}$ and satisfy the hypotheses of Theorem 1.4.1. Let ϕ and ψ be solutions of $x' = f(t,x)$ and $x' = g(t,x)$, respectively, such that $\phi(t_0) = x_0$, $\psi(t_0) = \hat{x}_0$, existing on a common interval $\alpha < t < \beta$. Suppose that $\|f(t,x) - g(t,x)\| \leq \varepsilon$ for $(t,x) \in \mathcal{U}$. Then the solutions ϕ and ψ satisfy*

$$\|\phi(t) - \psi(t)\| \leq \|x_0 - \hat{x}_0\| e^{K|t-t_0|} + \varepsilon(\beta - \alpha)e^{K|t-t_0|}$$

for all t, $\alpha < t < \beta$.

The following theorem [6, p. 58] is less restrictive in its hypotheses than the previous one, requiring only uniqueness of the solution of the IVP.

Theorem 1.4.3. *Let $\mathcal{U}$ be a domain of (t,x) space, $\mathcal{I}_\mu$ the domain $|\mu - \mu_0| < c$, with $c > 0$, and $\mathcal{U}_\mu$ the set of all (t,x,μ) satisfying $(t,x) \in \mathcal{U}$, $\mu \in \mathcal{I}_\mu$. Suppose f is a continuous function on U_μ, bounded by a constant M there. For $\mu = \mu_0$, let*

$$\begin{aligned} x' &= f(t,x,\mu) \\ x(t_0) &= x_0 \end{aligned} \tag{1.24}$$

have a unique solution ϕ_0 on the interval $[a,b]$, where $t_0 \in [a,b]$. Then there exists a $\delta > 0$ such that, for any fixed μ such that $|\mu - \mu_0| < \delta$, every solution ϕ_μ of (1.24) exists over $[a,b]$ and as $\mu \to \mu_0$

$$\phi_\mu \to \phi_0$$

uniformly over $[a,b]$.

Proof. We begin by considering $t_0 \in (a,b)$. First, choose an $\alpha > 0$ small enough that the region $\mathcal{R} = \{|t - t_0| \leq \alpha,\ \|x - x_0\| \leq M\alpha\}$ is in $\mathcal{U}$; note that $\mathcal{R}$ is a slight modification of the usual security domain. All solutions of (1.24) with $\mu \in \mathcal{I}_\mu$ exist over $[t_0 - \alpha, t_0 + \alpha]$ and remain in $\mathcal{R}$. Let ϕ_μ denote a solution. Then the set of functions $\{\phi_\mu\}$, $\mu \in \mathcal{I}_\mu$ is an uniformly bounded and equicontinuous set in $|t - t_0| \leq \alpha$. This follows from the integral equation

$$\phi_\mu(t) = x_0 + \int_{t_0}^t f(s, \phi_\mu(s), \mu)ds \quad (|t - t_0| \leq \alpha) \tag{1.25}$$

and the inequality $\|f\| \leq M$.

Suppose that for some $\tilde{t} \in [t_0 - \alpha, t_0 + \alpha]$, $\phi_\mu(\tilde{t})$ does not tend to $\phi_0(\tilde{t})$. Then there exists a sequence $\{\mu_k\}$, $k = 1, 2, \ldots$, for which $\mu_k \to \mu_0$, and corresponding solutions ϕ_{μ_k} such that ϕ_{μ_k} converges uniformly over $[t_0 - \alpha, t_0 + \alpha]$ as $k \to \infty$ to a limit function ψ, with $\psi(\tilde{t}) \neq \phi_0(\tilde{t})$. From the fact that $f \in C$ on $\mathcal{U}_\mu$, that $\psi \in C$ on $[t_0 - \alpha, t_0 + \alpha]$, and that ϕ_{μ_k} converges uniformly to ψ, (1.25) for the solutions ϕ_{μ_k} yields

$$\psi(t) = x_0 + \int_{t_0}^t f(s, \psi(s), \mu_0)ds \quad (|t - t_0| \leq \alpha)$$

Thus ψ is a solution of (1.24) with $\mu = \mu_0$. By the uniqueness hypothesis, it follows that $\psi(t) = \phi_0(t)$ on $|t - t_0| \leq \alpha$. Thus $\psi(\tilde{t}) = \phi_0(\tilde{t})$. Thus all solutions ϕ_μ on $|t - t_0| \leq \alpha$ tend to ϕ_0 as $\mu \to \mu_0$. Because of the equicontinuity, the convergence is uniform.

Let us now prove that the result holds over $[a, b]$. For this, let us consider the interval $[t_0, b]$. Let $\tau \in [t_0, b)$, and suppose that the result is valid for every small $h > 0$ over $[t_0, \tau - h]$ but not over $[t_0, \tau + h]$. It is clear that $\tau \geq t_0 + \alpha$. By the above assumption, for any small $\varepsilon > 0$, there exists a $\delta_\varepsilon > 0$ such that

$$\|\phi_\mu(\tau - \varepsilon) - \phi_0(\tau - \varepsilon)\| < \varepsilon \tag{1.26}$$

for $|\mu - \mu_0| < \delta_\varepsilon$. Let $\mathcal{H} \subset \mathcal{U}$ be defined as the region

$$\mathcal{H} = \{|t - \tau| \leq \gamma, \quad \|x - \phi_0(\tau - \gamma)\| \leq \gamma + M|t - \tau + \gamma|\}$$

with γ small enough that $\mathcal{H} \subset \mathcal{U}$. Any solution of $x' = f(t, x, \mu)$ starting on $t = \tau - \gamma$ with initial value ξ_0, $|\xi_0 - \phi_0(\tau - \gamma)| \leq \gamma$ will remain in $\mathcal{H}$ as t increases. Thus all solutions can be continued to $\tau + \gamma$.

By choosing $\varepsilon = \gamma$ in (1.26), it follows that for $|\mu - \mu_0| < \delta_\varepsilon$, the solutions ϕ_μ can all be continued to $\tau + \varepsilon$. Thus over $[t_0, \tau + \varepsilon]$ these solutions are in $\mathcal{U}$ so that the argument that $\phi_\mu \to \phi_0$ which has been given for $|t - t_0| \leq \alpha$, also applies over $[t_0, \tau + \varepsilon]$. Thus the assumption about the existence of $\tau < b$ is false. The case $\tau = b$ is treated in similar fashion on $\tau - \gamma \leq t \leq \tau$.

A similar argument applies to the left of t_0 and therefore the result is valid over $[a, b]$. □

Definition 1.4.4 (Well-posedness)**.** *A problem is said to be* well-posed *if solutions exist, are unique, and that there is continuous dependence on initial conditions.*

1.5 Generality of first order systems

Consider an nth order differential equation in normal form

$$x^{(n)} = f\left(t, x, x', \ldots, x^{(n-1)}\right) \tag{1.27}$$

This equation can be reduced to a system of n first order ordinary differential equations, by proceeding as follows. Let $y_0 = x, y_1 = x', y_2 = x'', \ldots, y_{n-1} = x^{(n)}$. Then (1.27) is equivalent to

$$y' = F(t, y) \tag{1.28}$$

with $y = (y_0, y_1, \ldots, y_{n-1})^T$ and

$$F(t, z) = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_{n-1} \\ f(t, y_0, \ldots, y_{n-1}) \end{bmatrix}$$

Similarly, the IVP associated to (1.27) is given by

$$\begin{aligned} x^{(n)} &= f\left(t, x, x', \ldots, x^{(n-1)}\right) \\ x(t_0) = x_0,\ x'(t_0) &= x_1,\ \ldots, x^{(n-1)}(t_0) = x_{n-1} \end{aligned} \tag{1.29}$$

is equivalent to the IVP

$$\begin{aligned} y' &= F(t, y) \\ y(t_0) &= y_0 = (x_0, \ldots, x_{n-1})^T \end{aligned} \tag{1.30}$$

As a consequence, all results in this chapter are true for equations of order higher than 1.

Example – Consider the second order IVP

$$\begin{aligned} x'' &= -2x' + 4x - 3 \\ x(0) &= 2, x'(0) = 1 \end{aligned}$$

To transform it into a system of first-order differential equations, we let $y = x'$. Substituting (where possible) y for x' in the equation gives

$$y' = -2y + 4x - 3$$

The initial condition becomes $x(0) = 2, y(0) = 1$. So finally, the following IVP is equivalent to the original one:

$$\begin{aligned} x' &= y \\ y' &= 4x - 2y - 3 \\ x(0) &= 2, y(0) = 1 \end{aligned}$$

Note that the linearity of the initial problem is preserved. ◇

Example – The differential equation

$$x^{(n)}(t) = a_{n-1}(t)x^{(n-1)}(t) + \cdots + a_1(t)x'(t) + a_0(t)x(t) + b(t)$$

is an nth order nonhomogeneous linear differential equation. Together with the initial condition

$$x^{(n-1)}(t_0) = x_0^{(n-1)}, \ldots, x'(t_0) = x_0', x(t_0) = x_0$$

where $x_0, x_0', \ldots, x_0^{(n-1)} \in \mathbb{R}$, it forms an IVP. We can transform it to a system of linear first order equations by setting

$$\begin{aligned} y_0 &= x \\ y_1 &= x' \\ &\vdots \\ y_{n-1} &= x^{(n-1)} \\ y_n &= x^{(n)} \end{aligned}$$

The nth order linear equation is then equivalent to the following system of n first order linear equations

$$\begin{aligned}
y_0' &= y_1 \\
y_1' &= y_2 \\
&\vdots \\
y_{n-2}' &= y_{n-1} \\
y_{n-1}' &= y_n \\
y_n' &= a_{n-1}(t)y_n(t) + a_{n-2}(t)y_{n-1}(t) + \cdots + a_1(t)y_1(t) + a_0(t)y_0(t) + b(t)
\end{aligned}$$

under the initial conditions

$$y_{n-1}(t_0) = x_0^{(n-1)}, \ldots, y_1(t_0) = x_0', y_0(t_0) = x_0$$

◇

1.6 Generality of autonomous systems

A nonautonomous system

$$x'(t) = f(t, x(t))$$

can be transformed into an autonomous system of equations by setting an auxiliary variable, say y, equal to t, giving

$$\begin{aligned}
x' &= f(y, x) \\
y' &= 1.
\end{aligned}$$

However, this transformation does not always make the system any easier to study.

1.7 Suggested reading, Further problems

Most of these results are treated one way or another in Coddington and Levinson [6] (first edition published in 1955), and the current text, as many others, does little but paraphrase them.

We have not seen here any results specific to complex valued differential equations. As complex numbers are two-dimensional real vectors, the results carry through to the complex case by simply assuming that if, in (1.2), we consider an n-dimensional complex vector, then this is equivalent to a $2n$-dimensional problem. Furthermore, if $f(t, x)$ is analytic in t and x, then analytic solutions can be constructed. See Section I-4 in [12], ..., for example.

Chapter 2

Linear systems

Let $\mathcal{I}$ be an interval of $\mathbb{R}$, E a normed vector space over a field $\mathbb{K}$ ($E = \mathbb{K}^n$, with $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$), and $\mathcal{L}(E)$ the space of continuous linear maps from E to E. Let $\|\ \|$ be a norm on E, and $|||\ |||$ be the induced supremum norm on $\mathcal{L}(E)$ (see Appendix A.1). Consider a map $A : \mathcal{I} \to \mathcal{L}(E)$ and a map $B : \mathcal{I} \to E$. A linear system of first order equations is defined by

$$x'(t) = A(t)x(t) + B(t) \tag{2.1}$$

where the unknown x is a map on $\mathcal{I}$, taking values in E, defined differentiable on a subinterval of $\mathcal{I}$. We restrict ourselves to the finite dimensional case ($E = \mathbb{K}^n$). Hence we consider $A \in \mathcal{M}_n(\mathbb{K})$, $n \times n$ matrices over the field $\mathbb{K}$, and $B \in \mathbb{K}^n$. We suppose that A and B have continuous entries. In most of what follows, we assume $\mathbb{K} = \mathbb{R}$.

The name *linear* for system (2.1) is an abuse of language. System (2.1) should be called an *affine* system, with associated linear system

$$x'(t) = A(t)x(t). \tag{2.2}$$

Another way to distinguish systems (2.1) and (2.2) is to refer to the former as a *nonhomogeneous* linear system and the latter as an *homogeneous* linear system. In order to lighten the language, since there will be other qualificatives added to both (2.1) and (2.2), we use in this chapter the names *affine system* for (2.1) and *linear system* for (2.2).

The exception to this naming convention is that we refer to (2.1) as a linear system if we consider the generic properties of (2.1), with (2.2) as a particular case, as in this chapter's title or in the next section, for example.

2.1 Existence and uniqueness of solutions

Theorem 2.1.1. *Let A and B be defined and continuous on $\mathcal{I} \ni t_0$. Then, for all $x_0 \in E$, there exists a unique solution $\phi_t(x_0)$ of (2.1) through (t_0, x_0), defined on the interval $\mathcal{I}$.*

Proof. Let $k(t) = \|\!|A(t)|\!\| = \sup_{\|x\|\leq 1} \|A(t)x\|$. Then for all $t \in \mathcal{I}$ and all $x_1, x_2 \in \mathbb{K}$,

$$\begin{aligned}\|f(t,x_1) - f(t,x_2)\| &= \|A(t)(x_1 - x_2)\| \\ &\leq \|\!|A(t)|\!\| \, \|x_1 - x_2\| \\ &\leq k(t)\|x_1 - x_2\|,\end{aligned}$$

where the inequality

$$\|A(t)(x_1 - x_2)\| \leq \|\!|A(t)|\!\| \, \|x_1 - x_2\|$$

results from the nature of the norm $\|\!| \; |\!\|$ (see Appendix A.1). Furthermore, k is continuous on $\mathcal{I}$. Therefore the conditions of Theorem 1.2.2 hold, leading to existence and uniqueness on the interval $\mathcal{I}$. □

With linear systems, it is possible to extend solutions easily, as is shown by the next theorem.

Theorem 2.1.2. *Suppose that the entries of $A(t)$ and the entries of $B(t)$ are continuous on an open interval $\mathcal{I}$. Then every solution of (2.1) which is defined on a subinterval $\mathcal{J}$ of the interval $\mathcal{I}$ can be extended uniquely to the entire interval $\mathcal{I}$ as a solution of (2.1).*

Proof. Suppose that $\mathcal{I} = (t_1, t_2)$, and that a solution ϕ of (2.1) is defined on $\mathcal{J} = (\tau_1, \tau_2)$, with $\mathcal{J} \subsetneq \mathcal{I}$. Then

$$\|\phi(t)\| \leq \|\phi(t_0)\| + \left\| \int_{t_0}^t A(s)\phi(s) + B(s)ds \right\|$$

for all $t \in \mathcal{J}$, where $t_0 \in \mathcal{J}$. Let

$$\begin{cases} K = \|\phi(t_0)\| + (\tau_2 - \tau_1) \max_{\tau_1 \leq t \leq \tau_2} \|B(t)\| \\ L = \max_{\tau_1 \leq t \leq \tau_2} \|A(t)\| \end{cases}$$

Then, for $t_0, t \in \mathcal{J}$,

$$\|\phi(t)\| \leq K + L \left\| \int_{t_0}^t \phi(s)ds \right\| \leq K + L \int_{t_0}^t \|\phi(s)\| ds.$$

Thus, using Gronwall's Lemma (Lemma A.7), the following estimate holds in $\mathcal{J}$,

$$\|\phi(t)\| \leq K e^{L|t-t_0|} \leq K e^{L(\tau_2 - \tau_1)} < \infty$$

This implies that case ii) in Corollary 1.3.4 is ruled out, leaving only the possibility for ϕ to be extendable over $\mathcal{I}$, since the vector field in (2.1) is Lipschitz. □

2.2 Linear systems

We begin our study of linear systems of ordinary differential equations by considering homogeneous systems of the form (2.2) (linear systems), with $x \in \mathbb{R}^n$ and $A \in \mathcal{M}_n(\mathbb{R})$, the set of square matrices over the field $\mathbb{R}$, A having continuous entries on an interval $\mathcal{I}$.

2.2.1 The vector space of solutions

Theorem 2.2.1 (Superposition principle). *Let $\mathcal{S}^0$ be the set of solutions of (2.2) that are defined on some interval $\mathcal{I} \subset \mathbb{R}$. Let $\phi_1, \phi_2 \in \mathcal{S}^0$, and $\lambda_1, \lambda_2 \in \mathbb{R}$. Then $\lambda_1\phi_1 + \lambda_2\phi_2 \in \mathcal{S}^0$.*

Proof. Let $\phi_1, \phi_2 \in \mathcal{S}^0$ be two solutions of (2.2), $\lambda_1, \lambda_2 \in \mathbb{R}$. Then for all $t \in \mathcal{I}$,

$$\begin{aligned}\phi_1' &= A(t)\phi_1\\ \phi_2' &= A(t)\phi_2,\end{aligned}$$

from which it comes that

$$\frac{d}{dt}(\lambda_1\phi_1 + \lambda_2\phi_2) = A(t)[\lambda_1\phi_1 + \lambda_2\phi_2],$$

implying that $\lambda_1\phi_1 + \lambda_2\phi_2 \in \mathcal{S}^0$. □

Thus the linear combination of any two solutions of (2.2) is in $\mathcal{S}^0$. This is a hint that $\mathcal{S}^0$ must be a vector space of dimension n on $\mathbb{K}$. To show this, we need to find a basis of $\mathcal{S}^0$. We proceed in the classical manner, with the notable difference from classical linear algebra that the basis is here composed of time-dependent functions.

Definition 2.2.2 (Fundamental set of solutions). *A set of n solutions of the linear differential equation* (2.2)*, all defined on the same open interval $\mathcal{I}$, is called a* fundamental set *of solutions on $\mathcal{I}$ if the solutions are linearly independent functions on $\mathcal{I}$.*

Proposition 2.2.3. *If $A(t)$ is defined and continuous on the interval $\mathcal{I}$, then the system (2.2) has a fundamental set of solutions defined on $\mathcal{I}$.*

Proof. Let $t_0 \in \mathcal{I}$, and $e_1, \ldots, e_n$ denote the canonical basis of $\mathbb{K}^n$. Then, from Theorem 2.1.1, there exists a unique solution $\phi(t_0) = (\phi_1(t_0), \ldots, \phi_n(t_0))$ such that $\phi_i(t_0) = e_i$, for $i = 1, \ldots, n$. Furthermore, from Theorem 2.1.1, each function ϕ_i is defined on the interval $\mathcal{I}$.

Assume that $\{\phi_i\}$, $i = 1, \ldots, n$, is linearly dependent. Then there exists $\alpha_i \in \mathbb{R}$, $i = 1, \ldots, n$, not all zero, such that $\sum_{i=1}^n \alpha_i\phi_i(t) = 0$ for all t. In particular, this is true for $t = t_0$, and thus $\sum_{i=1}^n \alpha_i\phi_i(t_0) = \sum_{i=1}^n \alpha_i e_i = 0$, which implies that the canonical basis of $\mathbb{K}^n$ is linearly dependent. Hence a contradiction, and the ϕ_i are linearly independent. □

Proposition 2.2.4. *If $\mathcal{F}$ is a fundamental set of solutions of the linear system (2.2) on the open interval $\mathcal{I}$, then every solution defined on $\mathcal{I}$ can be expressed as a linear combination of the elements of $\mathcal{F}$.*

Let $t_0 \in \mathcal{I}$, we consider the application

$$\begin{aligned}\Phi_{t_0} : \mathcal{S}^0 &\to \mathbb{K}^n\\ Y &\mapsto \Phi_{t_0}(x) = x(t_0)\end{aligned}$$

Lemma 2.2.5. Φ_{t_0} *is a linear isomorphism.*

Proof. Φ_{t_0} is bijective. Indeed, let $v \in \mathbb{K}^n$, from Theorem 2.1.1, there exists a unique solution passing through (t_0, v), *i.e.*,

$$\forall v \in \mathbb{K}^n,\ \exists! x \in \mathcal{S}^0,\ \ x(t_0) = v \Rightarrow \Phi_{t_0}(x) = v,$$

so Φ_{t_0} is surjective. That Φ_{t_0} is injective follows from uniqueness of solutions to an ODE. Furthermore, $\Phi_{t_0}(\lambda_1 x_1 + \lambda_2 x_2) = \lambda_1 \Phi_{t_0}(x_1) + \lambda_2 \Phi_{t_0}(x_2)$. Therefore $\dim \mathcal{S}^0 = \dim \mathbb{K}^n = n$. □

2.2.2 Fundamental matrix solution

Definition 2.2.6. *An $n \times n$ matrix function $t \mapsto \Phi(t)$, defined on an open interval $\mathcal{I}$, is called a* matrix solution *of the homogeneous linear system (2.2) if each of its columns is a (vector) solution. A matrix solution Φ is called a* fundamental matrix solution *if its columns form a fundamental set of solutions. If in addition $\Phi(t_0) = I$, a fundamental matrix solution is called the* principal fundamental matrix solution.

An important property of fundamental matrix solutions is the following, known as Abel's formula.

Theorem 2.2.7 (Abel's formula)**.** *Let $A(t)$ be continuous on $\mathcal{I}$ and $\Phi \in \mathcal{M}_n(\mathbb{K})$ be such that $\Phi'(t) = A(t)\Phi(t)$ on $\mathcal{I}$. Then $\det \Phi$ satisfies on $\mathcal{I}$ the differential equation*

$$(\det \Phi)' = (\mathrm{tr} A)(\det \Phi),$$

or, in integral form, for $t, \tau \in \mathcal{I}$,

$$\det \Phi(t) = \det \Phi(\tau) \exp\left(\int_\tau^t \mathrm{tr} A(s) ds \right). \tag{2.3}$$

Proof. Writing the differential equation $\Phi'(t) = A(t)\Phi(t)$ in terms of the elements φ_{ij} and a_{ij} of, respectively, Φ and A,

$$\varphi'_{ij}(t) = \sum_{k=1}^n a_{ik}(t) \varphi_{kj}(t), \tag{2.4}$$

for $i, j = 1, \ldots, n$. Writing

$$\det \Phi = \begin{vmatrix} \varphi_{11}(t) & \varphi_{12}(t) & \ldots & \varphi_{1n}(t) \\ \varphi_{21}(t) & \varphi_{22}(t) & \ldots & \varphi_{2n}(t) \\ \\ \varphi_{n1}(t) & \varphi_{n2}(t) & \ldots & \varphi_{nn}(t) \end{vmatrix},$$

we see that

$$(\det \Phi)' = \begin{vmatrix} \varphi'_{11} & \varphi'_{12} & \cdots & \varphi'_{1n} \\ \varphi_{21} & \varphi_{22} & \cdots & \varphi_{2n} \\ \\ \varphi_{n1} & \varphi_{n2} & \cdots & \varphi_{nn} \end{vmatrix} + \begin{vmatrix} \varphi_{11} & \varphi_{12} & \cdots & \varphi_{1n} \\ \varphi'_{21} & \varphi'_{22} & \cdots & \varphi'_{2n} \\ \\ \varphi_{n1} & \varphi_{n2} & \cdots & \varphi_{nn} \end{vmatrix} + \cdots + \begin{vmatrix} \varphi_{11} & \varphi_{12} & \cdots & \varphi_{1n} \\ \varphi_{21} & \varphi_{22} & \cdots & \varphi_{2n} \\ \\ \varphi'_{n1} & \varphi'_{n2} & \cdots & \varphi'_{nn} \end{vmatrix}.$$

Indeed, write $\det \Phi(t) = \Gamma(r_1, r_2, \ldots, r_n)$, where r_i is the ith row in $\Phi(t)$. Γ is then a linear function of each of its arguments, if all other rows are constant, which implies that

$$\frac{d}{dt}\det \Phi(t) = \Gamma\left(\frac{d}{dt}r_1, r_2, \ldots, r_n\right) + \Gamma\left(r_1, \frac{d}{dt}r_2, \ldots, r_n\right) + \cdots + \Gamma\left(r_1, r_2, \ldots, \frac{d}{dt}r_n\right).$$

(To show this, use the definition of the derivative as a limit.) Using (2.4) on the first of the n determinants in $(\det \Phi)'$ gives

$$\begin{vmatrix} \sum_k a_{1k}\varphi_{k1} & \sum_k a_{1k}\varphi_{k2} & \ldots & \sum_k a_{1k}\varphi_{kn} \\ \varphi_{21} & \varphi_{22} & \ldots & \varphi_{2n} \\ & & & \\ \varphi_{n1} & \varphi_{n2} & \ldots & \varphi_{nn} \end{vmatrix}.$$

Adding $-a_{12}$ times the second row, $-a_{13}$ times the first row, etc., $-a_{1n}$ times the nth row, to the first row, does not change the determinant, and thus

$$\begin{vmatrix} \sum_k a_{1k}\varphi_{k1} & \sum_k a_{1k}\varphi_{k2} & \ldots & \sum_k a_{1k}\varphi_{kn} \\ \varphi_{21} & \varphi_{22} & \ldots & \varphi_{2n} \\ & & & \\ \varphi_{n1} & \varphi_{n2} & \ldots & \varphi_{nn} \end{vmatrix} = \begin{vmatrix} a_{11}\varphi_{11} & a_{11}\varphi_{12} & \ldots & a_{11}\varphi_{1n} \\ \varphi_{21} & \varphi_{22} & \ldots & \varphi_{2n} \\ & & & \\ \varphi_{n1} & \varphi_{n2} & \ldots & \varphi_{nn} \end{vmatrix} = a_{11}\det \Phi.$$

Repeating this for each of the terms in $(\det \Phi)'$, we obtain $(\det \Phi)' = (a_{11} + a_{22} + \cdots + a_{nn})\det \Phi$, giving finally $(\det \Phi)' = (\mathrm{tr} A)(\det \Phi)$. Note that this equation takes the form $u' - \alpha(t)u = 0$, which implies that

$$u \exp\left(\int_\tau^t \alpha(s)ds\right) = \text{constant},$$

which in turn implies the integral form of the formula. □

Remark – Consider (2.3). Suppose that $\tau \in \mathcal{I}$ is such that $\det \Phi(\tau) \neq 0$. Then, since $e^a \neq 0$ for any a, it follows that $\det \Phi \neq 0$ for all $t \in \mathcal{I}$. In short, *linear independence of solutions for a $t \in \mathcal{I}$ is equivalent to linear independence of solutions for all $t \in \mathcal{I}$.* As a consequence, the column vectors of a fundamental matrix are linearly independent at every $t \in \mathcal{I}$. ◦

Theorem 2.2.8. *A solution matrix Φ of* (2.2) *is a fundamental solution matrix on $\mathcal{I}$ if, and only if,* $\det \Phi(t) \neq 0$ *for all $t \in \mathcal{I}$*

Proof. Let Φ be a fundamental matrix with column vectors ϕ_i, and suppose that ϕ is any nontrivial solution of (2.2). Then there exists $c_1, \ldots, c_n$, not all zero, such that

$$\phi = \sum_{j=1}^n c_j \phi_j,$$

or, writing this equation in terms of Φ, $\phi = \Phi c$, if $c = (c_1, \ldots, c_n)^T$. At any point $t_0 \in \mathcal{I}$, this is a system of n linear equations with n unknowns $c_1, \ldots, c_n$. This system has a unique solution for any choice of $\phi(t_0)$. Thus $\det \Phi(t_0) \neq 0$, and by the remark above, $\det \Phi(t) \neq 0$ for all $t \in \mathcal{I}$.

Reciproqually, let Φ be a solution matrix of (2.2), and suppose that $\det \Phi(t) \neq 0$ for $t \in \mathcal{I}$. Then the column vectors are linearly independent at every $t \in \mathcal{I}$. $\square$

From the remark above, the condition "$\det \Phi(t) \neq 0$ for all $t \in \mathcal{I}$" in Theorem 2.2.8 is equivalent to the condition "there exists $t \in \mathcal{I}$ such that $\det \Phi(t) \neq 0$". A frequent candidate for this role is t_0.

To conclude on fundamental solution matrices, remark that there are infinitely many of them, for a given linear system. However, since each fundamental solution matrix can provide a basis for the vector space of solutions, it is clear that the fundamental matrices associated to a given problem must be linked. Indeed, we have the following result.

Theorem 2.2.9. *Let Φ be a fundamental matrix solution to* (2.2)*. Let $C \in \mathcal{M}_n(\mathbb{K})$ be a constant nonsingular matrix. Then ΦC is a fundamental matrix solution to* (2.2)*. Conversely, if Ψ is another fundamental matrix solution to* (2.2)*, then there exists a constant nonsingular $C \in \mathcal{M}_n(\mathbb{K})$ such that $\Psi(t) = \Phi(t)C$ for all $t \in \mathcal{I}$.*

Proof. Since Φ is a fundamental matrix solution to (2.2), we have

$$(\Phi C)' = \Phi' C = (A(t)\Phi)C = A(t)(\Phi C),$$

and thus ΦC is a matrix solution to (2.2). Since Φ is a fundamental matrix solution to (2.2), Theorem 2.2.8 implies that $\det \Phi \neq 0$. Also, since C is nonsingular, $\det C \neq 0$. Thus, $\det \Phi C = \det \Phi \det C \neq 0$, and by Theorem 2.2.8, ΦC is a fundamental matrix solution to (2.2).

Conversely, assume that Φ and Ψ are two fundamental matrix solutions. Since $\Phi\Phi^{-1} = I$, taking the derivative of this expression gives $\Phi'\Phi^{-1} + \Phi\left(\Phi^{-1}\right)' = 0$, and therefore $\left(\Phi^{-1}\right)' = -\Phi^{-1}\Phi'\Phi^{-1}$. We now consider the product $\Phi^{-1}\Psi$. There holds

$$\begin{aligned}
\left(\Phi^{-1}\Psi\right)' &= \left(\Phi^{-1}\right)'\Psi + \Phi^{-1}\Psi' \\
&= -\Phi^{-1}\Phi'\Phi^{-1}\Psi + \Phi^{-1}A(t)\Psi \\
&= \left(-\Phi^{-1}A(t)\Phi\Phi^{-1} + \Phi^{-1}A(t)\right)\Psi \\
&= \left(-\Phi^{-1}A(t) + \Phi^{-1}A(t)\right)\Psi \\
&= 0.
\end{aligned}$$

Therefore, integrating $(\Phi^{-1}\Psi)'$ gives $\Phi^{-1}\Psi = C$, with $C \in \mathcal{M}_n(\mathbb{K})$ is a constant. Thus, $\Psi = C\Phi$. Furthermore, as Φ and Ψ are fundamental matrix solutions, $\det \Phi \neq 0$ and $\det \Psi \neq 0$, and therefore $\det C \neq 0$. $\square$

Remark – Note that if Φ is a fundamental matrix solution to (2.2) and $C \in \mathcal{M}_n(\mathbb{K})$ is a constant nonsingular matrix, then it is not necessarily true that $C\Phi$ is a fundamental matrix solution to (2.2). See Exercise 2.3. $\circ$

2.2.3 Resolvent matrix

If $t \mapsto \Phi(t)$ is a matrix solution of (2.2) on the interval $\mathcal{I}$, then $\Phi'(t) = A(t)\Phi(t)$ on $\mathcal{I}$. Thus, by Proposition 2.2.3, there exists a fundamental matrix solution.

Definition 2.2.10 (Resolvent matrix)**.** *Let $t_0 \in \mathcal{I}$ and $\Phi(t)$ be a fundamental matrix solution of* (2.2) *on $\mathcal{I}$. Since the columns of Φ are linearly independent, it follows that $\Phi(t_0)$ is invertible. The* resolvent *(or* state transition matrix*) of* (2.2) *is then defined as*

$$\mathcal{R}(t,t_0) = \Phi(t)\Phi(t_0)^{-1}.$$

It is evident that $\mathcal{R}(t,t_0)$ is the principal fundamental matrix solution at t_0 (since $\mathcal{R}(t_0,t_0) = \Phi(t_0)\Phi(t_0)^{-1} = I$). Thus system (2.2) has a principal fundamental matrix solution at each point in $\mathcal{I}$.

Proposition 2.2.11. *The resolvent matrix satisfies the Chapman-Kolmogorov identities*

1) $\mathcal{R}(t,t) = I$,

2) $\mathcal{R}(t,s)\mathcal{R}(s,u) = \mathcal{R}(t,u)$,

as well as the identities

3) $\mathcal{R}(t,s)^{-1} = \mathcal{R}(s,t)$,

4) $\frac{\partial}{\partial s}\mathcal{R}(t,s) = -\mathcal{R}(t,s)A(s)$,

5) $\frac{\partial}{\partial t}\mathcal{R}(t,s) = A(t)\mathcal{R}(t,s)$.

Proof. First, for the Chapman-Kolmogorov identities. 1) is $\mathcal{R}(t,t) = \Phi(t)\Phi^{-1}(t) = I$. Also, 2) gives

$$\mathcal{R}(t,s)\mathcal{R}(s,u) = \Phi(t)\Phi^{-1}(s)\Phi(s)\Phi^{-1}(u) = \Phi(t)\Phi^{-1}(u) = \mathcal{R}(t,u).$$

The other equalities are equally easy to establish. Indeed,

$$\mathcal{R}(t,s)^{-1} = \left(\Phi(t)\Phi^{-1}(s)\right)^{-1} = \left(\Phi^{-1}(s)\right)^{-1}\Phi(t)^{-1} = \Phi(s)\Phi^{-1}(t) = \mathcal{R}(s,t),$$

whence 3). Also,

$$\begin{aligned} \frac{\partial}{\partial s}\mathcal{R}(t,s) &= \frac{\partial}{\partial s}\left(\Phi(t)\Phi^{-1}(s)\right) \\ &= \Phi(t)\left(\frac{\partial}{\partial s}\Phi^{-1}(s)\right) \end{aligned}$$

As Φ is a fundamental matrix solution, Φ' exists and Φ is nonsingular, and differentiating $\Phi\Phi^{-1} = I$ gives

$$\begin{aligned} \frac{\partial}{\partial s}\left(\Phi(s)\Phi^{-1}(s)\right) = 0 &\Leftrightarrow \left(\frac{\partial}{\partial s}\Phi(s)\right)\Phi^{-1}(s) + \Phi(s)\left(\frac{\partial}{\partial s}\Phi^{-1}(s)\right) = 0 \\ &\Leftrightarrow \Phi(s)\left(\frac{\partial}{\partial s}\Phi^{-1}(s)\right) = -\left(\frac{\partial}{\partial s}\Phi(s)\right)\Phi^{-1}(s) \\ &\Leftrightarrow \frac{\partial}{\partial s}\Phi^{-1}(s) = -\Phi^{-1}(s)\left(\frac{\partial}{\partial s}\Phi(s)\right)\Phi^{-1}(s). \end{aligned}$$

Therefore,

$$\frac{\partial}{\partial s}\mathcal{R}(t,s) = -\Phi(t)\Phi^{-1}(s)\left(\frac{\partial}{\partial s}\Phi(s)\right)\Phi^{-1}(s) = -\mathcal{R}(t,s)\left(\frac{\partial}{\partial s}\Phi(s)\right)\Phi^{-1}(s).$$

Now, since $\Phi(s)$ is a fundamental matrix solution, it follows that $\partial\Phi(s)/\partial s = A(s)\Phi(s)$, and thus

$$\frac{\partial}{\partial s}\mathcal{R}(t,s) = -\mathcal{R}(t,s)A(s)\Phi(s)\Phi^{-1}(s) = -\mathcal{R}(t,s)A(s),$$

giving 4). Finally,

$$\begin{aligned}\frac{\partial}{\partial t}\mathcal{R}(t,s) &= \frac{\partial}{\partial t}\Phi(t)\Phi^{-1}(s)\\ &= A(t)\Phi(t)\Phi^{-1}(s) \quad \text{since } \Phi \text{ is a fundamental matrix solution}\\ &= A(t)\mathcal{R}(t,s),\end{aligned}$$

giving 5). □

The role of the resolvent matrix is the following. Recall that, from Lemma 2.2.5, Φ_{t_0} defined by

$$\begin{aligned}\Phi_{t_0}:\ &\mathcal{S}\to\mathbb{K}^n\\ &x\mapsto x(t_0),\end{aligned}$$

is a $\mathbb{K}$-linear isomorphism from the space $\mathcal{S}$ to the space $\mathbb{K}^n$. Then $\mathcal{R}$ is an application from $\mathbb{K}^n$ to $\mathbb{K}^n$,

$$\begin{aligned}\mathcal{R}(t,t_0):\ &\mathbb{K}^n\to\mathbb{K}^n\\ &v\mapsto \mathcal{R}(t,t_0)v = w\end{aligned}$$

such that

$$\mathcal{R}(t,t_0) = \Phi_t\circ\Phi_{t_0}^{-1}$$

i.e.,

$$(\mathcal{R}(t,t_0)v = w) \Leftrightarrow (\exists x\in\mathcal{S},\ w = x(t),\ v = x(t_0)).$$

Since Φ_t and Φ_{t_0} are $\mathbb{K}$-linear isomorphisms, $\mathcal{R}$ is a $\mathbb{K}$-linear isomorphism on $\mathbb{K}^n$. Thus $\mathcal{R}(t,t_0)\in\mathcal{M}_n(\mathbb{K})$ and is invertible.

Proposition 2.2.12. *$\mathcal{R}(t,t_0)$ is the only solution in $\mathcal{M}_n(\mathbb{K})$ of the initial value problem*

$$\begin{aligned}\frac{d}{dt}M(t) &= A(t)M(t)\\ M(t_0) &= \mathbb{I},\end{aligned}$$

with $M(t)\in\mathcal{M}_n(\mathbb{K})$.

Proof. Since $d(\mathcal{R}(t,t_0)v)/dt = A(t)\mathcal{R}(t,t_0)v$,

$$\left(\frac{d}{dt}\mathcal{R}(t,t_0)\right)v = \left(A(t)R(t,t_0)\right)v,$$

for all $v \in \mathbb{R}^n$. Therefore, $\mathcal{R}(t,t_0)$ is a solution to $M' = A(t)M$. But, by Theorem 2.1.1, we know the solution to the associated IVP to be unique, hence the result. □

From this, the following theorem follows immediately.

Theorem 2.2.13. *The solution to the IVP consisting of the linear homogeneous nonautonomous system* (2.2) *with initial condition* $x(t_0) = x_0$ *is given by*

$$\phi(t) = \mathcal{R}(t,t_0)x_0.$$

2.2.4 Wronskian

Definition 2.2.14. *The* Wronskian *of a system* $\{x_1,\ldots,x_n\}$ *of solutions to* (2.2) *is given by*

$$W(t) = \det(x_1(t),\ldots,x_n(t)).$$

Let $v_i = x_i(t_0)$. Then we have

$$x_i(t) = \mathcal{R}(t,t_0)v_i,$$

and it follows that

$$\begin{aligned} W(t) &= \det(\mathcal{R}(t,t_0)v_1,\ldots,\mathcal{R}(t,t_0)v_n) \\ &= \det\mathcal{R}(t,t_0)\det(v_1,\ldots,v_n). \end{aligned}$$

The following formulae hold

$$\Delta(t,t_0) := \det\mathcal{R}(t,t_0) = \exp\left(\int_{t_0}^t \mathrm{tr}A(s)ds\right) \tag{2.5a}$$

$$W(t) = \exp\left(\int_{t_0}^t \mathrm{tr}A(s)ds\right)\det(v_1,\ldots,v_n). \tag{2.5b}$$

2.2.5 Autonomous linear systems

At this point, we know that solutions to (2.2) take the form $\phi(t) = \mathcal{R}(t,t_0)x_0$, but this was obtained formally. We have no indication whatsoever as to the precise form of $\mathcal{R}(t,t_0)$. Typically, finding $\mathcal{R}(t,t_0)$ can be difficult, if not impossible. There are however cases where the resolvent can be explicitly computed. One such case is for autonomous linear systems, which take the form

$$x'(t) = Ax(t), \tag{2.6}$$

that is, where $A(t) \equiv A$. Our objective here is to establish the following result.

Lemma 2.2.15. *If $A(t) \equiv A$, then $\mathcal{R}(t, t_0) = e^{(t-t_0)A}$ for all $t, t_0 \in \mathcal{I}$.*

This result is deduced easily as a corollary to another result developped below, namely Theorem 2.2.16. Note that in Lemma 2.2.15, the notation $e^{(t-t_0)A}$ involves the notion of exponential of a matrix, which is detailed in Appendix A.10.

Because the reasoning used in constructing solutions to (2.6) is fairly straightforward, we now detail this derivation. Using the intuition from one-dimensional linear equations, we seek a $\lambda \in \mathbb{K}$ such that $\phi_\lambda(t) = e^{\lambda t} v$ be a solution to (2.6) with $v \in \mathbb{K}^n \setminus \{0\}$. We have

$$\phi'_\lambda = \lambda e^{\lambda t} v,$$

and thus ϕ_λ is a solution if, and only if,

$$\begin{aligned} \lambda e^{\lambda t} v &= A e^{\lambda t} v \\ &= e^{\lambda t} A v \\ \Leftrightarrow \lambda v &= A v \\ \Leftrightarrow (A - \lambda I) v &= 0 \quad \text{(with } I \text{ the identity matrix).} \end{aligned}$$

As $v = 0$ is not the only solution, this implies that $A - \lambda I$ must not be invertible, and so

$$\phi_\lambda \text{ is a solution} \Leftrightarrow \det(A - \lambda I) = 0,$$

i.e., λ is an eigenvalue of A.

In the simple case where A is diagonalizable, there exists a basis $(v_1, \ldots, v_n)$ of $\mathbb{K}^n$, with $v_1, \ldots, v_n$ the eigenvectors of A corresponding to the eigenvalues $\lambda_1, \ldots, \lambda_n$. We then obtain n linearly independent solutions $\phi_{\lambda_i}(t) = e^{\lambda_i (t-t_0)}$, $i = 1, \ldots, n$. The general solution is given by

$$\phi(t) = \left(e^{\lambda_1 (t-t_0)} x_{01}, \ldots, e^{\lambda_n (t-t_0)} x_{0n} \right),$$

where x_{0i} is the ith component of x_0, $i = 1, \ldots, n$. In the general case, we need the notion of matrix exponentials. Defining the exponential of matrix A as

$$e^A = \sum_{k=0}^{\infty} \frac{A^n}{n!}$$

(see Appendix A.10), we have the following result.

Theorem 2.2.16. *The global solution ϕ on $\mathbb{K}$ of (2.6) such that $\phi(t_0) = x_0$ is given by*

$$\phi(t) = e^{(t-t_0)A} x_0.$$

Proof. Assume $\phi = e^{(t-t_0)A} x_0$. Then $\phi(t_0) = e^{0A} x_0 = I x_0 = x_0$. Also,

$$\begin{aligned} \phi(t) &= \left(\sum_{n=0}^{\infty} \frac{1}{n!} (t-t_0)^n A^n \right) x_0 \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} (t-t_0)^n A^n x_0, \end{aligned}$$

so ϕ is a power series with radius of convergence $R = \infty$. Therefore, ϕ is differentiable on $\mathbb{R}$ and

$$\begin{aligned}
\phi'(t) &= \sum_{n=1}^{\infty} \frac{1}{n!} n(t-t_0)^{n-1} A^n x_0 \\
&= \sum_{n=0}^{\infty} \frac{1}{(n+1)!} (n+1)(t-t_0)^n A^{n+1} x_0 \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} (t-t_0)^n A^{n+1} x_0 \\
&= A \left(\sum_{n=0}^{\infty} \frac{1}{n!} (t-t_0)^n A^n x_0 \right) \\
&= A\phi(t)
\end{aligned}$$

so ϕ is solution of (2.6). Since (2.6) is linear, solutions are unique and global. $\square$

The problem is now to evaluate the matrix e^{tA}. We have seen that in the case where A is diagonalizable, solutions take the form

$$\phi(t) = \left(e^{\lambda_1 (t-t_0)} x_{01}, \ldots, e^{\lambda_n (t-t_0)} x_{0n} \right),$$

which implies that, in this case, the matrix $\mathcal{R}(t, t_0)$ takes the form

$$\mathcal{R}(t, t_0) = \begin{pmatrix} e^{\lambda_1 (t-t_0)} & 0 & & 0 \\ 0 & e^{\lambda_2 (t-t_0)} & & 0 \\ \vdots & & \ddots & \\ 0 & 0 & & e^{\lambda_n (t-t_0)} \end{pmatrix}.$$

In the general case, we need the notion of generalized eigenvectors.

Definition 2.2.17 (Generalized eigenvectors)**.** *Let λ be an eigenvalue of the $n \times n$ matrix A, with multiplicity $m \leq n$. Then, for $k = 1, \ldots, m$, any nonzero solution v of*

$$(A - \lambda \mathbb{I})^k v = 0$$

is called a generalized eigenvector *of A.*

Theorem 2.2.18. *Let A be a real $n \times n$ matrix with real eigenvalues $\lambda_1, \ldots, \lambda_n$ repeated according to their multiplicity. Then there exists a basis of generalized eigenvectors for $\mathbb{R}^n$. And if $\{v_1, \ldots, v_n\}$ is any basis of generalized eigenvectors for $\mathbb{R}^n$, the matrix $P = [v_1 \cdots v_n]$ is invertible,*

$$A = D + N,$$

where

$$P^{-1} D P = \operatorname{diag}(\lambda_j),$$

the matrix $N = A - D$ is nilpotent of order $k \leq n$, and D and N commute.

2.3 Affine systems

We consider the general (affine) problem (2.1), which we restate here for convenience. Let $x \in \mathbb{R}^n$, $A : \mathcal{I} \to \mathcal{L}(E)$ and $B : \mathcal{I} \to E$, where $\mathcal{I} \subset \mathbb{R}$ and E is a normed vector space, we consider the system

$$x'(t) = A(t)x(t) + B(t) \tag{2.1}$$

2.3.1 The space of solutions

The first problem that we are faced with when considering system (2.1) is that the set of solutions does not constitute a vector space; in particular, the superposition principle does not hold. However, we have the following result.

Proposition 2.3.1. *Let x_1, x_2 be two solutions of (2.1). Then $x_1 - x_2$ is a solution of the associated homogeneous equation (2.2).*

Proof. Since x_1 and x_2 are solutions of (2.1),

$$\begin{aligned} x_1' &= A(t)x_1 + B(t) \\ x_2' &= A(t)x_2 + B(t) \end{aligned}$$

Therefore

$$\frac{d}{dt}(x_1 - x_2) = A(t)(x_1 - x_2)$$ □

Theorem 2.3.2. *The global solutions of (2.1) that are defined on $\mathcal{I}$ form an n dimensional affine subspace of the vector space of maps from $\mathcal{I}$ to $\mathbb{K}^n$.*

Theorem 2.3.3. *Let V be the vector space over $\mathbb{R}$ of solutions to the linear system $x' = A(t)x$. If ψ is a particular solution of the affine system* (2.1)*, then the set of all solutions of* (2.1) *is precisely*

$$\{\phi + \psi,\ \phi \in V\}.$$

Practical rules:

1. To obtain all solutions of (2.1), all solutions of (2.2) must be added to a particular solution of (2.1).

2. To obtain all solutions of (2.2), it is sufficient to know a basis of $\mathcal{S}^0$. Such a basis is called a *fundamental system of solutions* of (2.2).

2.3.2 Construction of solutions

We have the following variation of constants formula.

Theorem 2.3.4. *Let $R(t, t_0)$ be the resolvent of the homogeneous equation $x' = A(t)x$ associated to (2.1). Then the solution x to (2.1) is given by*

$$x(t) = R(t, t_0) + \int_{t_0}^{t} R(t, s)B(s)ds \tag{2.7}$$

Proof. Let $R(t, t_0)$ be the resolvent of $x' = A(t)x$. Any solution of the latter equation is given by

$$x(t) = R(t, t_0)v, \quad v \in \mathbb{R}^n$$

Let us now seek a particular solution to (2.1) of the form $x(t) = R(t, t_0)v(t)$, *i.e.*, using a variation of constants approach. Taking the derivative of this expression of x, we have

$$\begin{aligned} x'(t) &= \frac{d}{dt}[R(t, t_0)]v(t) + R(t, t_0)v'(t) \\ &= A(t)R(t, t_0)v(t) + R(t, t_0)v'(t) \end{aligned}$$

Thus x is a solution to (2.1) if

$$\begin{aligned} A(t)R(t, t_0)v(t) + R(t, t_0)v'(t) &= A(t)R(t, t_0)v(t) + B(t) \\ \Leftrightarrow R(t, t_0)v'(t) &= B(t) \\ \Leftrightarrow v'(t) &= R(t_0, t)B(t) \end{aligned}$$

since $R(t, s)^{-1} = R(s, t)$. Therefore, $v(t) = \int_{t_0}^{t} R(t_0, s)B(s)ds$. A particular solution is given by

$$\begin{aligned} x(t) &= R(t, t_0) \int_{t_0}^{t} R(t_0, s)B(s)ds \\ &= \int_{t_0}^{t} R(t, t_0)R(t_0, s)B(s)ds \\ &= \int_{t_0}^{t} R(t, s)B(s)ds \end{aligned}$$

□

2.3.3 Affine systems with constant coefficients

We consider the affine equation (2.1), but with the matrix $A(t) \equiv A$.

Theorem 2.3.5. *The general solution to the IVP*

$$\begin{aligned} x'(t) &= Ax(t) + B(t) \\ x(t_0) &= x_0 \end{aligned} \tag{2.8}$$

is given by

$$x(t) = e^{(t-t_0)A}x_0 + \int_{t_0}^{t} e^{(t-t_0)A}B(s)ds \tag{2.9}$$

Proof. Use Lemma 2.2.15 and the variation of constants formula (2.7). □

2.4 Systems with periodic coefficients

2.4.1 Linear systems: Floquet theory

We consider the linear system (2.2) in the following case,

$$\begin{aligned} x' &= A(t)x \\ A(t+\omega) &= A(t), \quad \forall t, \end{aligned} \tag{2.10}$$

with entries of $A(t)$ continuous on $\mathbb{R}$.

Definition 2.4.1 (Monodromy operator)**.** *Associated to system* (2.10) *is the resolvent* $\mathcal{R}(t,s)$*. For all $s \in \mathbb{R}$, the operator*

$$C(s) := \mathcal{R}(s+\omega, s)$$

is called the monodromy *operator.*

Theorem 2.4.2. *If $X(t)$ is a fundamental matrix for (2.10), then there exists a nonsingular constant matrix V such that, for all t,*

$$X(t+\omega) = X(t)V.$$

This matrix takes the form

$$V = X^{-1}(0)X(\omega),$$

and is called the monodromy matrix*.*

Proof. Since X is a fundamental matrix solution, there holds that $X'(t) = A(t)X(t)$ for all t. Therefore $X'(t+\omega) = A(t+\omega)X(t+\omega)$, and by periodicity of $A(t)$, $X'(t+\omega) = A(t)X(t+\omega)$, which implies that $X(t+\omega)$ is a fundamental matrix of (2.10). As a consequence, by Theorem 2.2.9, there exists a matrix V such that $X(t+\omega) = X(t)V$.

Since at $t = 0$, $X(\omega) = X(0)V$, it follows that $V = X^{-1}(0)X(\omega)$. □

Theorem 2.4.3 (Floquet's theorem, complex case)**.** *Any fundamental matrix solution Φ of (2.10) takes the form*

$$\Phi(t) = P(t)e^{tB} \tag{2.11}$$

where $P(t)$ and B are $n \times n$ complex matrices such that

i) $P(t)$ is invertible, continuous, and periodic of period ω in t,

ii) B is a constant matrix such that $\Phi(\omega) = e^{\omega B}$.

Proof. Let Φ be a fundamental matrix solution. From 2.4.2, the monodromy matrix $V = \Phi^{-1}(0)\Phi(\omega)$ is such that $\Phi(t+\omega) = \Phi(t)V$. By Theorem A.11.1, there exists $B \in \mathcal{M}_n(\mathbb{C})$

such that $e^{B\omega} = V$. Let $P(t) = \Phi(t)e^{-Bt}$, so $\Phi(t) = P(t)e^{Bt}$. It is clear that P is continuous and nonsingular. Also,

$$\begin{aligned}
P(t+\omega) &= \Phi(t+\omega)e^{-B(t+\omega)} \\
&= \Phi(t)Ve^{-B(\omega+t)} \\
&= \Phi(t)e^{B\omega}e^{-B\omega}e^{-Bt} \\
&= \Phi(t)e^{-Bt} \\
&= P(t),
\end{aligned}$$

proving the P is ω-periodic. □

Theorem 2.4.4 (Floquet's theorem, real case)**.** *Any fundamental matrix solution Φ of (2.10) takes the form*

$$\Phi(t) = P(t)e^{tB} \tag{2.12}$$

where $P(t)$ and B are $n \times n$ real matrices such that

i) $P(t)$ is invertible, continuous, and periodic of period 2ω in t,

ii) B is a constant matrix such that $\Phi(\omega)^2 = e^{2\omega B}$.

Proof. The proof works similarly as in the complex case, except that here, Theorem A.11.1 implies that there exists $B \in \mathcal{M}_n(\mathbb{R})$ such that $e^{2\omega B} = V^2$. Let $P(t) = \Phi(t)e^{-Bt}$, so $\Phi(t) = P(t)e^{tB}$. It is clear that P is continuous and nonsingular. Also,

$$\begin{aligned}
P(t+2\omega) &= \Phi(t+2\omega)e^{-(t+2\omega)B} \\
&= \Phi(t+\omega)Ve^{-(2\omega+t)B} \\
&= \Phi(t)V^2e^{-(2\omega+t)B} \\
&= \Phi(t)e^{2\omega B}e^{-2\omega B}e^{-tB} \\
&= \Phi(t)e^{-tB} \\
&= P(t),
\end{aligned}$$

proving the P is ω-periodic. □

See [12, p. 87-90], [4, p. 162-179].

Theorem 2.4.5 (Floquet's theorem, [4])**.** *If $\Phi(t)$ is a fundamental matrix solution of the ω-periodic system (2.10), then, for all $t \in \mathbb{R}$,*

$$\Phi(t+\omega) = \Phi(t)\Phi^{-1}(0)\Phi(\omega).$$

In addition, for each possibly complex matrix B such that

$$e^{\omega B} = \Phi^{-1}(0)\Phi(\omega),$$

there is a possibly complex ω-periodic matrix function $t \mapsto P(t)$ such that $\Phi(t) = P(t)e^{tB}$ for all $t \in \mathbb{R}$. Also, there is a real matrix R and a real 2ω-periodic matrix function $t \to Q(t)$ such that $\Phi(t) = Q(t)e^{tR}$ for all $t \in \mathbb{R}$.

Definition 2.4.6 (Floquet normal form)**.** *The representation $\Phi(t) = P(t)e^{tR}$ is called a* Floquet normal form*.*

In the case where $\Phi(t) = P(t)e^{tB}$, we have $dP(t)/dt = A(t)P(t) - P(t)B$. Therefore, letting $x = P(t)z$, we obtain $x' = P(t)x' + dP(t)/dtx = P(t)A(t)x + A(t)P(t)x - P(t)Bx$

$z = P^{-1}(t)x$, so $z' = \frac{dP^{-1}(t)}{dt}x + P^{-1}(t)x' = \frac{dP^{-1}(t)}{dt}P(t)z + P^{-1}(t)A(t)P(t)z$

Definition 2.4.7 (Characteristic multipliers)**.** *The eigenvalues $\lambda_1, \ldots, \lambda_n$ of a monodromy matrix B are called the* characteristic multipliers *of equation (2.10).*

Definition 2.4.8 (Characteristic exponents)**.** *Numbers μ such that $e^{\mu\omega}$ is a characteristic multiplier of* (2.10) *are called the* Floquet exponents *of* (2.10)*.*

Theorem 2.4.9 (Spectral mapping theorem)**.** *Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. If $C \in \mathbf{GL}_n(\mathbb{K})$ is written $C = e^B$, then the eigenvalues of C coincide with the exponentials of the eigenvalues of B, with same multiplicity.*

Definition 2.4.10 (Characteristic exponents)**.** *The eigenvalues $\lambda_1, \ldots, \lambda_n$ of a monodromy matrix B are called the characteristic exponents of equation (2.10). The exponents $\rho_1 = \exp(2\omega\lambda_1), \ldots, \rho_n = \exp(2\omega\lambda_n)$ of the matrix $\Phi(\omega)^2$ are called the (Floquet) multipliers of (2.10).*

Proposition 2.4.11. *Suppose that X, Y are fundamental matrices for (2.10) and that $X(t+\omega) = X(t)V$, $Y(t+\omega) = Y(t)U$. Then the monodromy matrices U and V are similar.*

Proof. Suppose that $X(t+\omega) = X(t)V$ and $Y(t+\omega) = Y(t)U$. But, by Theorem 2.2.9, since X and Y are fundamental matrices for (2.10), there exists an invertible matrix C such that $X(t) = Y(t)C$ for all t. Thus, in particular, $X(t+\omega) = Y(t+\omega)C$, and so

$$C^{-1}UCX(t+\omega) = Y(t+\omega)C = Y(t)UC = X(t)C^{-1}UC,$$

since $Y(t) = X(t)C^{-1}$. It follows that $V = C^{-1}UC$, so U and V are similar. □

From this Proposition, it follows that monodromy matrices share the same spectrum.

Corollary 2.4.12. *All solutions of (2.10) tend to 0 as $t \to \infty$ if and only if $|\rho_j| < 1$ for all j (or $\Re(\lambda_j) < 0$ for all j).*

Let p be an eigenvector of $\Phi(\omega)^2$ associated with a multiplier ρ. Then the solution $\phi(t) = \Phi(t)p$ of (2.10) satisfies the condition $\phi(t+2\omega) = \rho\phi(t)$. This is the origin of the term *multiplier*.

2.4.2 Affine systems: the Fredholm alternative

We discuss here an extension of a theorem that was proved implicitly in Exercise 4, Assignment 2. Let us start by stating the result in question. We consider here the system

$$x' = A(t)x + b(t), \tag{2.13}$$

where $x \in \mathbb{R}^n$, $A \in \mathcal{M}_n(\mathbb{R})$ and $b \in \mathbb{R}^n$, with A and b continuous and ω-periodic.

Theorem 2.4.13. *If the homogeneous equation*

$$x' = A(t)x \tag{2.14}$$

associated to (2.13) has no nonzero solution of period ω, then (2.13) has for each function f, a unique ω-periodic solution.

The Fredholm alternative concerns the case where there exists a nonzero periodic solution of (2.14). We give some needed results before going into details. Consider (2.14). Associated to this system is the so-called *adjoint system*, which is defined by the following differential equation,

$$y' = -A^T(t)y \tag{2.15}$$

Proposition 2.4.14. *The adjoint equation has the following properties.*

i) Let $R(t,t_0)$ be the resolvent matrix of (2.14). Then, the resolvent matrix of (2.15) is $R^T(t_0,t)$.

ii) There are as many independent periodic solutions of (2.14) as there are of (2.15).

iii) If x is a solution of (2.14) and y is a solution of (2.15), then the scalar product $\langle x(t), y(t)\rangle$ is constant.

Proof. i) We know that $\frac{\partial}{\partial s}R(t,s) = -R(t,s)A(s)$. Therefore, $\frac{\partial}{\partial s}R^T(t,s) = -A^T(s)R^T(t,s)$. As $R(s,s) = I$, the first point is proved.

ii) The solution of (2.15) with initial value y_0 is $R^T(0,t)y_0$. The initial value of a periodic solution of (2.15) is y_0 such that

$$R^T(0,\omega)y_0 = y_0$$

This can also be written as

$$\left[R^T(0,\omega) - I\right] y_0 = 0$$

or, taking the transpose,

$$y_0^T \left[R(0,\omega) - I\right] = 0$$

Now, since $R(0,\omega) = R^{-1}(\omega,0)$, it follows that

$$y_0^T \left[R(0,\omega) - I\right] = 0 \Leftrightarrow y_0^T \left[R^{-1}(\omega,0) - I\right] = 0$$

This is equivalent to $y_0^T \left[R(0,\omega) - I\right] = 0$. The latter equation has as many solutions as $\left[R(0,\omega) - I\right] x_0 = 0$; the number of these depends on the rank of $R(\omega,0) - I$.

iii) Recall that for differentiable functions a, b,

$$\frac{d}{dt}\langle a(t), b(t)\rangle = \langle \frac{d}{dt}a(t), b(t)\rangle + \langle a(t), \frac{d}{dt}b(t)\rangle$$

Thus

$$\frac{d}{dt}\langle x(t), y(t)\rangle = \langle A(t)x(t), y(t)\rangle + \langle x(t), -A^T(t)y(t)\rangle = 0$$ □

Before we carry on to the actual Fredholm alternative in the context of ordinary differential equations, let us consider the problem in a more general setting. Let H be a Hilbert space. If $A \in \mathcal{L}(H, H)$, the adjoint operator A^* of A is the element of $\mathcal{L}(H, H)$ such that

$$\forall u, v \in H, \quad \langle Au, v\rangle = \langle u, A^* v\rangle$$

Let Img(A) be the image of A, Ker(A^*) be the kernel of A^*. Then we have $H = \text{Img}(A) \oplus \text{Ker}(A^*)$.

Theorem 2.4.15 (Fredholm alternative). *For the equation $Af = g$ to have a solution, it is necessary and sufficient that g be orthogonal to every element of Ker(A^*).*

We now use this very general setting to prove the following theorem, in the context of ODEs.

Theorem 2.4.16 (Fredholm alternative for ODEs). *Consider (2.13) with A and f continuous and ω-periodic. Suppose that the homogeneous equation (2.14) has p independent solutions of period ω. Then the adjoint equation (2.15) also has p independent solutions of period p, which we denote $y_1, \ldots, y_p$. Then*

i) If

$$\int_0^\omega \langle y_k(t), b(t)\rangle dt = 0, \quad k = 1, \ldots, p \tag{2.16}$$

then there exist p independent solutions of (2.13) of period ω, and,

ii) if this condition is not fulfilled, (2.13) has no nontrivial solution of period ω.

Proof. First, remark that x_0 is the initial condition of a periodic solution of (2.13) if, and only if,

$$[R(0, \omega) - I]\, x_0 = \int_0^\omega R(0, s)b(s)ds \tag{2.17}$$

By Theorem 2.3.4, the solution of (2.13) through $(0, x_0)$ is given by

$$x(t) = R(t, 0)x_0 + \int_0^t R(t, s)b(s)ds$$

Hence, at time ω,

$$x(\omega) = R(\omega, 0)x_0 + \int_0^\omega R(\omega, s)b(s)ds$$

If x_0 is the initial condition of a ω-periodic solution, then $x(\omega) = x_0$, and so

$$x_0 - R(\omega, 0) = \int_0^\omega R(\omega, s)b(s)ds$$

On the other hand, $y_k(0)$ is the initial condition of an ω-periodic solution y_k if, and only if,

$$\left[R^T(0, \omega) - I\right] y_k(0) = 0$$

Let $C = R(0, \omega) - I$. We have that $\mathbb{R}^n = \text{Img}(C) \oplus \text{Ker}(C^T)$. We now use the Fredholm alternative in this context. There exists x_0 such that

$$Cx_0 = \int_0^\omega R(0, s)b(s)ds$$

if, and only if,

$$\int_0^\omega R(0, s)b(s)ds \in \text{Img}(C)$$

Indeed, from the Fredholm alternative, setting $f = x_0$ and $g = \int_0^\omega R(0, s)b(s)ds$, we have that $Cf = g$ has a solution if, and only if, g is orthogonal to every element of $\text{Ker}(C^T)$, *i.e.*, since $\mathbb{R}^n = \text{Img}(C) \oplus \text{Ker}(C^T)$, if, and only if, $g \in \text{Img}(C)$.

Now, $y_1(0), \ldots, y_p(0)$ is a basis of $\text{Ker}(C^T)$. It follows that there exists a solution of (2.13) if, and only if, for all $k = 1, \ldots, p$,

$$\begin{aligned}
&\forall k = 1, \ldots, p, \quad \langle \int_0^\omega R(0, s)b(s)ds, y_k(0)\rangle = 0 \\
\Leftrightarrow\ &\forall k = 1, \ldots, p, \quad \int_0^\omega \langle R(0, s)b(s), y_k(0)\rangle ds = 0 \\
\Leftrightarrow\ &\forall k = 1, \ldots, p, \quad \int_0^\omega \langle b(s), R^T(0, s)y_k(0)\rangle ds = 0 \\
\Leftrightarrow\ &\forall k = 1, \ldots, p, \quad \int_0^\omega \langle b(s), y_k(s)\rangle ds = 0
\end{aligned}$$

If these relations are satisfied, the set of vectors v such that

$$Av = \int_0^\omega R(0, s)b(s)ds$$

is of the form $v_0 + \text{Ker}(C^T)$, where v_0 is one of these vectors; hence there exist p of them which are independent and are initial conditions of the p independent ω-periodic solutions of (2.13). $\square$

Example – The equation

$$x'' = f(t) \tag{2.18}$$

where f is ω-periodic, has solutions of period ω if, and only if,

$$\int_0^\omega f(s)ds = 0$$

Let $y = x'$. Then, differentiating y and substituting into (2.18), we have

$$y' = f(t)$$

Hence the system is

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ f(t) \end{pmatrix}$$

Hence,

$$A^T = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

and the adjoint equation $\xi' = A^T \xi$ has the periodic solution $(0, a)^T$. ⋄

2.5 Further developments, bibliographical notes

2.5.1 A variation of constants formula for a nonlinear system with a linear component

The variation of constants formula given in Theorem 2.3.4 can be extended.

Theorem 2.5.1 (Variation of constants formula). *Consider the IVP*

$$x' = A(t)x + g(t, x) \tag{2.19a}$$
$$x(t_0) = x_0, \tag{2.19b}$$

where $g : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ a smooth function, and let $\mathcal{R}(t, t_0)$ be the resolvent associated to the homogeneous system $x' = A(t)x$, with $\mathcal{R}$ defined on some interval $\mathcal{I} \ni t_0$. Then the solution ϕ of (2.19) *is given by*

$$\phi(t) = \mathcal{R}(t, t_0)x_0 + \int_{t_0}^{t} \mathcal{R}(t, s)g(\phi(s), s)ds, \tag{2.20}$$

on some subinterval of $\mathcal{I}$.

Proof. We proceed using a variation of constants approach. It is known that the general solution to the homogeneous equation $x' = A(t)x$ associated to (2.19) is given by

$$\phi(t) = \mathcal{R}(t, t_0)x_0.$$

We seek a solution to (2.19) by assuming that $\phi(t) = \mathcal{R}(t, t_0)v(t)$. We have

$$\begin{aligned} \phi'(t) &= \left(\frac{d}{dt}\mathcal{R}(t, t_0)\right) v(t) + \mathcal{R}(t, t_0)v'(t) \\ &= A(t)\mathcal{R}(t, t_0)v(t) + \mathcal{R}(t, t_0)v'(t), \end{aligned}$$

from Proposition 2.2.11. For ϕ to be solution, it must satisfy the differential equation (2.19), and thus

$$\begin{aligned}
\phi'(t) = A(t)\phi(t) + g(t,\phi(t)) &\Leftrightarrow A(t)\mathcal{R}(t,t_0)v(t) + \mathcal{R}(t,t_0)v'(t) = A(t)\mathcal{R}(t,t_0)v(t) + g(t,\phi(t)) \\
&\Leftrightarrow \mathcal{R}(t,t_0)v'(t) = g(t,\phi(t)) \\
&\Leftrightarrow v'(t) = \mathcal{R}(t,t_0)^{-1}g(t,\phi(t)) \\
&\Leftrightarrow v'(t) = \mathcal{R}(t_0,t)g(t,\phi(t)) \\
&\Leftrightarrow v(t) = \int_{t_0}^{t} \mathcal{R}(t_0,s)g(s,\phi(s))ds + C,
\end{aligned}$$

using Proposition 2.2.11 again. Therefore,

$$\phi(t) = \mathcal{R}(t,t_0)\left(\int_{t_0}^{t} \mathcal{R}(t_0,s)g(s,\phi(s))ds + C\right).$$

Evaluating this expression at $t = t_0$ gives $\phi(t_0) = C$, so $C = x_0$. Therefore,

$$\begin{aligned}
\phi(t) &= R(t,t_0)x_0 + \mathcal{R}(t,t_0)\int_{t_0}^{t} \mathcal{R}(t_0,s)g(s,\phi(s))ds \\
&= R(t,t_0)x_0 + \int_{t_0}^{t} \mathcal{R}(t,t_0)\mathcal{R}(t_0,s)g(s,\phi(s))ds \\
&= R(t,t_0)x_0 + \int_{t_0}^{t} \mathcal{R}(t,s)g(s,\phi(s))ds,
\end{aligned}$$

from Proposition 2.2.11. □

Chapter 3

Stability of linear systems

3.1 Stability at fixed points

We consider here the autonomous equation (not necessarily linear),

$$x' = f(x). \tag{3.1}$$

To emphasize the fact that we are dealing with flows, we write $x(t, x_0)$ the solution to (3.1), at time t and satisfying at time $t = 0$ the initial condition $x(0) = x_0$.

Definition 3.1.1 (Fixed point)**.** *A fixed point of* (3.1) *is a point x^* such that $f(x^*) = 0$.*

This is evident, as a point such that $f(x^*) = 0$ satisfies $(x^*)' = f(x^*) = 0$, so that the solution is constant when $x = x^*$. Note also that this implies that $x(t) = x^*$ is a solution defined on $\mathbb{R}$.

Definition 3.1.2 (Stable equilibrium point)**.** *The fixed point x^* is (*positively*)* stable *if the following two conditions hold:*

i) There exists $r > 0$ such that if $\|x_0 - x^*\| < r$, then the solution $x(t, x_0)$ is defined for all $t \geq 0$. (This is automatically satisfied for flows).

ii) For any $\varepsilon > 0$, there exists $\delta > 0$ such that $\|x_0 - x^*\| < \delta$ implies $\|x(t, x_0) - x^*\| < \varepsilon$.

Definition 3.1.3 (Asymptotically stable equilibrium point)**.** *If the equilibrium x^* is (positively) stable and that additionally, there exists $\gamma > 0$ such that $\|x_0 - x^*\| < \gamma$ implies* $\lim_{t\to\infty} x(t, x_0) = x^*$*, then x^* is (positively) asymptotically stable.*

3.2 Affine systems with small coefficients

We consider here a linear system of the form

$$x' = Ax + b(t)x \tag{3.2}$$

where b is continuous and small, *i.e.*, $\lim_{t\to\infty} b(t) = 0$, with $A \in M_n(\mathbb{R})$ and $b \in \mathbb{R}$.

Theorem 3.2.1. *Suppose that all eigenvalues of A have negative real parts, and that b is continuous and such that $\lim_{t\to\infty} b(t) = 0$. Then 0 is a g.a.s. equilibrium of (3.2).*

The proof comes from [2, p. 156-157].

Proof. For any given (t_0, x_0), $t_0 > 0$, we have, from [2, Th 2.1, p. 37] about the existence and uniqueness of the solutions to the linear equation $x' = A(t)x + g(t)$, that the (unique) solution $\phi_t(x_0)$ satisfying the initial condition $\phi_{t_0}(x_0) = x_0$ exists for all $t \geq t_0$.

by the variation of constants formula, using $b(t)x$ as the inhomogeneous term, we can express the solution by means of the equivalent integral equation, for $t_0 \leq t < \infty$,

$$\phi_t(x_0) = e^{(t-t_0)A}x_0 + \int_{t_0}^{t} e^{(t-s)A}b(s)\phi_s(x_0)ds \tag{3.3}$$

by the hypothesis on A, $e^{(t-t_0)A}$ is such that for $t_0 \leq t < \infty$, $\|\Psi(t, t_0)\| \leq Ke^{-\sigma(t-t_0)}$ for $K > 0$, $\sigma > 0$, where $R(t, t_0) = e^{(t-t_0)A}$ is the fundamental matrix of the homogeneous part of (3.2).

Since $\lim_{t\to\infty} b(t) = 0$, given any $\eta > 0$, there exists a number $T \geq t_0$ such that $|b(t)| < \eta$ for $t \geq T$. We now use the variation of constants formula (3.3) with the point $(T, \phi_T(x_0))$ for initial condition. We have, for $T \leq t < \infty$,

$$\phi_t(x_0) = e^{(t-T)A}\phi_T(x_0) + \int_T^t e^{(t-s)A}b(s)\phi_s(x_0)ds$$

Thus, using $\|\Phi(t)\| \leq Ke^{-\sigma(t-t_0)}$ (with $t_0 = T$) and $|b(t)| < \eta$ for $t \geq T$, we obtain, for $T \leq t < \infty$,

$$\|\phi_t(x_0)\| \leq Ke^{-\sigma(t-T)}\|\phi_T(x_0)\| + K\eta \int_T^t e^{-\sigma(t-s)}\|\phi_s(x_0)\|ds$$

Multiplying both sides of this inequality by $e^{\sigma t}$ and using Gronwall's inequality (Appendix A.7) with the function $\|\phi_t(x_0)\|e^{\sigma t}$, we obtain, for $T \leq t < \infty$,

$$\|\phi_t(x_0)\| \leq K\|\phi_T(x_0)\|e^{-(\sigma-K\eta)(t-T)} \tag{3.4}$$

From this we conclude that if $0 < \eta < \sigma/K$, the solution $\phi_t(x_0)$ will approach zero exponentially. This does not yet prove that the zero solution of (3.2) is stable. To do this, we compute a bound on $\|\phi_T(x_0)\|$. Returning to (3.3) and restricting t to the interval $t_0 \leq t \leq T$, we have

$$\|\phi_t(x_0)\| \leq Ke^{-\sigma(t-t_0)}\|x_0\| + K_1K \int_{t_0}^t e^{-\sigma(t-s)}\|\phi(s, t_0, x_0)\|ds$$

where $K_1 = \max_{t_0\leq t\leq T} |b(t)|$. Multiplying by $e^{\sigma t}$ and applying the Gronwall inequality we obtain

$$\begin{aligned}\|\phi_t(x_0)\| &\leq K\|x_0\|e^{-\sigma(t-t_0)+K_1K(t-t_0)} \\ &\leq K\|x_0\|e^{K_1K(t-t_0)}, \quad t_0 \leq t \leq T\end{aligned} \tag{3.5}$$

Therefore,

$$\|\phi_T(x_0)\| \leq K\|x_0\|e^{K_1K(T-t_0)}, \quad t_0 \leq T \tag{3.6}$$

Thus we can make $|\phi_T(x_0)|$ small by choosing $|x_0|$ sufficiently small. This together with (3.4) gives the stability.

Indeed, substituting (3.6) into (3.4) gives, for $T \leq t < \infty$,

$$\|\phi_t(x_0)\| \leq K^2\|x_0\|e^{K_1K(T-t_0)}e^{-(\sigma-K\eta)(t-T)} \tag{3.7}$$

Let then $K_2 = \max\left(Ke^{K_1K(T-t_0)}, K^2e^{K_1K(T-t_0)}\right)$. From (3.5) and (3.7) we have

$$\|\phi_t(x_0)\| \leq \begin{cases} K_2\|x_0\| & \text{if } t_0 \leq t \leq T \\ K_2\|x_0\|e^{-(\sigma-K\eta)(t-T)} & \text{if } T \leq t < \infty \end{cases} \tag{3.8}$$

For a given matrix A, we can compute K and σ; we next pick any $0 < \eta < \sigma/K$ and then $T \geq t_0$ so that $|b(t)| < \eta$ for $t \geq T$. We then compute K_1 and K_2. Now, given any $\varepsilon > 0$, choose $\delta < \varepsilon/K_2$. Then from (3.8), if $\|x_0\| < \delta$, $\|\phi_t(x_0)\| < \varepsilon$ for all $t \geq t_0$ so that the zero solution is stable. From (3.8), it is clear that the zero solution is globally asymptotically stable. □

Corollary 3.2.2. *Let all eigenvalues of A have negative real part, so that $|e^{At}| \leq Ke^{-\sigma t}$ for some constants $K > 0$, $\sigma > 0$ and all $t \geq 0$. Let $b(t)$ be continuous for $0 \leq t < \infty$ and suppose that there exists $T > 0$ such that $|b(t)| < \sigma/K$ for $t \geq T$. Then the zero solution of (3.2) is globally asymptotically stable.*

Theorem 3.2.3. *Let all eigenvalues of A have negative real part, and let $b(t)$ be continuous for $0 \leq t < \infty$ and such that $\int_0^\infty |b(s)|ds < \infty$. Then the zero solution of (3.2) is globally asymptotically stable.*

We give some notions of linear stability theory, in the case of the autonomous linear system (2.6), repeated here for convenience:

$$x'(t) = Ax(t). \tag{2.6}$$

We let $w_j = u_j + iv_j$ be a generalized eigenvector of A corresponding to an eigenvalue $\lambda_j = a_j + ib_j$, with $v_j = 0$ if $b_j = 0$, and

$$B = \{u_1, \ldots, u_k, u_{k+1}, v_{k+1}, \ldots, u_m, v_m\}$$

be a basis of $\mathbb{R}^n$, with $n = 2m - k$.

Definition 3.2.4 (Stable, unstable and center subspaces)**.** *The* stable*,* unstable *and* center *subspaces of the linear system* (2.6) *are given, respectively, by*

$$E^s = \mathit{Span}\{u_j, v_j : a_j < 0\},$$

$$E^u = \mathit{Span}\{u_j, v_j : a_j > 0\}$$

and

$$E^c = \mathit{Span}\{u_j, v_j : a_j = 0\}.$$

Definition 3.2.5. *The mapping $e^{At} : \mathbb{R}^n \to \mathbb{R}^n$ is called the* flow *of the linear system* (2.6)*.*

The term *flow* is used since e^{At} describes the motion of points $x_0 \in \mathbb{R}^n$ along trajectories of (2.6).

Definition 3.2.6. *If all eigenvalues of A have nonzero real part, that is, if $E^c = \emptyset$, then the flow e^{At} of system* (2.6) *is called a* hyperbolic *flow, and the system* (2.6) *is a* hyperbolic linear system*.*

Definition 3.2.7. *A subspace $E \subset \mathbb{R}^n$ is* invariant with respect to the flow e^{At}*, or* invariant under the flow *of* (2.6)*, if $e^{At}E \subset E$ for all $t \in \mathbb{R}$.*

Theorem 3.2.8. *Let E be the generalized eigenspace of A associated to the eigenvalue λ. Then $AE \subset E$.*

Theorem 3.2.9. *Let $A \in \mathcal{M}_n(\mathbb{R})$. Then*

$$\mathbb{R}^n = E^s \oplus E^u \oplus E^c.$$

Furthermore, if the matrix A is the matrix of the linear autonomous system (2.6)*, then E^s, E^u and E^c are invariant under the flow of* (2.6)*.*

Definition 3.2.10. *If all the eigenvalues of A have negative (resp. positive) real parts, then the origin is a* sink *(resp.* source*) for the linear system* (2.6)*.*

Theorem 3.2.11. *The stable, center and unstable subspaces E^S, E^C and E^U, respectively, are invariant with respect to e^{At},* i.e.*, let $x_0 \in E^S$, $y_0 \in E^C$ and $z_0 \in E^U$, then $e^{At}x_0 \in E^S$, $e^{At}y_0 \in E^C$ and $e^{At}z_0 \in E^U$.*

Definition 3.2.12 (Homeomorphism)**.** *Let X be a metric space and let A and B be subsets of X. A* homeomorphism *$h : A \to B$ of A onto B is a continuous one-to-one map of A onto B such that $h^{-1} : B \to A$ is continuous. The sets A and B are called* homeomorphic *or* topologically equivalent *if there is a homeomorphism of A onto B.*

Definition 3.2.13 (Differentiable manifold)**.** *An n-dimensional* differentiable manifold *M (or a manifold of class C^k) is a connected metric space with an open covering $\{U_\alpha\}$ (*i.e.*, $M = \cup_\alpha U_\alpha$) such that*

i) for all α, U_α is homeomorphic to the open unit ball in $\mathbb{R}^n$, $B = \{x \in \mathbb{R}^n : |x| < 1\}$, i.e.*, for all α there exists a homeomorphism of U_α onto B, $h_\alpha : U_\alpha \to B$,*

ii) if $U_\alpha \cap U_\beta \neq \emptyset$ and $h_\alpha : U_\alpha \to B$, $h_\beta : U_\beta \to B$ are homeomorphisms, then $h_\alpha(U_\alpha \cap U_\beta)$ and $h_\beta(U_\alpha \cap U_\beta)$ are subsets of $\mathbb{R}^n$ and the map

$$h = h_\alpha \circ h_\beta^{-1} : \; h_\beta(U_\alpha \cap U_\beta) \to h_\alpha(U_\alpha \cap U_\beta)$$

is differentiable (or of class C^k) and for all $x \in h_\beta(U_\alpha \cap U_\beta)$, the determinant of the Jacobian, $\det Dh(x) \neq 0$.

Remark – The manifold is *analytic* if the maps $h = h_\alpha \circ h_\beta^{-1}$ are analytic. ∘

Next, recall that x^* is an *equilibrium point* of (4.1) if $f(x^*) = 0$. An equilibrium point x^* is *hyperbolic* if the Jacobian matrix Df of (4.1) evaluated at x^*, denoted $Df(x^*)$, has no eigenvalues with zero real part. Also, recall that the solutions of (4.1) form a one-parameter group that defines the *flow* of the nonlinear differential equation (4.1). To be more precise, consider the IVP consisting of (4.1) and an initial condition $x(t_0) = x_0$. Let $\mathcal{I}(x_0)$ be the maximal interval of existence of the solution to the IVP. Let then $\phi : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ be defined as follows: For $x_0 \in \mathbb{R}^n$ and $t \in \mathcal{I}(x_0)$, $\phi(t, x_0) = \phi_t(x_0)$ is the solution of the IVP defined on its maximal interval of existence $\mathcal{I}(x_0)$.

Example – Consider the (linear) ordinary differential equation $x' = ax$, with $a, x \in \mathbb{R}$. The solution is $\phi(t, x_0) = e^{at} x_0$, and satisfies the group property

$$\phi(t + s, x_0) = e^{a(t+s)} x_0 = e^{at}(e^{as} x_0) = \phi(t, e^{as} x_0) = \phi(t, \phi(s, x_0))$$

◇

For simplicity and without loss of generality since both results are local results, we assume hereforth that $x^* = 0$, *i.e.*, that a change of coordinates has been performed translating x^* to the origin. We also assume that $t_0 = 0$.

Chapter 4

Linearization

We consider here the autonomous nonlinear system in $\mathbb{R}^n$

$$x' = f(x) \tag{4.1}$$

The object of this chapter is to show two results which link the behavior of (4.1) near a hyperbolic equilibrium point x^* to the behavior of the linearized system

$$x' = Df(x^*)(x - x^*) \tag{4.2}$$

about that same equilibrium.

4.1 Some linear stability theory

We now give some notions of linear stability theory, in the case of the autonomous linear system (2.6), repeated here for convenience:

$$x'(t) = Ax(t). \tag{2.6}$$

We let $w_j = u_j + iv_j$ be a generalized eigenvector of A corresponding to an eigenvalue $\lambda_j = a_j + ib_j$, with $v_j = 0$ if $b_j = 0$, and

$$B = \{u_1, \dots, u_k, u_{k+1}, v_{k+1}, \dots, u_m, v_m\}$$

be a basis of $\mathbb{R}^n$, with $n = 2m - k$.

Definition 4.1.1 (Stable, unstable and center subspaces)**.** *The* stable, unstable *and* center *subspaces of the linear system* (2.6) *are given, respectively, by*

$$E^s = Span\{u_j, v_j : a_j < 0\},$$

$$E^u = Span\{u_j, v_j : a_j > 0\}$$

and

$$E^c = Span\{u_j, v_j : a_j = 0\}.$$

Definition 4.1.2. *The mapping $e^{At} : \mathbb{R}^n \to \mathbb{R}^n$ is called the* flow *of the linear system* (2.6)*.*

The term *flow* is used since e^{At} describes the motion of points $x_0 \in \mathbb{R}^n$ along trajectories of (2.6).

Definition 4.1.3. *If all eigenvalues of A have nonzero real part, that is, if $E^c = \emptyset$, then the flow e^{At} of system* (2.6) *is called a* hyperbolic *flow, and the system* (2.6) *is a* hyperbolic linear system*.*

Definition 4.1.4. *A subspace $E \subset \mathbb{R}^n$ is* invariant with respect to the flow e^{At}*, or* invariant under the flow *of* (2.6)*, if $e^{At}E \subset E$ for all $t \in \mathbb{R}$.*

Theorem 4.1.5. *Let E be the generalized eigenspace of A associated to the eigenvalue λ. Then $AE \subset E$.*

Theorem 4.1.6. *Let $A \in \mathcal{M}_n(\mathbb{R})$. Then*

$$\mathbb{R}^n = E^s \oplus E^u \oplus E^c.$$

Furthermore, if the matrix A is the matrix of the linear autonomous system (2.6)*, then E^s, E^u and E^c are invariant under the flow of* (2.6)*.*

Definition 4.1.7. *If all the eigenvalues of A have negative (resp. positive) real parts, then the origin is a* sink *(resp.* source*) for the linear system* (2.6)*.*

Theorem 4.1.8. *The stable, center and unstable subspaces E^S, E^C and E^U, respectively, are invariant with respect to e^{At},* i.e.*, let $x_0 \in E^S$, $y_0 \in E^C$ and $z_0 \in E^U$, then $e^{At}x_0 \in E^S$, $e^{At}y_0 \in E^C$ and $e^{At}z_0 \in E^U$.*

Definition 4.1.9 (Homeomorphism)**.** *Let X be a metric space and let A and B be subsets of X. A* homeomorphism *$h : A \to B$ of A onto B is a continuous one-to-one map of A onto B such that $h^{-1} : B \to A$ is continuous. The sets A and B are called* homeomorphic *or* topologically equivalent *if there is a homeomorphism of A onto B.*

Definition 4.1.10 (Differentiable manifold)**.** *An n-dimensional* differentiable manifold *M (or a manifold of class C^k) is a connected metric space with an open covering $\{U_\alpha\}$ (*i.e.*, $M = \cup_\alpha U_\alpha$) such that*

i) for all α, U_α is homeomorphic to the open unit ball in $\mathbb{R}^n$, $B = \{x \in \mathbb{R}^n : \ |x| < 1\}$, i.e.*, for all α there exists a homeomorphism of U_α onto B, $h_\alpha : U_\alpha \to B$,*

ii) if $U_\alpha \cap U_\beta \neq \emptyset$ and $h_\alpha : U_\alpha \to B$, $h_\beta : U_\beta \to B$ are homeomorphisms, then $h_\alpha(U_\alpha \cap U_\beta)$ and $h_\beta(U_\alpha \cap U_\beta)$ are subsets of $\mathbb{R}^n$ and the map

$$h = h_\alpha \circ h_\beta^{-1} : \ h_\beta(U_\alpha \cap U_\beta) \to h_\alpha(U_\alpha \cap U_\beta)$$

is differentiable (or of class C^k) and for all $x \in h_\beta(U_\alpha \cap U_\beta)$, the determinant of the Jacobian, $\det Dh(x) \neq 0$.

Remark – The manifold is *analytic* if the maps $h = h_\alpha \circ h_\beta^{-1}$ are analytic. ◦

Next, recall that x^* is an *equilibrium point* of (4.1) if $f(x^*) = 0$. An equilibrium point x^* is *hyperbolic* if the Jacobian matrix Df of (4.1) evaluated at x^*, denoted $Df(x^*)$, has no eigenvalues with zero real part. Also, recall that the solutions of (4.1) form a one-parameter group that defines the *flow* of the nonlinear differential equation (4.1). To be more precise, consider the IVP consisting of (4.1) and an initial condition $x(t_0) = x_0$. Let $\mathcal{I}(x_0)$ be the maximal interval of existence of the solution to the IVP. Let then $\phi : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ be defined as follows: For $x_0 \in \mathbb{R}^n$ and $t \in \mathcal{I}(x_0)$, $\phi(t, x_0) = \phi_t(x_0)$ is the solution of the IVP defined on its maximal interval of existence $\mathcal{I}(x_0)$.

Example – Consider the (linear) ordinary differential equation $x' = ax$, with $a, x \in \mathbb{R}$. The solution is $\phi(t, x_0) = e^{at}x_0$, and satisfies the group property

$$\phi(t + s, x_0) = e^{a(t+s)}x_0 = e^{at}(e^{as}x_0) = \phi(t, e^{as}x_0) = \phi(t, \phi(s, x_0))$$

◇

For simplicity and without loss of generality since both results are local results, we assume hereforth that $x^* = 0$, *i.e.*, that a change of coordinates has been performed translating x^* to the origin. We also assume that $t_0 = 0$.

4.2 The stable manifold theorem

Theorem 4.2.1 (Stable manifold theorem)**.** *Let E be an open subset of $\mathbb{R}^n$ containing the origin, let $f \in C^1(E)$, and let ϕ_t be the flow of the nonlinear system (4.1). Suppose that $f(0) = 0$ and that $Df(0)$ has k eigenvalues with negative real part and $n - k$ eigenvalues with positive real part. Then there exists a k-dimensional differentiable manifold S tangent to the stable subspace E^s of the linear system (4.2) at 0 such that for all $t \geq 0$, $\phi_t(S) \subset S$ and for all $x_0 \in S$,*

$$\lim_{t\to\infty} \phi_t(x_0) = 0$$

and there exists an $(n - k)$-dimensional differentiable manifold U tangent to the unstable subspace E^u of (4.2) at 0 such that for all $t \leq 0$, $\phi_t(U) \subset U$ and for all $x_0 \in U$,

$$\lim_{t\to-\infty} \phi_t(x_0) = 0$$

There are several approaches to the proof of this result. Hale [10] gives a proof which uses functional analysis. The proof we give here comes from [18, p. 108-111], who derives it from [6, p. 330-335]. It consists in showing that there exists a real nonsingular constant matrix C such that if $y = C^{-1}x$ then there are $n - k$ real continuous functions $y_j = \psi_j(y_1, \dots, y_k)$ defined for small $|y_i|$, $i \leq k$, such that

$$y_j = \psi_j(y_1, \dots, y_k) \quad (j = k + 1, \dots, n)$$

define a k-dimensional differentiable manifold $\tilde{S}$ in y space. The stable manifold in S space is obtained by applying the transformation P^{-1} to y so that $x = P^{-1}y$ defines S in terms of k curvilinear coordinates $y_1, \dots, y_k$.

Proof. System (4.1) can be written as

$$x' = Df(0)x + F(x)$$

with $F(x) = f(x) - Df(0)x$. Since $f \in C^1(E)$, $F \in C^1(E)$, and $F(0) = f(0) = 0$ since $f(0) = 0$. Also, $DF(x) = Df(x) - Df(0)$ and so $DF(0) = 0$. To continue, we use the following lemma (which we will not prove).

Lemma 4.2.2. *Let E be an open subset of $\mathbb{R}^n$ containing the origin. If $F \in C^1(E)$, then for all $x, y \in N_\delta(0) \subset E$, there exists a $\xi \in N_\delta(0)$ such that*

$$|F(x) - F(y)| \leq \|DF(\xi)\| \, |x - y|$$

From Lemma 4.2.2, it follows that for all $\varepsilon > 0$ there exists a $\delta > 0$ such that $|x| \leq \delta$ and $|y| \leq \delta$ imply that

$$|F(x) - F(y)| \leq \varepsilon |x - y|$$

Let C be an invertible $n \times n$ matrix such that

$$B = C^{-1} Df(0) C = \begin{bmatrix} P & 0 \\ 0 & Q \end{bmatrix}$$

where the eigenvalues $\lambda_1, \ldots, \lambda_k$ of the $k \times k$ matrix P have negative real part and the eigenvalues $\lambda_{k+1}, \ldots, \lambda_n$ of the $(n-k) \times (n-k)$ matrix Q have positive real part. Let $\alpha > 0$ be chosen small enough that for $j = 1, \ldots, k$, $\Re(\lambda_j) < -\alpha < 0$. Let $y = C^{-1}x$, we have

$$\begin{aligned} y' &= C^{-1}x' \\ &= C^{-1}Df(0)x + C^{-1}F(x) \\ &= C^{-1}Df(0)Cy + C^{-1}F(Cy) \\ &= By + G(y) \end{aligned}$$

where $G(y) = C^{-1}F(Cy)$. Since $F \in C^1(E)$, $G \in C^1(\tilde{E})$, where $\tilde{E} = C^{-1}(E)$. Also, Lemma 4.2.2 applies to G.

Now consider the system

$$y' = By + G(y) \tag{4.3}$$

and let

$$U(t) = \begin{bmatrix} e^{Pt} & 0 \\ 0 & 0 \end{bmatrix} \text{ and } V(t) = \begin{bmatrix} 0 & 0 \\ 0 & e^{Qt} \end{bmatrix}$$

Then $U' = BU$, $V' = BV$ and $e^{Bt} = U(t) + V(t)$. Choosing as previously noted α sufficiently small, we can then choose $K > 0$ large enough and $\sigma > 0$ small enough that

$$\|U(t)\| \leq K e^{-(\alpha+\sigma)t} \text{ for all } t \geq 0$$

and

$$\|V(t)\| \leq K e^{-\sigma t} \text{ for all } t \leq 0$$

Consider now the integral equation

$$u(t,a) = U(t)a + \int_0^t U(t-s)G(u(s,a))ds - \int_t^\infty V(t-s)G(u(s,a))ds \tag{4.4}$$

where $a, u \in \mathbb{R}^n$ and a is a constant vector. We can solve this equation using the method of successive approximations. Indeed, let

$$u^{(0)}(t,a) = 0$$

and

$$u^{(j+1)}(t,a) = U(t)a + \int_0^t U(t-s)G(u^{(j)}(s,a))ds - \int_t^\infty V(t-s)G(u^{(j)}(s,a))ds \tag{4.5}$$

We show by induction that for all $j = 1, \ldots$ and $t \geq 0$,

$$|u^{(j)}(t,a) - u^{(j-1)}(t,a)| \leq \frac{K|a|e^{-\alpha t}}{2^{j-1}} \tag{4.6}$$

Clearly, (4.6) holds for $j = 1$ since

$$|u^{(1)}(t,a)| \leq \|U(t)a\| + \int_0^t \|U(t-s)G(u(s,a))\|ds + \int_t^\infty \|V(t-s)G(u(s,a))\|ds$$

Now suppose that (4.6) holds for $j = k$. We have

$$\begin{aligned}
|u^{(k+1)}(t,a) - u^{(k)}(t,a)| &= \left| \int_0^t U(t-s)\left[G(u^{(k+1)}(s,a)) - G(u^{(k)}(s,a))\right] ds \right. \\
&\quad \left. - \int_t^\infty V(t-s)\left[G(u^{(k+1)}(s,a)) - G(u^{(k)}(s,a))\right] ds \right| \\
&\leq \int_0^t \|U(t-s)\| \left|G(u^{(k+1)}(s,a)) - G(u^{(k)}(s,a))\right| ds \\
&\quad + \int_t^\infty \|V(t-s)\| \left|G(u^{(k+1)}(s,a)) - G(u^{(k)}(s,a))\right| ds
\end{aligned}$$

which, since G verifies a Lipschitz-type condition as given by Lemma 4.2.2, implies that there exists $\varepsilon > 0$ such that

$$\begin{aligned}
|u^{(k+1)}(t,a) - u^{(k)}(t,a)| &\leq \varepsilon \int_0^t \|U(t-s)\| \left|u^{(k+1)}(s,a) - u^{(k)}(s,a)\right| ds \\
&\quad + \varepsilon \int_t^\infty \|V(t-s)\| \left|u^{(k+1)}(s,a) - u^{(k)}(s,a)\right| ds
\end{aligned}$$

Using the bounds on $\|U\|$ and $\|V\|$ as well as the induction hypothesis (4.6), it follows that

$$\begin{aligned}
|u^{(k+1)}(t,a) - u^{(k)}(t,a)| &\leq \varepsilon \int_0^t K e^{-(\alpha+\sigma)(t-s)} \frac{K|a|e^{-\alpha s}}{2^{k-1}} ds \\
&\quad + \varepsilon \int_t^\infty K e^{\sigma(t-s)} \frac{K|a|e^{-\alpha s}}{2^{k-1}} ds \\
&\leq \frac{\varepsilon K^2 |a| e^{-\alpha t}}{\sigma 2^{k-1}} + \frac{\varepsilon K^2 |a| e^{-\alpha t}}{\sigma 2^{k-1}}
\end{aligned}$$

which, if we choose $\varepsilon < \sigma/(4K)$, *i.e.*, $\varepsilon K/\sigma < 1/4$, implies that

$$|u^{(k+1)}(t,a) - u^{(k)}(t,a)| < \left(\frac{1}{4} + \frac{1}{4}\right) \frac{K|a|e^{-\alpha t}}{2^{k-1}} = \frac{K|a|e^{-\alpha t}}{2^k} \tag{4.7}$$

Note that for G to satisfy a Lipschitz-type condition, we must choose $K|a| < \delta/2$, *i.e.*, $|a| < \delta/(2K)$. Then, by induction, (4.6) holds for all $t \geq 0$ and $j = 1, \ldots$.

As a consequence, for $t \geq 0$, $n > m > N$,

$$\begin{aligned}
|u^{(n)}(t,a) - u^{(m)}(t,a)| &\leq \sum_{j=N}^{\infty} |u^{(j+1)}(t,a) - u^{(j)}(t,a)| \\
&\leq K|a| \sum_{j=N}^{\infty} \frac{1}{2^j} = \frac{K|a|}{2^{N-1}}
\end{aligned}$$

As this last quantity approaches 0 as $N \to \infty$, it follows that $\{u^{(j)}(t,a)\}$ is a Cauchy sequence (of continuous functions).

It follows that

$$\lim_{j\to\infty} u^{(j)}(t,a) = u(t,a)$$

uniformly for all $t \geq 0$ and $|a| < \delta/(2K)$. From the uniform convergence, we deduce that $u(t,a)$ is continuous. Now taking the limit as $j \to \infty$ in both sides of (4.5), it follows that $u(t,a)$ satisfies the integral equation (4.4) and as a consequence, the differential equation (4.3).

Since $G \in C^1(\tilde{E})$, it follows from induction on (4.5) that $u^{(j)}(t,a)$ is a differentiable function of a for $|a| < \delta/(2K)$ and $t \geq 0$. Since $u^{(j)}(t,a) \to u(t,a)$ uniformly, it then follows that $u(t,a)$ is differentiable for $t \geq 0$ and $|a| < \delta/(2K)$. The estimate (4.7) implies that

$$|u(t,a)| \leq 2K|a|e^{-\alpha t} \tag{4.8}$$

for $t \geq 0$ and $|a| < \delta/(2K)$.

It is clear from (4.4) that the last $n - k$ components of a do not enter the computation of $u(t_0, a)$ and may thus be taken as zero. So the components $u_j(t,a)$ of the solution $u(t,a)$ satisfy the initial conditions

$$u_j(0,a) = a_j \text{ for } j = 1, \ldots, k$$

and

$$u_j(0,a) = -\left(\int_0^\infty V(-s)G(u(s,a_1,\ldots,a_k,0,\ldots,0))ds\right)_j \quad \text{for } j = k+1,\ldots,n$$

where $(\ )_j$ denotes the jth component. For $j = k+1,\ldots,n$, define the functions

$$\psi_j(a_1,\ldots,a_k) = u_j(0,a_1,\ldots,a_k,0,\ldots,0) \tag{4.9}$$

The initial values $y_j = u_j(0,a_1,\ldots,a_k,0,\ldots,0)$ then satisfy

$$y_j = \psi_j(y_1,\ldots,y_k) \text{ for } j = k+1,\ldots,n$$

which defines a differentiable manifold $\tilde{S}$ in y space for $\sqrt{y_1^2+\cdots+y_k^2} < \delta/(2K)$. Furthermore, if $y(t)$ is a solution of the differential equation (4.3) with $y(0) \in \tilde{S}$, *i.e.*, with $y(0) = u(0,a)$, then

$$y(t) = u(t,a)$$

It follows from the estimate (4.8) that if $y(t)$ is a solution of (4.3) with $y(0) \in \tilde{S}$, then $y(t) \to 0$ as $t \to \infty$. It can also be shown that if $y(t)$ is a solution of (4.3) with $y(0) \notin \tilde{S}$, then $y(t) \not\to 0$ as $t \to \infty$; see [6, p. 332].

This implies, as ϕ_t satisfies the group property $\phi_{s+t}(x_0) = \phi_s(\phi_t)(x_0)$, that if $y(0) \in \tilde{S}$, then $y(t) \in \tilde{S}$ for all $t \geq 0$. And it can be shown as in [6, Th 4.2, p. 333] that

$$\frac{\partial \psi_j}{\partial y_i}(0) = 0$$

for $i = 1,\ldots,k$ and $j = k+1,\ldots,n$, *i.e.*, that the differentiable manifold $\tilde{S}$ is tangent to the stable subspace $E^s = \{y \in \mathbb{R}^n :\ y_1 = \cdots = y_k = 0\}$ of the linear system $y' = By$ at 0.

The existence of the unstable manifold $\tilde{U}$ of (4.3) is established the same way, but considering a reversal of time, $t \to -t$, *i.e.*, considering the system

$$y' = -By - G(y)$$

The stable manifold for this system is the unstable manifold $\tilde{U}$ of (4.3). In order to determine the $(n-k)$-dimensional manifold $\tilde{U}$ using the above process, the vector y has to be replaced by the vector $(y_{k+1},\ldots,y_n,y_1,\ldots,y_k)$. □

4.3 The Hartman-Grobman theorem

Adapted from [4, p. 311].

Theorem 4.3.1 (Hartman-Grobman)**.** *Suppose that 0 is an equilibrium point of the nonlinear system (4.1). Let φ_t be the flow of (4.1), and ψ_t be the flow of the linearized system $x' = Df(0)x$. If 0 is a hyperbolic equilibrium, then there exists an open subset $\mathcal{D}$ of $\mathbb{R}^n$ containing 0, and a homeomorphism G with domain in $\mathcal{D}$ such that $G(\varphi_t(x)) = \psi_t(G(x))$ whenever $x \in \mathcal{D}$ and both sides of the equation are defined.*

We follow here [18].

Theorem 4.3.2 (Hartman-Grobman). *Let E be an open subset of $\mathbb{R}^n$ containing the origin, let $f \in C^1(E)$, and let ϕ_t be the flow of the nonlinear system (4.1). Suppose that $f(0) = 0$ and that the matrix $A = Df(0)$ has no eigenvalue with zero real part.*

Then there exists a homeomorphism H of an open set U containing the origin onto an open set V containing the origin such that for each $x_0 \in U$, there is an open interval $\mathcal{I}_0 \subset \mathbb{R}$ containing 0 such that for all $x_0 \in U$ and $t \in \mathcal{I}_0$,

$$H \circ \phi_t(x_0) = e^{At} H(x_0);$$

i.e., *H maps trajectories of (4.1) near the origin onto trajectories of $x' = Df(0)x$ near the origin and preserves the parametrization by time.*

Proof. Suppose that $f \in C^1(E)$, $f(0) = 0$ (*i.e.*, 0 is an equilibrium) and $A = Df(0)$ the jacobian matrix of f at 0.

1. As we have assumed that the matrix A has no eigenvalues with zero real part (*i.e.*, 0 is an hyperbolic equilibrium point), we can write A in the form

$$A = \begin{bmatrix} P & 0 \\ 0 & Q \end{bmatrix}$$

where P has only eigenvalues with negative real parts and Q has only eigenvalues with positive real parts.

2. Let ϕ_t be the flow of the nonlinear system (4.1). Let us write the solution as

$$x(t, x_0) = \phi_t(x_0) = \begin{bmatrix} y(t, y_0, z_0) \\ z(t, y_0, z_0) \end{bmatrix}$$

where

$$x_0 = \begin{bmatrix} y_0 \\ z_0 \end{bmatrix} \in \mathbb{R}^n$$

has been decomposed as $y_0 \in E^s$, the stable subspace of A, and $z_0 \in E^u$, the unstable subspace of A.

3. Let

$$\tilde{Y}(y_0, z_0) = y(1, y_0, z_0) - e^P y_0$$

and

$$\tilde{Z}(y_0, z_0) = z(1, y_0, z_0) - e^Q z_0$$

Then $\tilde{Y}(0) = \tilde{Y}(0, 0) = y(1, 0, 0) = 0$. The same is true of $\tilde{Z}(0) = 0$. Also, $D\tilde{Y}(0) = D\tilde{Z}(0) = 0$. Since $f \in C^1(E)$, $\tilde{Y}(y_0, z_0)$ and $\tilde{Z}(y_0, z_0)$ are continuously differentiable. Thus

$$\|D\tilde{Y}(y_0, z_0)\| \leq a$$

and

$$\|D\tilde{Z}(y_0, z_0)\| \leq a$$

on the compact set $|y_0|^2 + |z_0|^2 \leq s_0^2$. By choosing s_0 sufficiently small, we can make a as small as we like. We let $Y(y_0, z_0)$ and $Z(y_0, z_0)$ be smooth functions, defined by

$$Y(y_0, z_0) = \begin{cases} \tilde{Y}(y_0, z_0) & \text{for } |y_0|^2 + |z_0|^2 \leq \left(\frac{s_0}{2}\right)^2 \\ 0 & \text{for } |y_0|^2 + |z_0|^2 \geq s_0^2 \end{cases}$$

and

$$Z(y_0, z_0) = \begin{cases} \tilde{Z}(y_0, z_0) & \text{for } |y_0|^2 + |z_0|^2 \leq \left(\frac{s_0}{2}\right)^2 \\ 0 & \text{for } |y_0|^2 + |z_0|^2 \geq s_0^2 \end{cases}$$

By the mean value theorem,

$$|Y(y_0, z_0)| \leq a\sqrt{|y_0|^2 + |z_0|^2} \leq a(|y_0| + |z_0|)$$

and

$$|Z(y_0, z_0)| \leq a\sqrt{|y_0|^2 + |z_0|^2} \leq a(|y_0| + |z_0|)$$

for all $(y_0, z_0) \in \mathbb{R}^n$. Let $B = e^P$ and $C = e^Q$. Assuming that P and Q have been normalized in a proper way, we have

$$b = \|B\| < 1 \text{ and } c = \|C^{-1}\| < 1$$

4. For

$$x = \begin{bmatrix} y \\ z \end{bmatrix} \in \mathbb{R}^n$$

define the transformations

$$L(y, z) = \begin{bmatrix} By \\ Cz \end{bmatrix}$$

and

$$T(y, z) = \begin{bmatrix} By + Y(y, z) \\ Cz + Z(y, z) \end{bmatrix}$$

i.e., $L(x) = e^A x$ and, locally, $T(x) = \phi_1(x)$. Then the following lemma holds, which we prove later.

Lemma 4.3.3. *There exists a homeomorphism H of an open set U containing the origin onto an open set V containing the origin such that*

$$H \circ T = L \circ H$$

5. We let H_0 be the homeomorphism defined above and L^t and T^t be the one-parameter families of transformations defined by

$$L^t(x_0) = e^{At}x_0 \text{ and } T^t(x_0) = \phi_t(x_0)$$

Define

$$H = \int_0^1 L^{-s} H_0 T^s ds$$

It follows from the above lemma that there exists a neighborhood of the origin for which

$$
\begin{aligned}
L^t H &= \int_0^1 L^{t-s} H_0 T^{s-t} ds T^t \\
&= \int_{-t}^{1-t} L^{-s} H_0 T^s ds T^t \\
&= \left[\int_{-t}^0 L^{-s} H_0 T^s ds + \int_0^{1-t} L^{-s} H_0 T^s ds \right] T^t \\
&= \int_0^1 L^{-s} H_0 T^s ds T^t \\
&= H T^t
\end{aligned}
$$

since by the above lemma, $H_0 = L^{-1} H_0 T$ which implies that

$$
\begin{aligned}
\int_{-s}^0 L^{-s} H_0 T^s ds &= \int_{-t}^0 L^{-s-1} H_0 T^{s+1} ds \\
&= \int_{1-t}^1 L^{-s} H_0 T^s ds
\end{aligned}
$$

Thus $H \circ T^t = L^t H$ or equivalently

$$
H \circ \phi_t(x_0) = e^{At} H(x_0)
$$

and it can be shown that H is a homeomorphism on $\mathbb{R}^n$. The outline of the proof is complete. □

We now prove Lemma 4.3.3.

Proof. We use the method of successive approximations. For $x \in \mathbb{R}^n$, let

$$
H(x) = \begin{bmatrix} \Phi(y,z) \\ \Psi(y,z) \end{bmatrix}
$$

Then $H \circ T = L \circ H$ is equivalent to the pair of equations

$$
B\Phi(y,z) = \Phi(By + Y(y,z), Cz + Z(y,z)) \tag{4.10a}
$$

$$
C\Psi(y,z) = \Psi(By + Y(y,z), Cz + Z(y,z)) \tag{4.10b}
$$

Successive approximations for (4.10b) are defined by

$$
\begin{aligned}
\Psi_0(y,z) &= z \\
\Psi_{k+1}(y,z) &= C^{-1}\Psi_k(By + Y(y,z), Cz + Z(y,z))
\end{aligned} \tag{4.11}
$$

It can be shown by induction that for $k = 0, 1, \ldots$ the functions Ψ_k are continuous and such that $\Psi_k(y,z) = z$ for $|y| + |z| \geq 2s_0$.

Let us now prove that $\{\Psi_k\}$ is a Cauchy sequence. For this, we show by induction that for all $j \geq 1$,

$$|\Psi_j(y,z) - \Psi_{j-1}(y,z)| \leq Mr^j(|y|+|z|)^\delta \tag{4.12}$$

where $r = c[2\max(a,b,c)]^\delta$ with $\delta \in (0,1)$ chosen sufficiently small that $r < 1$ (which is possible since $c < 1$) and $M = ac(2s_0)^{1-\delta}/r$. Inequality (4.12) is satisfied for $j = 1$ since

$$\begin{aligned}
|\Psi_1(y,z) - \Psi_0(y,z)| &= |C^{-1}\Psi_0(By + Y(y,z), Cz + Z(y,z)) - z| \\
&= |C^{-1}(Cz + Z(y,z)) - z| \\
&= |C^{-1}Z(y,z)| \\
&\leq \|C^{-1}\|\,|Z(y,z)| \\
&\leq ca(|y|+|z|) \\
&\leq Mr(|y|+|z|)^\delta
\end{aligned}$$

since $Z(y,z) = 0$ for $|y|+|z| \geq 2s_0$. Now assuming that (4.12) holds for $j = k$ gives

$$\begin{aligned}
|\Psi_{k+1}(y,z) - \Psi_k(y,z)| &= |C^{-1}\Psi_k(By+Y(y,z), Cz+Z(y,z)) - C^{-1}\Psi_{k-1}(By+Y(y,z), Cz+Z(y,z))| \\
&= |C^{-1}(\Psi_k - \Psi_{k-1})| \\
&\leq \|C^{-1}\|\,|\Psi_k - \Psi_{k-1}|
\end{aligned}$$

which, using induction hypothesis (4.12) and $c = \|C^{-1}\|$ gives

$$\begin{aligned}
&\leq cMr^k\Big(|By+Y(y,z)| + |Cz+Z(y,z)|\Big)^\delta \\
&\leq cMr^k\Big(b|y| + 2a(|y|+|z|) + c|z|\Big)^\delta \\
&\leq cMr^k\big(2\max(a,b,c)\big)^\delta\big(|y|+|z|\big)^\delta \\
&\leq Mr^{k+1}\big(|y|+|z|\big)^\delta
\end{aligned}$$

Using the same type of argument as in the proof of the stable manifold theorem, Ψ_k is thus a Cauchy sequence of continuous functions that converges uniformly as $k \to \infty$ to a continuous function $\Psi(y,z)$. Also, $\Psi(y,z) = z$ for $|y|+|z| \geq 2s_0$. Taking limits in (4.11) and left-multiplying by C shows that $\Psi(y,z)$ is a solution of (4.10b).

Now for (4.10a). This equation can be written

$$B^{-1}\Phi(y,z) = \Phi(B^{-1}y + Y_1(y,z), C^{-1}z + Z_1(y,z)) \tag{4.13}$$

where Y_1 and Z_1 occur in the inverse of T, which exists provided that a is small enough (*i.e.*, s_0 is sufficiently small),

$$T^{-1}(y,z) = \begin{bmatrix} B^{-1}y + Y_1(y,z) \\ C^{-1}z + Z_1(y,z) \end{bmatrix}$$

Successive approximations with $\Phi_0(y,z) = y$ can then be used as above (since $b = \|B\| < 1$) to solve (4.13).

Therefore, we obtain the continuous map

$$H(y,z) = \begin{bmatrix} \Phi(y,z) \\ \Psi(y,z) \end{bmatrix}$$

and it follows as in [11, p. 248-249] that H is a homeomorphism of $\mathbb{R}^n$ onto $\mathbb{R}^n$. □

4.4 Example of application

4.4.1 A chemostat model

To illustrate the use of the theorems in this chapter, we take an example of nonlinear system, a system of two nonlinear differential equations modeling a biological device called a *chemostat*. Without going into details, the system is the following.

$$\frac{dS}{dt} = D(S^0 - S) - \mu(S)x \tag{4.14a}$$

$$\frac{dx}{dt} = (\mu(S) - D)x \tag{4.14b}$$

The parameters S^0 and D, respectively the *input concentration* and the *dilution* rate, are real and positive. The function μ is the *growth* function. It is generally assumed to satisfy $\mu(0) = 0$, $\mu' > 0$ and $\mu'' < 0$.

To be complete, one should verify that the positive quadrant is positively invariant under the flow of (4.14), *i.e.*, that for $S(0) \geq 0$ and $x(0) \geq 0$, solutions remain nonnegative for all positive times, and similar properties. But since we are here only interested in applications of the stable manifold theorem, we proceed to a very crude analysis, and will not deal with this point.

Note that in vector form, the system is noted

$$\xi' = f(\xi)$$

with $\xi = (S, x)^T$ and

$$f(\xi) = \begin{pmatrix} D(S^0 - S) - \mu(S)x \\ (\mu(S) - D)x \end{pmatrix}$$

Equilibria of the system are found by solving $f(\xi) = 0$. We find two, the first one situated on one of the boundaries of the positive quadrant,

$$\xi_T^* = (S_T^*, x_T^*) = (S^0, 0)$$

the second one in the interior,

$$\xi_I^* = (S^*, x^*) = (\lambda, S^0 - \lambda)$$

where λ is such that $\mu(\lambda) = D$. Note that this implies that if $\lambda \geq S^0$, ξ_T^* is the only equilibrium of the system.

At an arbitrary point $\xi = (S, x)$, the Jacobian matrix is given by

$$Df(\xi) = \begin{pmatrix} -D - \mu'(S)x & -\mu(S) \\ \mu'(S)x & \mu(S) - D \end{pmatrix}$$

Thus, at the trivial equilibrium ξ_T^*,

$$Df(\xi_T^*) = \begin{pmatrix} -D & -\mu(S^0) \\ 0 & \mu(S^0) - D \end{pmatrix}$$

We have two eigenvalues, $-D$ and $\mu(S^0) - D$. Let us suppose that $\mu(S^0) - D < 0$. Note that this implies that ξ_T^* is the only equilibrium, since, as we have seen before, ξ_I^* is not feasible if $\lambda > S^0$.

As the system has dimensionality 2, and that the matrix $Df(\xi_T^*)$ has two negative eigenvalues, the stable manifold theorem (Theorem 4.2.1) states that there exists a 2-dimensional differentiable manifold $\mathcal{M}$ such that

- $\phi_t(\mathcal{M}) \subset \mathcal{M}$
- for all $\xi_0 \in \mathcal{M}$, $\lim_{t\to\infty} \phi_t(\xi_0) = \xi_T^*$.
- At ξ_T^*, $\mathcal{M}$ is tangent to the stable subspace E^S of the linearized system $\xi' = Df(\xi_T^*)(\xi - \xi_T^*)$.

Since there are no eigenvalues with positive real part, there does not exist an unstable manifold in this case. Let us now caracterize the nature of the stable subspace E^S. It is obtained by studying the linear system

$$\begin{aligned} \xi' &= Df(\xi_T^*)(\xi - \xi_T^*) \\ &= \begin{pmatrix} -D & -\mu(S^0) \\ 0 & \mu(S^0) - D \end{pmatrix} \begin{pmatrix} S - S^0 \\ x \end{pmatrix} \\ &= \begin{pmatrix} -D(S - S^0) - \mu(S^0)x \\ (\mu(S^0) - D)x \end{pmatrix} \end{aligned} \tag{4.15}$$

Of course, the Jacobian matrix associated to this system is the same as that of the nonlinear system (at ξ_T^*). Associated to the eigenvalue $-D$ is the eigenvector $v_1 = (1, 0)^T$, to $\mu(S^0) - D$ is $v_2 = (-1, 1)^T$.

The stable subspace is thus given by Span (v_1, v_2), *i.e.*, the whole of $\mathbb{R}^2$.

In fact, the stable manifold of ξ_T^* is the whole positive quadrant, since all solutions limit to this equilibrium. But let us pretend that we do not have this information, and let us try to find an approximation of the stable manifold.

4.4.2 A second example

This example is adapted from [18, p. 111]. Consider the nonlinear system

$$\begin{aligned} x' &= -x - y^2 \\ y' &= x^2 + y \end{aligned} \tag{4.16}$$

From the nullclines equations, it is clear that $(x, y) = (0, 0)$ is the only equilibrium point. At $(0, 0)$, the Jacobian matrix of (4.16) is given by

$$J = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

The linearized system at 0 is

$$\begin{aligned} x' &= -x \\ y' &= y \end{aligned} \tag{4.17}$$

So the eigenvalues are 1 and -1, with associated eigenvectors $(1, 0)^T$ and $(0, 1)^T$, respectively. Therefore, the stable manifold theorem (Theorem 4.2.1) implies that there exists a 1-dimensional stable (differentiable) manifold S such that $\phi_t(S) \subset S$ and $\lim_{t\to\infty} \phi_t(x_0) = 0$ for all $x_0 \in S$, and a 1-dimensional unstable (differentiable) manifold U such that $\phi_t(U) \subset U$ and $\lim_{t\to-\infty} \phi_t(x_0)$ for all $x_0 \in U$. Furthermore, at 0, S is tangent to the stable subspace E^S of (4.17), and U is tangent to the unstable subspace E^U of (4.17).

The stable subspace E^S is given by Span (v_1), with $v_1 = (0, 1)^T$, *i.e.*, the y-axis. The unstable subspace E^U is Span (v_2), with $v_2 = (1, 0)^T$, *i.e.*, the x-axis. The behavior of this system is illustrated in Figure 4.1.

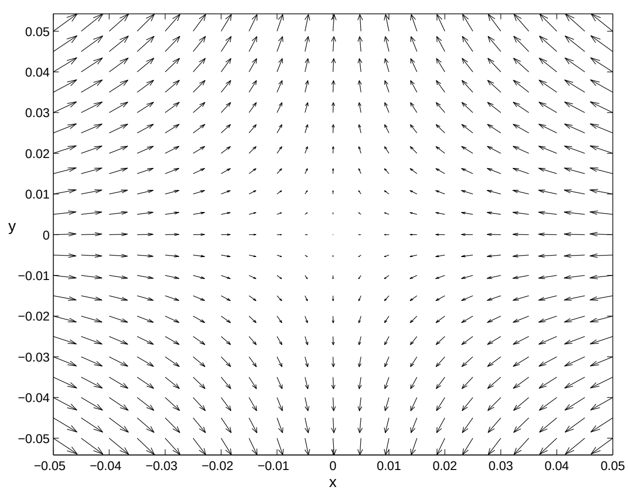

Figure 4.1: Vector field of system (4.16) in the neighborhood of 0.

To be more precise about the nature of the stable manifold S, we proceed as follows. First of all, as A is in diagonal form, we have

$$A = B = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

and $C = I$. Also, $F(\xi) = G(\xi) = \begin{pmatrix} -y^2 \\ x^2 \end{pmatrix}$. Here, the matrices P and Q are in fact scalars,

$P = -1$ and $Q = 1$. Thus

$$U(t) = \begin{pmatrix} e^{-t} & 0 \\ 0 & 0 \end{pmatrix}, \quad V(t) = \begin{pmatrix} 0 & 0 \\ 0 & e^{t} \end{pmatrix}$$

Finally, $a = (a_1, 0)^T$. So now we can use successive approximations to find an approximate solution to the integral equation (4.4), which here takes the form

$$u(t,a) = \begin{pmatrix} e^{-t}a_1 \\ 0 \end{pmatrix} + \int_0^t \begin{pmatrix} -e^{-(t-s)}u_2^2(s) \\ 0 \end{pmatrix} ds - \int_t^\infty \begin{pmatrix} 0 \\ e^{(t-s)}u_1^2(s) \end{pmatrix} ds$$

To construct the sequence of successive approximations, we start with $u(t,a) = (0,0)^T$, then compute the successive terms using equation (4.5), which takes the form

$$\begin{aligned} u^{(j+1)}(t,a) &= \begin{pmatrix} e^{-t}a_1 \\ 0 \end{pmatrix} + \int_0^t \begin{pmatrix} e^{-(t-s)} & 0 \\ 0 & 0 \end{pmatrix} G\left(u^{(j)}(s)\right) ds - \int_t^\infty \begin{pmatrix} 0 & 0 \\ 0 & e^{(t-s)} \end{pmatrix} G\left(u^{(j)}(s)\right) ds \\ &= \begin{pmatrix} e^{-t}a_1 \\ 0 \end{pmatrix} + \int_0^t \begin{pmatrix} e^{-(t-s)} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \left(u_2^{(j)}(s)\right)^2 \\ \left(u_1^{(j)}(s)\right)^2 \end{pmatrix} ds - \int_t^\infty \begin{pmatrix} 0 & 0 \\ 0 & e^{(t-s)} \end{pmatrix} \begin{pmatrix} \left(u_2^{(j)}(s)\right)^2 \\ \left(u_1^{(j)}(s)\right)^2 \end{pmatrix} ds \\ &= \begin{pmatrix} e^{-t}a_1 \\ 0 \end{pmatrix} + \int_0^t \begin{pmatrix} -e^{-(t-s)}\left(u_2^{(j)}(s)\right)^2 \\ 0 \end{pmatrix} ds - \int_t^\infty \begin{pmatrix} 0 \\ e^{(t-s)}\left(u_1^{(j)}(s)\right)^2 \end{pmatrix} ds \end{aligned}$$

Therefore,

$$u^{(1)}(t,a) = U(t)a = \begin{pmatrix} e^{-a}a_1 \\ 0 \end{pmatrix}$$

since $u^{(0)}(t,a) = \begin{pmatrix} u_1^{(0)}(t,a) \\ u_2^{(0)}(t,a) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

Then,

$$\begin{aligned} u^{(2)}(t,a) &= \begin{pmatrix} e^{-t}a_1 \\ 0 \end{pmatrix} - \int_t^\infty \begin{pmatrix} 0 \\ e^{(t-s)}\left(e^{-s}a_1\right)^2 \end{pmatrix} ds \\ &= \begin{pmatrix} e^{-t}a_1 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ \frac{1}{3}a_1^2 e^{-2t} \end{pmatrix} \\ &= \begin{pmatrix} e^{-t}a_1 \\ -\frac{1}{3}a_1^2 e^{-2t} \end{pmatrix} \end{aligned}$$

and continuing this process, we find

$$u^{(3)}(t,a) = \begin{pmatrix} e^{-t}a_1 + \frac{1}{27}(e^{-4t} - e^{-t})a_1^4 \\ -\frac{1}{3}a_1^2 e^{-2t} \end{pmatrix}$$

Also, it is possible to show that $u^{(4)}(t,a) - u^{(3)}(t,a) = \mathcal{O}(a_1^5)$.

The stable manifold $\mathcal{S}$ is 1-dimensional, so here it has the form $\psi_2(a_1) = u_2(0, a_1, 0)$, and is here approximated by

$$\psi_2(a_1) = -\frac{1}{3}a_1^2 + \mathcal{O}(a_1^5)$$

as $a_1 \to 0$. Thus $\mathcal{S}$ is approximated by

$$y = -\frac{x^2}{3} + \mathcal{O}(x^5)$$

as $x \to 0$.

Chapter 5

Exponential dichotomy

Our aim here is to show the equivalent of the Hartman-Grobman theorem for linear systems with variable coefficients. Compared to other results we have seen so far, this is a much more recent field. The first results were shown in the 60s by Lin. We give here only the most elementary results. For more details, see, *e.g.*, [13].

We consider the linear system of differential equations

$$\frac{dx}{dt} = A(t)x \tag{5.1}$$

where the $n \times n$ matrix $A(t)$ is continuous on the real axis.

5.1 Exponential dichotomy

Definition 5.1.1 (Exponential dichotomy)**.** *Let $X(t)$ be a the fundamental matrix solution of (5.1). If $X(t)$ and $X^{-1}(s)$ can be decomposed into the following forms*

$$\begin{aligned} X(t) &= X_1(t) + X_2(t) \\ X^{-1}(s) &= Z_1(s) + Z_2(s) \\ X(t)Z^{-1}(s) &= X_1(t)Z_1(s) + X_2(t)Z_2(s) \end{aligned}$$

and satisfy the conditions that there exists α, β, positive constants such that

$$\begin{aligned} \|X_1(t)Z_1(s)\| &\leq \beta e^{-\alpha(t-s)}, \quad t \geq s \\ \|X_2(t)Z_2(s)\| &\leq \beta e^{\alpha(t-s)}, \quad\ \ s \geq t \end{aligned} \tag{5.2}$$

where

$$X_1(t) = (X_{11}(t), 0), \quad X_2(t) = (0, X_{12}(t)),$$
$$Z_1(s) = \begin{pmatrix} Z_{11}(s) \\ 0 \end{pmatrix}, \quad Z_2(s) = \begin{pmatrix} 0 \\ Z_{21}(s) \end{pmatrix},$$

or there is a projection P on the stable manifold such that

$$\begin{aligned} \|X(t)PX^{-1}(s)\| &\leq \beta e^{-\alpha(t-s)}, \quad t \geq s \\ \|X(t)(I-P)X^{-1}(s)\| &\leq \beta e^{\alpha(t-s)}, \quad s \geq t \end{aligned} \tag{5.3}$$

then we say that the system (5.1) admits exponential dichotomy*.*

Remark – A matrix P defines a projection if it is such that $P^2 = P$. ∘

Another definition [1].

Definition 5.1.2. *Let $\Phi(t,s)$, $\Phi(t,t) = I$, be the principal matrix solution of (5.1). We say that (5.1) has an exponential dichotomy on the interval $\mathcal{I}$ if there are projections $P(t) : \mathbb{R}^n \to \mathbb{R}^n$, $t \in \mathcal{I}$, continuous in t, such that if $Q(t) = I - P(t)$, then*

i) $\Phi(t,s)P(s) = P(t)\phi(t,s)$*, for* $t, s \in \mathcal{I}$*.*

ii) $\|\Phi(t,s)P(s)\| \leq Ke^{-\alpha(t-s)}$*, for* $t \geq s \in \mathcal{I}$*.*

iii) $\|\Phi(t,s)Q(s)\| \leq Ke^{\alpha(t-s)}$*, for* $s \geq t \in \mathcal{I}$*.*

A more general definition is the following (see, *e.g.*, [16]).

Definition 5.1.3 (**($\boldsymbol{\mu_1, \mu_2}$)**-dichotomy)**.** *If μ_1, μ_2 are continuous real-valued functions on the real interval $\mathcal{I} = (\omega_-, \omega_+)$, system (5.1) is said to have a* (μ_1, μ_2)-dichotomy *if there exist supplementary projections P_1, P_2 on $\mathbb{R}^n$ such that*

$$\|X(t)P_iX^{-1}(s)\| \leq K_i \exp(\int_s^t \mu_i), \text{ if } (-1)^i(s-t) \geq 0, \quad i = 1, 2.$$

where K_1, K_2 are positive constants.

In the case that μ_1, μ_2 are constants, the system (5.1) is said to have an exponential dichotomy *if $\mu_1 < 0 < \mu_2$ and an* ordinary dichotomy *if $\mu_1 = \mu_2 = 0$.*

The following remark is from [7].

Remark – The autonomous system

$$x' = Ax$$

has an exponential dichotomy on $\mathbb{R}_+$ if and only if no eigenvalue of A has zero real part. It has ordinary dichotomy is and only if all eigenvalues of A with zero real part are semisimple (their algebraic multiplicities are equal to their geometric multiplicities, i.e., the dimension of their eigenspace). In each case, $X(t) = e^{tA}$ and we can take P to be the spectral projection defined by

$$P = \frac{1}{2\pi i} \int_\gamma (zI - A)^{-1} dz$$

with γ any rectifiable simple closed curve in the open left half-plane which contains in its interior all eigenvalues of A with negative real part. ∘

5.2 Existence of exponential dichotomy

To check that the previous definitions hold can be a very tedious task. Some authors have thus worked on deriving simpler conditions that imply exponential dichotomy.

Theorem 5.2.1. *If the matrix $A(t)$ in (5.1) is continuous and bounded on $\mathbb{R}$, and there exists a quadratic form $V(t,x) = x^T G(t)x$, where the matrix $G(t)$ is symmetric, regular, bounded and C^1, such that the derivative of $V(t,x)$ with respect to (5.1) is positive definite, then (5.1) admits exponential dichotomy.*

The converse is true, without the requirement that $A(t)$ be bounded.

A result of [7].

Theorem 5.2.2. *Let $A(t)$ be a continuous $n \times n$ matrix function defined on an interval $\mathcal{I}$ such that*

i) $A(t)$ has k eigenvalues with real part $\leq -\alpha < 0$ and $n-k$ eigenvalues with real part $\geq \beta > 0$ for all $t \in \mathcal{I}$,

ii) $\|A(t)\| \leq M$ for all $t \in \mathcal{I}$.

For any positive constant $\varepsilon < \min(\alpha, \beta)$, there exists a positive constant $\delta = \delta(M, \alpha+\beta, \varepsilon)$ such that, if

$$\|A(t_2) - A(t_1)\| \leq \delta \text{ for } |t_2 - t_1| \leq h$$

where $h > 0$ is a fixed number not greater than the length of $\mathcal{I}$, then the equation

$$x' = A(t)x$$

has a fundamental matrix $X(t)$ satisfying the inequalities

$$\|X(t)\tilde{P}X^{-1}(s)\| \leq Ke^{-(\alpha-\varepsilon)(t-s)} \text{ for } t \geq s$$

$$\|X(t)(I-\tilde{P})X^{-1}(s)\| \leq Le^{-(\beta-\varepsilon)(s-t)} \text{ for } s \geq t$$

where K, L are positive constants depending only on $M, \alpha+\beta, \varepsilon$ and

$$\tilde{P} = \begin{pmatrix} I_k & 0 \\ 0 & 0 \end{pmatrix}$$

The following result, due to Muldowney [16], gives a criterion for the existence of a (μ_1, μ_2)-dichotomy.

Proposition 5.2.3. *Suppose there is a continuous real-valued function ρ on $\mathcal{I}$ and constants l_i, $0 \leq l_i < 1$, $i = 1, 2$, such that for some m, $0 \leq m \leq n$,*

$$\max\left\{ l_1 \Re(a_{jj}) + l_1 \sum_{i=1, i\neq j}^{m} |a_{ij}| + \sum_{i=m+1}^{n} |a_{ij}| : j = 1, \ldots, m \right\} \leq l_1 \rho,$$

$$\min\left\{l_2\Re(a_{jj}) - \sum_{i=1}^{m}|a_{ij}| - l_2\sum_{i=m+1,i\neq j}^{n}|a_{ij}| : j=m+1,\ldots,n\right\} \geq l_2\rho.$$

Then the system (5.1) has a (μ_1,μ_2)-dichotomy, where

$$\mu_1 = \max\left\{l_1\Re(a_{jj}) + \sum_{i=1,i\neq j}^{m}|a_{ij}| + l_2\sum_{i=m+1}^{n}|a_{ij}| : j=1,\ldots,m\right\},$$

$$\mu_2 = \min\left\{\Re(a_{jj}) - l_1\sum_{i=1}^{m}|a_{ij}| - \sum_{i=m+1,i\neq j}^{n}|a_{ij}| : j=m+1,\ldots,n\right\}.$$

The same sort of theorem can be proved with sums of the columns replaced by sums of the rows.

Example – Consider

$$A(t) = \begin{pmatrix} -1 & 0 & 1/2 \\ t/2 & t & t^2 \\ t/2 & -t^2 & t \end{pmatrix}, \quad t>0$$

◇

5.3 First approximate theory

We consider the nonlinear system

$$\frac{dx}{dt} = A(t)x + f(t,x) \tag{5.4}$$

where $f:\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}^n$ is continuous, $f(t,x)=\mathcal{O}(\|x\|^2)$, $\|x\|=o(1)$, $\|f(t,x_1)-f(t,x_2)\|\leq L\|x_1-x_2\|$ with L small enough.

Let $x(t)$ be a non trivial solution of (5.1); define

$$\bar\lambda_u(x(t)) = \limsup_{t-s\to\infty}\frac{1}{t-s}\log\frac{\|x(t)\|}{\|x(s)\|}$$

and

$$\underline{\lambda}_u(x(t)) = \liminf_{t-s\to\infty}\frac{1}{t-s}\log\frac{\|x(t)\|}{\|x(s)\|}$$

The numbers $\bar\lambda_u(x(t))$ and $\underline{\lambda}_u(x(t))$ are called the *uniform upper characteristic exponent* and *uniform lower characteristic exponent* of $x(t)$, respectively.

Remark – If $\bar\lambda(x)\leq-\alpha<0$, then $\lim_{s\to-\infty}\|x(s)\|=\infty$. If $\underline{\lambda}(x)\geq\alpha>0$, then $\lim_{t\to\infty}\|x(t)\|=\infty$. ◦

Then we have the following theorem.

Theorem 5.3.1. *If (5.1) admits the exponential dichotomy, then the linear system (5.1) and the nonlinear system (5.4) are topologically equivalent,* i.e.,

i) if the solution $x(t)$ of (5.4) remains in a neighborhood of the origin for $t \geq 0$, or $t \leq 0$, then $\lim_{t\to\infty} x(t) = 0$, or $\lim_{t\to-\infty} x(t) = 0$, respectively;

ii) there exists positive constants α_0 and β_0 such that if a solution $x(t)$ of (5.4) is such that $\lim_{t\to\infty} x(0) = 0$, or $\lim_{t\to-\infty} x(t) = 0$, then

$$\|x(t)\| \leq \beta_0 \|x(s)\| e^{-\alpha_0(t-s)}, \quad t \geq s$$

or

$$\|x(t)\| \leq \beta_0 \|x(s)\| e^{\alpha_0(t-s)}, \quad s \geq t$$

respectively. At this time, $\bar{\lambda}_u(x(t)) \leq -\alpha_0 < 0$, or $\underline{\lambda}_u(x(t)) \geq \alpha_0 > 0$;

iii) for a k-dimensional solution x of (5.1) with $\bar{\lambda}_u(x(t)) \leq -\alpha < 0$, or an $(n-k)$-dimensional solution $x(t)$ of (5.1) with $\underline{\lambda}_u(x(t)) \geq \alpha > 0$, there is a unique k-dimensional or $(n-k)$-dimensional $y(t)$, solution of (5.4), such that $\bar{\lambda}_u(x(t)) \leq -\alpha < 0$, or $\underline{\lambda}_u(x(t)) \geq \alpha > 0$ respectively.

A different statement of the same sort of result is given by Palmer [17].

Theorem 5.3.2. *Suppose that $A(t)$ is a continuous matrix function such that the linear equation $x' = A(t)x$ has an exponential dichotomy. Suppose that $f(t,x)$ is a continuous function of $\mathbb{R} \times \mathbb{R}^n$ into $\mathbb{R}^n$ such that*

$$\|f(t,x)\| \leq \mu, \qquad \|f(t,x_1) - f(t,x_2)\| \leq L\|x_1 - x_2\|$$

for all t, x, x_1, x_2. Then if

$$4LK \leq \alpha$$

there is a unique function $H(t,x)$ of $\mathbb{R} \times \mathbb{R}^n$ into $\mathbb{R}^n$ satisfying

i) $H(t,x) - x$ is bounded in $\mathbb{R} \times \mathbb{R}^n$,

ii) if $x(t)$ is any solution of the differential equation $x' = A(t)x + f(t,x)$, then $H(t,x(t))$ is a solution of $x' = A(t)x$.

Moreover, H is continuous in $\mathbb{R} \times \mathbb{R}^n$ and

$$\|H(t,x) - x\| \leq 4K\mu\alpha^{-1}$$

for all t, x. For each fixed t, $H_t(x) = H(t,x)$ is a homeomorphism of $\mathbb{R}^n$. $L(t,x) = H_t^{-1}(x)$ is continuous in $\mathbb{R} \times \mathbb{R}^n$ and if $y(t)$ is any solution of $x' = A(t)x$, then $L(t,y(t))$ is a solution of $x' = A(t)x + f(t,x)$.

5.4 Stability of exponential dichotomy

Theorem 5.4.1. *Suppose that the linear system (5.1) admits exponential dichotomy. Then there exists a constant $\eta > 0$ such that the linear system*

$$\frac{dx}{dt} = (A(t) + B(t))x \tag{5.5}$$

also admits exponential dichotomy, when $B(t)$ is continuous on $\mathbb{R}$ and $\|B(t)\| \leq \eta$.

Another version of this theorem.

Theorem 5.4.2. *Let $B : \mathcal{M}_n(\mathbb{R}_+)$ be a bounded, continuous matrix function. Suppose that (5.1) has an exponential dichotomy on $\mathbb{R}_+$. If $\delta = \sup \|B(t)\| < \alpha/(4K^2)$, then the perturbed equation*

$$x' = (A(t) + B(t))z$$

also has an exponential dichotomy on $\mathbb{R}_+$ with constants $\tilde{K}$ and $\tilde{\alpha}$ determined by K, α and δ. Moreover, if $\tilde{P}(t)$ is the corresponding projection, then $\|P(t) - \tilde{P}(t)\| = \mathcal{O}(\delta)$ uniformly in $t \in \mathbb{R}_+$. Also, $|\alpha - \tilde{\alpha}| = \mathcal{O}(\delta)$.

5.5 Generality of exponential dichotomy

The exposition has been done here in the case of a system of ODEs. But it is important to realize that exponential dichotomies exist in a much more general setting.

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Appendix A

A few useful definitions and results

Here, some results that are important for the course are given with a somewhat random ordering.

A.1 Vector spaces, norms

A.1.1 Norm

Consider a vector space E over a field $\mathbb{K}$. A norm is an application, denoted $\| \;\|$, from E to $\mathbb{R}_+$ that satisfies the following:

1) $\forall x \in E$, $\|x\| \geq 0$, with $\|x\| = 0$ if and only if $x = 0$.

2) $\forall x \in E$, $\forall a \in \mathbb{K}$, $\|ax\| = |a| \, \|x\|$.

3) $\forall x, y \in E$, $\|x + y\| \leq \|x\| + \|y\|$.

A vector space E equiped with a norm $\| \;\|$ is a *normed vector space.*

A.1.2 Matrix norms

A.1.3 Supremum (or operator) norm

The supremum norm is defined by

$$\forall L \in \mathcal{L}(E), \quad |||L||| = \sup_{x \in E - \{0\}} \frac{\|L(x)\|}{\|x\|} = \sup_{\|x\| \leq 1} \|L(x)\|.$$

The inequality

$$\|A(t)(x_1 - x_2)\| \leq |||A(t)||| \, \|x_1 - x_2\|$$

results from the nature of the norm $\|\| \;\; \|\|$. See appendix A.1. is best understood by indicating the spaces in which the various norms are defined. We have

$$\begin{aligned}
\|Ax\|_a &= \|x\|_b \left\| A\left(\frac{x}{\|x\|_b}\right)\right\|_a \\
&\leq \|x\|_b \, |\!|\!| A |\!|\!| \\
&= |\!|\!| A |\!|\!| \|x\|_b,
\end{aligned}$$

since

$$\left\| \frac{x}{\|x\|_b} \right\|_b = 1.$$

A.2 An inequality involving norms and integrals

Lemma A.2.1. *Suppose $f : \mathbb{R}^n \to \mathbb{R}^n$ is a continuous function. Then*

$$\left\| \int_a^b f(x)dx \right\| \leq \int_a^b \|f(x)\|\, dx.$$

Proof. First, note that we have

$$\left\| \int_a^b f(x)dx \right\| = \left(\left\| \int_a^b f_1(x)dx \right\|, \ldots, \left\| \int_a^b f_n(x)dx \right\| \right).$$

For a given component function f_i, $i = 1, \ldots, n$, using the definition of the Riemann integral,

$$\int_a^b f_i(x)dx = \lim_{k\to\infty} \sum_{j=1}^k f_i(x_j^*)\Delta x_j,$$

where x_j^* is the sample point in the interval $[x_{j-1}, x_j]$ with width Δx_j. Therefore,

$$\left\| \int_a^b f_i(x)dx \right\| = \left\| \lim_{k\to\infty} \sum_{j=1}^k f_i(x_j^*)\Delta x \right\| = \lim_{k\to\infty} \left\| \sum_{j=1}^k f_i(x_j^*)\Delta x \right\|,$$

since the norm is a continuous function. The result then follows from the triangle inequality. □

A.3 Types of convergences

Definition A.1 (Pointwise convergence)**.** *Let X be any set, and let Y be a topological space. A sequence $f_1, f_2, \ldots$ of mappings from X to Y is said to be* pointwise convergent *(or* simply convergent*) to a mapping $f : X \to Y$, if the sequence $f_n(x)$ converges to $f(x)$ for each x in X. This is usually denoted by $f_n \to f$. In other words,*

$$(f_n \to f) \Leftrightarrow (\forall x \in X, \forall \varepsilon > 0,\ \exists N \geq 0,\ \forall n \geq N,\ d(f_n(x), f(x)) < \varepsilon).$$

Definition A.2 (Uniform convergence)**.** *Let X be any set, and let Y be a topological space. A sequence $f_1, f_2, \dots$ of mappings from X to Y is said to be* uniformly convergent *to a mapping $f : X \to Y$, if given $\varepsilon > 0$, there exists N such that for all $n \geq N$ and all $x \in X$,*

$$d(f_n(x), f(x)) < \varepsilon.$$

This is usually denoted by $f_n \xrightarrow{u} f$. In other words,

$$(f_n \xrightarrow{u} f) \Leftrightarrow (\forall \varepsilon > 0,\ \exists N \geq 0,\ \forall n \geq N, \forall x \in X,\ d(f_n(x), f(x)) < \varepsilon).$$

An important theorem follows.

Theorem A.3.1. *If the sequence of maps $\{f_n\}$ is uniformly convergent to the map f, then f is continuous.*

A.4 Asymptotic Notations

Let n be a integer variable which tends to ∞ and let x be a continuous variable tending to some limit. Also, let $\phi(n)$ or $\phi(x)$ be a positive function and $f(n)$ or $f(x)$ any function. Then

i) $f = \mathcal{O}(\phi)$ means that $|f| < A\phi$ for some constant A and all values of n and x,

ii) $f = o(\phi)$ mean that $f/\phi \to 0$,

iii) $f \sim \phi$ means that $f/\phi \to 1$.

Note that $f = o(\phi)$ implies $f = \mathcal{O}(\phi)$.

A.5 Types of continuities

Definition A.3 (Uniform continuity)**.** *Let (X, d_X) and (Y, d_Y) be two metric spaces, $E \subseteq X$ and $F \subseteq Y$. A function $f : E \to F$ is* uniformly continuous *on the set $E \subset X$ if for every $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$d_Y(f(x), f(y)) < \varepsilon \text{ whenever } x, y \in E \text{ and } d_X(x, y) < \delta.$$

In other words,

$$\begin{gathered}(f : E \subseteq (X, d_X) \to F \subseteq (Y, d_Y) \text{ uniformly continuous on } E) \\ \Leftrightarrow (\forall \varepsilon > 0,\ \exists \delta > 0,\ \forall x, y \in E,\ d_X(x, y) < \delta \Rightarrow d_Y(f(x), f(y)) < \varepsilon).\end{gathered}$$

Definition A.4 (Equicontinuous set)**.** *A set of functions $F = \{f\}$ defined on a real interval $\mathcal{I}$ is said to be* equicontinuous *on $\mathcal{I}$ if, given any $\varepsilon > 0$, there exists a $\delta_\varepsilon > 0$, independent of $f \in F$ and also $t, \tilde{t} \in \mathcal{I}$ such that*

$$\|f(t) - f(\tilde{t})\| < \varepsilon \text{ whenever } |t - \tilde{t}| < \delta_\varepsilon$$

An interpretation of equicontinuity is that a sequence of functions is equicontinuous if all the functions are continuous and they have equal variation over a given neighbourhood. Equicontinuity of a sequence of functions has important consequences.

Theorem A.5.1. *Let $\{f_n\}$ be an equicontinuous sequence of functions. If $f_n(x) \to f(x)$ for every $x \in X$, then the function f is continuous.*

Lemma A.5 (Ascoli)**.** *On a bounded interval $\mathcal{I}$, let $F = \{f\}$ be an infinite, uniformly bounded, equicontinuous set of functions. Then F contains a sequence $\{f_n\}$, $n = 1, 2, \ldots$, which is uniformly convergent on $\mathcal{I}$.*

Theorem A.6. *Let $C(X)$ be the space of continuous functions on the complete metric space X with values in $\mathbb{R}^n$. If a sequence $\{f_n\}$ in $C(X)$ is bounded and equicontinuous, then it has a uniformly convergent subsequence.*

A.6 Lipschitz function

Definition A.6.1 (Lipschitz function)**.** *A map $f : \mathcal{U} \subset \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is* Lipschitz *in x if there exists a real number L such that for all $(t, x_1) \in \mathcal{U}$ and $(t, x_2) \in \mathcal{U}$,*

$$\|f(t, x_1) - f(t, x_2)\| \leq L\|x_1 - x_2\|.$$

In the case of a Lipschitz function as defined above, the constant L is independent of x_1 and x_2, it is given for $\mathcal{U}$. A weaker version is local Lipschitz functions.

Definition A.6.2 (Locally Lipschitz function)**.** *A map $f : \mathcal{U} \subset \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is* locally Lipschitz *in x if, for all $(t_0, x_0) \in \mathcal{U}$, there exists a neighborhood $V \subset \mathcal{U}$ of (t_0, x_0) and a real number L, such that for all $(t, x_1), (t, x_2) \in V$,*

$$\|f(t, x_1) - f(t, x_2)\| \leq L\|x_1 - x_2\|.$$

In other words, f is locally Lipschitz if the restriction of f to V is Lipschitz.

Thus, a locally Lipschitz function is Lipschitz if it is locally Lipschitz on $\mathcal{U}$ with everywhere the same Lipschitz constant L. Another definition of a locally Lipschitz function is as follows.

Definition A.6.3. *A function $f : \mathcal{U} \subset \mathbb{R}^{n+1} \to \mathbb{R}^n$ is locally Lipschitz continuous if for every compact set $V \subset \mathcal{U}$, the number*

$$L = \sup_{(t,x) \neq (t,y) \in V} \frac{\|f(t, x) - f(t, y)\|}{\|x - y\|}$$

is finite, with L depending on V.

Property A.6.4. *Let $f(t, x)$ be a function. The following properties hold.*

i) f Lipschitz $\Rightarrow f$ uniformly continuous in x.

ii) f uniformly continuous $\not\Rightarrow f$ Lipschitz.

iii) $f(t,x)$ has continuous partial derivative $\partial f/\partial x$ on a bounded closed domain $\mathcal{D} \Rightarrow f$ is locally Lipschitz on $\mathcal{D}$.

Proof. *i*) Suppose that f is Lipschitz, *i.e.*, there exists $L > 0$ such that $\|f(t,x_1) - f(t,x_2)\| \leq L\|x_1 - x_2\|$. Recall that f is uniformly continuous if for every $\varepsilon > 0$, there exists $\delta > 0$ such that for all x_1, x_2, $\|x_1 - x_2\| < \delta$ implies that $\|f(t,x_1) - f(t,x_2)\| < \varepsilon$. So, given an $\varepsilon > 0$, choose $\delta < \varepsilon/L$. Then $\|x_1 - x_2\| < \delta$ implies that

$$\|f(t,x_1) - f(t,x_2)\| < L\|x_1 - x_2\| < L\delta < L\varepsilon/L = \varepsilon.$$

Thus f is uniformly continuous (see Definition A.3).

ii) This is left as an exercise. Consider for example the function f defined by $f(x) = 1/\ln x$ on $(0, \frac{1}{2}]$, $f(0) = 0$.

iii) Notice that $\partial f/\partial x$ continuous on $\mathcal{D}$ implies that $\|\partial f/\partial x\|$ is continuous on (the bounded closed domain) $\mathcal{D}$, and thus $\|\partial f/\partial x\|$ is bounded on $\mathcal{D}$. Let

$$L = \sup_{(t,x)\in\mathcal{D}} \left\| \frac{\partial f(t,x)}{\partial x} \right\|$$

If $(t,x_1), (t,x_2) \in \mathcal{U}$, by the mean-value theorem, there exists $\eta \in [x_1, x_2]$ such that $f(t,x_2) - f(t,x_1) = (x_2 - x_1)\frac{\partial f}{\partial x}(t,\eta)$. As $\eta \in \mathcal{U}$, it follows that $\|\frac{\partial f}{\partial x}(t,\eta)\| \leq L$, and thus $\|f(t,x_2) - f(t,x_1)\| \leq L\|x_2 - x_1\|$. □

A.7 Gronwall's lemma

The name of Gronwall is associated to a certain number of inequalities. We give a few of them here. We prove the most simple one (as it is an easy proof to remember), as well as the most general one (Lemma A.9). In [12, p. 3], the lemma is stated as follows.

Lemma A.7 (Gronwall)**.** *If*

i) $g(t)$ is continuous on $t_0 \leq t \leq t_1$,

ii) for $t_0 \leq t \leq t_1$, $g(t)$ satisfies the inequality

$$0 \leq g(t) \leq K + L\int_{t_0}^{t} g(s)ds.$$

Then

$$0 \leq g(t) \leq Ke^{L(t-t_0)},$$

for $t_0 \leq t \leq t_1$.

Proof. Let $v(t) = \int_{t_0}^t g(s)ds$. Then $v'(t) = g(t)$, which implies that the assumption on g can be written

$$0 \le v'(t) \le K + Lv(t).$$

The right inequality is a linear differential inequality, with integrating factor $\exp(-\int_{t_0}^t Lds)$ Also, $v(t_0) = 0$. Hence,

$$\frac{d}{dt}\left(e^{-L(t-t_0)}v(t)\right) \le Ke^{-L(t-t_0)}$$

and therefore,

$$e^{-L(t-t_0)}v(t) \le \frac{K}{L}\left(1 - e^{-L(t-t_0)}\right).$$

Thus $Lv(t) \le K\left(e^{L(t-t_0)} - 1\right)$, and $g(t) \le K + Lv(t) \le Ke^{L(t-t_0)}$. □

In [4, p. 128-130], Gronwall's inequality is stated as

Lemma A.8. *Suppose that $a < b$ and let g, K and L be nonnegative continuous functions defined on the interval $[a, b]$. Moreover, suppose that either K is a constant function, or K is differentiable on $[a, b]$ with positive derivative K'. If, for all $t \in [a, b]$*

$$g(t) \le K(t) + \int_a^t L(t)g(s)ds,$$

then

$$g(t) \le K(t)\exp\left(\int_a^t L(s)ds\right),$$

for all $t \in [a, b]$.

Finally, the most general formulation is in [8, p. 286].

Lemma A.9. *If φ and ψ are two nonnegative regulated functions on the interval $[0, c]$, then for every nonnegative regulated function w in $[0, c]$ satisfying the inequality*

$$w(t) \le \varphi(t) + \int_0^t \psi(s)w(s)ds,$$

we have, in $[0, c]$,

$$w(t) \le \varphi(t) + \int_0^t \varphi(s)\psi(s)\exp\left(\int_s^t \psi(\xi)d\xi\right)ds. \tag{A.1}$$

Before proving the result, let us recall that a function f from an interval $\mathcal{I} \subset \mathbb{R}$ to a Banach space F is *regulated* if it admits in each point in $\mathcal{I}$ a left limit and a right limit. In particular, every continuous mapping from $\mathcal{I} \subset \mathbb{R}$ to a Banach space is regulated, as well as monotone maps from $\mathcal{I}$ to $\mathbb{R}$; see, *e.g.*, [8, Section 7.6].

Proof. Let $y(t) = \int_0^t \psi(s)w(s)ds$; y is continuous, and since $w(t) \leq \varphi(t) + \int_0^t \psi(s)w(s)ds$, it follows that, except maybe at a denumerable number of points of $[0, c]$, we have

$$y'(t) - \psi(t)y(t) \leq \varphi(t)\psi(t) \tag{A.2}$$

from [8, Section 8.7]. Let $z(t) = y(t)\exp(-\int_0^t \psi(s)ds)$. Then (A.2) is equivalent to

$$z'(t) \leq \varphi(t)\psi(t)\exp\left(-\int_0^t \psi(s)ds\right).$$

Using a mean-value type theorem (see, *e.g.*, [8, Th. 8.5.3]) and using the fact that $z(0) = 0$, we get, for $t \in [0, c]$,

$$z(t) \leq \int_0^t \varphi(s)\psi(s)\exp\left(-\int_0^s \psi(\xi)d\xi\right)ds,$$

whence by definition

$$y(t) \leq \int_0^t \varphi(s)\psi(s)\exp\left(\int_s^t \psi(\xi)d\xi\right)ds,$$

and inequality (A.1) now follows from the relation $w(t) \leq \varphi(t) + y(t)$. □

A.8 Fixed point theorems

Definition A.10 (Contraction mapping)**.** *Let (X, d) be a metric space, and let $S \subset X$. A mapping $f : S \to S$ is a* contraction *on S if there exists $K < 1$ such that, for all $x, y \in S$,*

$$d(f(x), f(y)) \leq Kd(x, y)$$

Every contraction is uniformly continuous on X (from Proposition A.6.4, since a contraction is Lipschitz).

Theorem A.11 (Contraction mapping principle)**.** *Consider the complete metric space (X, d). Every contraction mapping $f : X \to X$ has one and only one $x \in X$ such that $f(x) = x$.*

Proof. **Existence** We use successive approximations. Let $x_0 \in X$. Define $x_1 = f(x_0), x_2 = f(x_1), \ldots, x_n = f(x_{n-1}), \ldots$ This defines an infinite sequences of elements of X. As f is a contraction,

$$\begin{aligned} d(x_2, x_1) &= d(f(x_1), f(x_0)) \\ &\leq Kd(x_1, x_0). \end{aligned}$$

Similarly,

$$\begin{aligned} d(x_3, x_2) &= d(f(x_2), f(x_1)) \\ &\leq Kd(x_2, x_1) \\ &\leq K^2 d(x_1, x_0). \end{aligned}$$

Iterating,

$$d(x_{n+1}, x_n) \leq K^n d(x_1, x_0).$$

Therefore,

$$\begin{aligned} d(x_{n+p}, x_n) &\leq d(x_{n+p}, x_{n+p-1}) + d(x_{n+p-1}, x_{n+p-2}) + \cdots + d(x_{n+1}, x_n) \\ &\leq (K^{p-1} + K^{p-2} + \cdots + K + 1) K^n d(x_1, x_0) \\ &\leq \frac{K^n}{1-K} d(x_1, x_0). \end{aligned}$$

Therefore $d(x_{n+p}, x_n)$ tends to 0 as $n \to \infty$, so $\{x_n\}$ is a Cauchy sequence. Since X is a complete space, it follows that $\{x_n\}$ admits a limit ℓ. As $\lim_{n\to\infty} x_n = \ell$, it follows from continuity of f that $x_{n+1} = f(x_n)$ tends to $f(\ell)$. But x_{n+1} also converges to ℓ, so $f(\ell) = \ell$, that is, ℓ is a fixed point of f.

Uniqueness Suppose ℓ_1 and ℓ_2 are two fixed points. Then there must hold that

$$d(\ell_1, \ell_2) \leq K d(\ell_1, \ell_2) < d(\ell_1, \ell_2),$$

if $d(\ell_1, \ell_2) \neq 0$. Therefore $d(\ell_1, \ell_2) = 0$, and $\ell_1 = \ell_2$. □

In the case that $f : S \subset X \to S$, the theorem takes the form of Theorem A.12. Closedness of S is an implicit requirement, since the set S in the complete metric space X is closed if, and only if, S is complete.

Theorem A.12. *Let S be a closed subset of the complete metric space (X, d). Every contraction mapping $f : S \to S$ has one and only one $x \in S$ such that $f(x) = x$.*

Theorem A.8.1. *Consider a mapping $f : X \to X$, where X is a complete metric space. Suppose that f is not necessarily a contraction, but that one of the iterates f^k of f, is a contraction. Then f has a unique fixed point.*

A.9 Jordan normal form

Theorem A.9.1. *Every complex $n \times n$ matrix A is similar to a matrix of the form*

$$J = \begin{bmatrix} J_0 & 0 & 0 & \cdots & 0 \\ 0 & J_1 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & J_s \end{bmatrix}$$

where J_0 is a diagonal matrix with diagonal $\lambda_1, \ldots, \lambda_n$, and, for $i = 1, \ldots, s$,

$$J_i = \begin{bmatrix} \lambda_{q+i} & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \lambda_{q+i} & 1 & 0 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdots & \lambda_{q+i} & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & \lambda_{q+i} \end{bmatrix}$$

The λ_j, $j = 1, \dots, q+s$, are the characteristic roots of A, which need not all be distinct. If λ_j is a simple root, then it occurs in J_0, and therefore, if all the roots are distinct, A is similar to the diagonal matrix

$$J = \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

An algorithm to compute the Jordan canonical form of an $n \times n$ matrix A [15].

i) Compute the eigenvalues of A. Let $\lambda_1, \dots, \lambda_m$ be the distinct eigenvalues of A with multiplicities $n_1, \dots, n_m$, respectively.

ii) Compute n_1 linearly independent generalized eigenvectors of A associated with λ_1 as follows. Compute
$$(A - \lambda_1 E_n)^i$$
for $i = 1, 2, \dots$ until the rank of $(A-\lambda_1 E_n)^k$ is equal to the rank of $(A-\lambda_1 E_n)^{k+1}$. Find a generalized eigenvector of rank k, say u. Define $u_i = (A-\lambda_1 E_n)^{k-1} u$, for $i = 1, \dots, k$. If $k = n_1$, proceed to step 3. If $k < n_1$, find another linearly independent generalized eigenvector with rank k. If this is not possible, try $k-1$, and so forth, until n_1 linearly independent generalized eigenvectors are determined. Note that if $\rho(A - \lambda_1 E_n) = r$, then there are totally $(n-r)$ chains of generalized eigenvectors associated with λ_1.

iii) Repeat step 2 for $\lambda_2, \dots, \lambda_m$.

iv) Let $u_1, \dots, u_k, \dots$ be the new basis. Observe that
$$\begin{aligned} Au_1 &= \lambda_1 u_1, \\ Au_2 &= u_1 + \lambda_1 u_2, \\ &\vdots \\ Au_k &= u_{k-1} + \lambda_1 u_k \end{aligned}$$
Thus in the new basis, A has the representation

$$J = \begin{pmatrix} \boxed{\begin{matrix} \alpha_1 & 1 & \\ & \ddots & 1 \\ & & \alpha_1 \end{matrix}} & & & \\ & \boxed{\begin{matrix} \alpha_2 & 1 & \\ & \ddots & 1 \\ & & \alpha_2 \end{matrix}} & & \\ & & \boxed{\begin{matrix} \alpha_3 & 1 & \\ & \ddots & 1 \\ & & \alpha_3 \end{matrix}} & \\ & & & \ddots \end{pmatrix}$$

where each chain of generalized eigenvectors generates a Jordan block whose order equals the length of the chain.

v) The similarity transformation which yields $J = Q^{-1}AQ$ is given by $Q = [u_1, \dots, u_k, \dots]$.

A.10 Matrix exponentials

Let $A \in \mathcal{M}_n(\mathbb{K})$. The exponential of A is defined by

$$e^A = \sum_{n=0}^{\infty} \frac{A^n}{n!} \tag{A.3}$$

We have $e^A \in \mathcal{M}_n(\mathbb{K})$. Also $\left\|\left\| \frac{1}{n!}A^n \right\|\right\| \leq \frac{1}{n!} \left\|\left\| A \right\|\right\|^n$, so that the series $\sum \frac{1}{n!}A^n$ is absolutely convergent in $\mathcal{M}_n(\mathbb{K})$. Thus e^A is well defined.

Property A.10.1. *Let $A, B \in \mathcal{M}_n(\mathbb{K})$. Then*

i) $\left\|\left\| e^A \right\|\right\| \leq e^{\|\|A\|\|}$.

*ii) If A and B commute (*i.e., *$AB = BA$), then $e^{A+B} = e^A e^B$.*

iii) $e^A = I + \sum_{n=1}^{\infty} \frac{1}{n!}A^n$.

iv) $e^0 = I$.

v) e^A is invertible with inverse e^{-A}.

vi) e^{At} is differentiable, and $\frac{d}{dt}e^{At} = Ae^{At}$.

vii) If P and T are linear transformations on $\mathbb{K}^n$, and $S = PTP^{-1}$, then $e^S = Pe^T P^{-1}$.

Proof. v) For any matrix A, we have $A(-A) = -AA = (-A)A$, so A and $-A$ commute. Therefore, $e^A e^{-A} = e^{A-A} = e^0 = I$. Therefore, for any A, e^A is invertible with inverse e^{-A}. □

There are several shortcuts to computing the exponential of a matrix A:

1) If A is nilpotent, that is, if there exists $q \in \mathbb{N}$ such that $A^q = 0$, then $e^A = \sum_{k=1}^{q} A^k/k!$. A nilpotent matrix has several interesting properties. A is nilpotent if and only if all its eigenvalues are zero. A nilpotent matrix has zero determinant and trace.

2) If A is such that there exists $q \in \mathbb{N}$ such that $A^q = A$, then this can sometimes be exploited to simplify the computation of e^A.

3) Any matrix A can be written in the form $A = D + N$, where D is diagonalizable, N is nilpotent, and D and N commute. Therefore, $e^A = e^{D+N} = e^D e^N$.

4) Other cases require the use of the Jordan normal form (explained below).

Use of the Jordan form to compute the exponential of a matrix. Suppose that $J = P^{-1}AP$ is the Jordan form of the matrix A. For a block diagonal matrix

$$B = \begin{pmatrix} B_1 & & 0 \\ & \ddots & \\ 0 & & B_s \end{pmatrix},$$

we have, for $k = 0, 1, \ldots$,

$$B^k = \begin{pmatrix} B_1^k & & 0 \\ & \ddots & \\ 0 & & B_s^k \end{pmatrix},$$

Therefore, for $t \in \mathbb{R}$,

$$e^{Jt} = \begin{pmatrix} e^{J_0 t} & & 0 \\ & \ddots & \\ 0 & & e^{J_s t} \end{pmatrix},$$

with

$$e^{J_0} = \begin{pmatrix} e^{\lambda_1 t} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_k t} \end{pmatrix}.$$

Now, since $J_i = \lambda_{k+i} I_i + N_i$, with N_i a nilpotent matrix, and since I_i and N_i commute, there holds

$$e^{J_i t} = e^{\lambda_{k+i} t} e^{N_i t}.$$

For $k \geq N_i$, $N_i^k = 0$, so

$$e^{tJ_i} = \begin{pmatrix} 1 & t & \cdots & \frac{t^{n_i-1}}{(n_i-1)!} \\ 0 & 1 & \cdots & \frac{t^{n_i-2}}{(n_i-2)!} \\ & & & \\ 0 & & \cdots & 1 \end{pmatrix}.$$

A.11 Matrix logarithms

Theorem A.11.1. *Suppose that M is a nonsingular $n \times n$ matrix. Then there is an $n \times n$ matrix B (possibly complex), such that $e^B = M$. If, additionally, $M \in \mathcal{M}_n(\mathbb{R})$, then there is $B \in \mathcal{M}_n(\mathbb{R})$ such that $e^B = M^2$.*

Let $u \in \mathbb{C}$, we have, for $|u| < 1$,

$$\frac{1}{1+u} = 1 - u + u^2 - u^3 + \cdots$$

and, for $|u| < 1$,

$$\begin{aligned}\ln(1+u) &= t - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} + \cdots \\ &= \sum_{k=1}^{\infty} (-1)^{k+1} \frac{t^k}{k}\end{aligned}$$

For any $z \in \mathbb{C}$,

$$\begin{aligned}\exp(z) &= 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots \\ &= \sum_{n=1}^{\infty} \frac{z^n}{n!}\end{aligned}$$

Therefore, for $|u| < 1$, $u \in \mathbb{C}$,

$$\begin{aligned}1 + u &= \exp(\ln(1+u)) \\ &= \sum_{n=1}^{\infty} \frac{1}{n!} \left[\sum_{k=1}^{\infty} (-1)^{k+1} \frac{u^k}{k} \right]^n\end{aligned}$$

Suppose that

$$B = (\ln \lambda) I + \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \frac{1}{\lambda^k} Z^k$$

where

$$Z = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ & \ddots & \ddots & \\ 0 & & 0 & 1 \\ 0 & & & 0 \end{bmatrix}$$

is an $m \times m$-matrix. Since Z is nilpotent ($Z^N = 0$ for all $N \geq m$), the above sum is finite. Observe that

$$\begin{aligned}\exp(B) &= \exp((\ln \lambda) I) \exp\left(\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \frac{1}{\lambda^k} Z^k \right) \\ &= \lambda \exp\left(\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \left(\frac{Z}{\lambda} \right)^k \right) \\ &= \lambda \left(I + \frac{Z}{\lambda} \right) \\ &= \lambda I + Z \\ &= J\end{aligned}$$

We say $\ln J = B$.

A.12 Spectral theorems

When studying systems of differential equations, it is very important to be able to compute the eigenvalues of a matrix, for instance in order to study the local asymptotic stability of an equilibrium point. This can be a very difficult problem, that often becomes intractable even for systems with low dimensionality. However, even if it is not possible to compute an explicit solution, it is often possible to use spectrum localization theorems. We here give two of the most famous ones: the Routh-Hurwitz criterion, and the Gershgorin theorem.

Let A be a $n \times n$ matrix, denote its elements by (a_{ij}). The set of all eigenvalues of A is called its *spectrum*, and is denoted $\text{Sp}(A)$.

Theorem A.12.1 (Routh-Hurwitz)**.** *If* $n = 2$*, suppose that* $\det A > 0$ *and* $\text{tr}A < 0$*. Then* A *has only eigenvalues with negative real part.*

Theorem A.12.2 (Gershgorin)**.** *Let*

$$R_j = \sum_{i=1, i\neq j}^{n} |a_{ij}|$$

Let $\lambda \in Sp(A)$*. Then*

$$\lambda \in \bigcup_j \{|\lambda - a_{jj}| \leq R_j\}$$

Gershgorin's theorem is extremely helpful in certain cases. Suppose that A is stictly diagonally dominant, *i.e.*, $|a_{ii}| > \sum_{i=1,i\neq j}^{n} |a_{ij}|$. Then A has no eigenvalues with zero real part. Also, the number of eigenvalues with negative real part is equal to the number of negative entries on the diagonal of A, and conversely for eigenvalues with positive real parts and the number of positive entries on the diagonal of A.

Appendix B

Problem sheets

Contents

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Homework Sheet 1

Exercise 1.1 – We consider here the equation

$$x'(t) = -\alpha x(t) + f(x(t)) \tag{B.1}$$

where $\alpha \in \mathbb{R}$ is constant and f is continuous on $\mathbb{R}$.

i) Show that x is a solution of (B.1) on $\mathbb{R}_+$ if, and only if,

$$\begin{cases} x \text{ is continuous on } \mathbb{R}_+ \\ \text{and} \\ x(t) = e^{-\alpha t}x(0) + e^{-\alpha t}\int_0^t e^{\alpha s} f(x(s))ds \quad \forall t \in \mathbb{R}_+ \end{cases} \tag{B.2}$$

Suppose now that $\alpha > 0$ and that f is such that

$$\exists a, k \in \mathbb{R}, a > 0, 0 < k < \alpha : \quad \forall u \in \mathbb{R},\ |u| \leq a \Rightarrow |f(u)| \leq k|u| \tag{B.3}$$

Part I. Suppose that there exists a solution x of (B.1), defined on $\mathbb{R}_+$ and satisfying the inequality

$$|x(t)| \leq a, \quad t \in \mathbb{R}_+ \tag{B.4}$$

i) Prove the inequality

$$|x(t)| \leq |x(0)|e^{-(\alpha-k)t}, \quad t \in \mathbb{R}_+$$

[Hint: Use Gronwall's lemma with the function $g(t) = e^{\alpha t}|x(t)|$].

ii) Deduce that x admits the limit 0 as $t \to \infty$.

Part II.

i) Show that any solution of (B.1) on $\mathbb{R}_+$ that satisfies $|x(0)| < a$, satisfies (B.4).

ii) Deduce from the preceding questions the two following properties.

   a) Any solution x of (B.1) on $\mathbb{R}_+$ satisfying the condition $|x(0)| < a$, admits the limit 0 when $t \to \infty$.

   b) The function $x \equiv 0$ is the unique solution of (B.1) on $\mathbb{R}_+$ such that $x(0) = 0$.

Part III. Application. Show that, for $\alpha > 1$, all solutions of the equation

$$x' = -\alpha x + \ln\left(1 + x^2\right)$$

tend to zero when $t \to \infty$. ○

Exercise 1.2 – Let $f : [0, +\infty) \to \mathbb{R}$, $f \in C^1$, and $a \in \mathbb{R}$. We consider the initial value problems

$$\begin{aligned} &x'(t) + ax(t) = f(t), \qquad t \geq 0 \\ &x(0) = 0 \end{aligned} \tag{B.5}$$

and

$$\begin{aligned} &x'(t) + ax(t) = f'(t), \qquad t \geq 0 \\ &x(0) = 0 \end{aligned} \tag{B.6}$$

As these equations are linear, the initial value problems (B.5) and (B.6) admit unique solutions. We denote ϕ the solution to (B.5) and ψ the solution to (B.6). Find a necessary and sufficient condition on f such that $\phi' = \psi$.

[Hint: Use a variation of constants formula]. ◦

Exercise 1.3 – Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be continuous. Consider the differential equation

$$x'(t) = f(x(t)) \tag{B.7}$$

i) Let x be a solution of (B.7) defined on a bounded interval $[0, \alpha)$, with $\alpha > 0$. Suppose that $t \mapsto f(x(t))$ is bounded on $[0, \alpha)$.

   a) Consider the sequence

   $$z_{\alpha,n} = x(\alpha - \frac{1}{n}), \qquad n \in \mathbb{N}^*$$

   Show that $(z_{\alpha,n})_{n \in \mathbb{N}^*}$ is a Cauchy sequence.

   b) Deduce that there exists $x_\alpha \in \mathbb{R}^n$ such that,

   $$\|x(t) - x_\alpha\| \leq M_\alpha |t - \alpha|$$

   with $M = \sup_{t \in [0,\alpha)} \|f(x(t))\|$.

   c) Show that x admits a finite limit when $t \to \alpha$, $t < \alpha$.

ii) Show that there exists an extension of x that is a solution of (B.7) on the interval $[0, \alpha]$.

◦

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Homework Sheet 1 – Solutions

Solution – Exercise 1 – 1) Let us first show that x solution of (B.1) implies (B.2). If x is a solution of (B.1) on $\mathbb{R}_+$, then x is differentiable on $\mathbb{R}_+$, and so x is continuous on $\mathbb{R}_+$. Furthermore,

$$
\begin{aligned}
x'(s) &= -\alpha x(s) + f(x(s)), \text{ for all } s \in \mathbb{R}_+ \\
\Leftrightarrow e^{\alpha s} x'(s) &= -\alpha e^{\alpha s} + e^{\alpha s} f(x(s)) \\
\Rightarrow \int_0^t e^{\alpha s} x'(s) ds &= -\alpha \int_0^t e^{\alpha s} x(s) ds + \int_0^t e^{\alpha s} f(x(s)) ds \\
[e^{\alpha s} x(s)]_0^t - \alpha \int_0^t e^{\alpha s} x(s) ds &= -\alpha \int_0^t e^{\alpha s} x(s) ds + \int_0^t e^{\alpha s} f(x(s)) ds \\
e^{\alpha t} x(t) - x(0) &= \int_0^t e^{\alpha s} f(x(s)) ds, \text{ for all } t \in \mathbb{R}_+ \\
x(t) &= e^{-\alpha t} x(0) + e^{-\alpha t} \int_0^t e^{\alpha s} f(x(s)) ds, \text{ for all } t \in \mathbb{R}_+
\end{aligned}
$$

Let us now prove the converse, *i.e.*, that if a function x satisfies (B.2), then it is a solution to the IVP (B.1). Since x and f are continuous on $\mathbb{R}_+$, $t \mapsto e^{\alpha t} f(x(t))$ is continuous on $\mathbb{R}_+$. This implies that

$$t \mapsto \int_0^t e^{\alpha s} f(x(s)) ds$$

is differentiable on $\mathbb{R}_+$, and, differentiating the expression for $x(t)$ as given in (B.2),

$$
\begin{aligned}
x'(t) &= -\alpha e^{-\alpha t} x(0) - \alpha e^{-\alpha t} \int_0^t e^{\alpha s} f(x(s)) ds + e^{-\alpha t} e^{\alpha t} f(x(t)) \\
\Rightarrow x'(t) &= -\alpha \underbrace{\left[e^{-\alpha t} x(0) + e^{-\alpha t} \int_0^t e^{\alpha s} f(x(s)) ds \right]}_{x(t)} + f(x(t))
\end{aligned}
$$

And thus

$$x'(t) = -\alpha x(t) + f(x(t))$$

which implies that x is a solution to (B.1).

Part I. We now assume that (B.3) is also satisfied, and that there exists a solution x on $\mathbb{R}_+$ satisfying (B.4).

1) If x is a solution of (B.1), then

$$x(t) = e^{-\alpha t} x(0) + e^{-\alpha t} \int_0^t e^{\alpha s} f(x(s)) ds$$

This implies that

$$|x(t)| \leq e^{-\alpha t}|x(0)| + e^{-\alpha t}\int_0^t e^{\alpha s}|f(x(s))|ds$$

From (B.3) and multiplying both sides by $e^{\alpha t}$,

$$e^{\alpha t}|x(t)| \leq |x(0)| + \int_0^t k e^{\alpha s}|x(s)|ds$$

We use Gronwall's Lemma (Lemma A.2) as follows,

$$\underbrace{e^{\alpha t}|x(t)|}_{g(t)} \leq \underbrace{|x(0)|}_{K(t)} + \int_0^t \underbrace{k}_{L(t)} \underbrace{e^{\alpha s}|x(s)|}_{g(s)} ds$$

Thus,

$$\begin{aligned} e^{\alpha t}|x(t)| &\leq |x(0)| \exp\left(\int_0^t k ds\right) \\ &\leq |x(0)|e^{kt} \end{aligned}$$

and so finally, for all $t \in \mathbb{R}_+$, we have

$$|x(t)| \leq |x(0)|e^{t(k-\alpha)} = |x(0)|e^{-(\alpha-k)t}$$

2) From (B.3), $0 < k < \alpha$, hence $\alpha - k > 0$, which implies, together with the result of the previous question, that $\lim_{t\to\infty} x(t) = 0$.

Part II.

1) Let us suppose that x is a solution of (B.1) that is such that $|x(0)| < 0$. Let $\mathcal{A} = \{t : |x(t)| \leq a\}$. Let us show that $\mathcal{A} = [0, +\infty)$.

First of all, notice that $|x(0)| < a$ and x continuous on $\mathbb{R}_+$ implies that there exists $t_0 \in \mathbb{R}_+ - \{0\}$ such that, for all $t \in [0, t_0]$, $|x(t)| \leq a$. Indeed, suppose this were not the case. Then, for all $\varepsilon > 0$, there exists $t_\varepsilon \in [0, \varepsilon]$ such that $|x(t_\varepsilon)| > a$. This means that for all $n \in \mathbb{N} - \{0\}$, there exists $u_n \in [0, \frac{1}{n}]$ such that $|x(u_n)| > a$. As $u_n \to 0$ as $n \to \infty$ and that x is continuous, $|x(0)| \geq a$, since taking the limit implies that strict inequalities become loose. This is a contradiction with $|x(0)| < a$. Thus $[0, t_0] \subset \mathcal{A}$.

Let us now show that for all $t_1 \in \mathcal{A}$, $[0, t_1] \subset \mathcal{A}$, *i.e.*, $\mathcal{A}$ is an interval. First, if $t_1 \leq t_0$ then $[0, t_1] \subset [0, t_0] \subset \mathcal{A}$. Secondly, in the case $t_1 > t_0$, suppose that $[0, t_1] \not\subset \mathcal{A}$. This means that $\exists \eta \in [0, t_1]$ such that $\eta \notin \mathcal{A}$. More precisely, $\exists \eta \in (t_0, t_1)$ such that $\eta \notin \mathcal{A}$, since $[0, t_0] \subset \mathcal{A}$ and $t_1 \in \mathcal{A}$. Let β be the smallest such η, that is, $\beta = \inf\{t \in (t_0, t_1);\ t \notin \mathcal{A}\}$. Note that β can also be defined as $\beta = \sup\{t \in (t_0, t_1);\ t \in \mathcal{A}\}$.

Thus

$$\beta = \inf\{t \in (t_0, t_1);\ |x(t)| > a\} = \sup\{t \in (t_0, t_1);\ |x(t)| < a\}$$

Since x is continuous, this implies that $|x(\beta)| \geq a$ and $|x(\beta)| \leq a$, hence $x(\beta) = \pm a$. But, with its sup definition, this implies that $\beta = t_1$, whereas with its inf definition, this implies that $\beta < t_1$.

Hence $\mathcal{A}$ is an interval. We now want to show that it is an unbounded interval. Assume it is bounded, hence let $c = \sup(\mathcal{A}) < \infty$. Since x is continuous, $\mathcal{A} = \{t \geq 0,\ |x(t)| \leq a\}$ implies that $c \in \mathcal{A}$. Thus $\mathcal{A} = [0, c]$. Therefore, for all $t \in [0, c]$,

$$|x(t)| \leq a$$

so, by Part I, 1),

$$\begin{aligned} |x(t)| &\leq |x(0)|e^{-(\alpha-k)t} \\ \Rightarrow |x(t)| &\leq |x(0)| < a \\ \Rightarrow |x(c)| &< a \end{aligned}$$

Since x is continuous on $\mathbb{R}_+$, there exists $t > c$ such that $|x(t)| \leq a$, and thus there exists $t > c$ such that $t \in \mathcal{A}$, which is a contradiction. Therefore, $\mathcal{A} = [0, \infty)$, and we can conclude that $\forall t \in \mathbb{R}_+$, $|x(t)| \leq a$.

Another proof, contributed by Guihong Fan, proceeds by contradiction, using the fact the (B.1) is an autonomous scalar equation. Notice that equation (B.1) can be written in the form $x' = g(x)$, with $g(u) = -\alpha u + f(u)$. Thus, since this mapping is continuous, we can apply Theorem 1.1.8 on the monotonicity of the solutions to an autonomous scalar differential equation. Assume that $x(t)$ is a solution of (B.1) on $\mathbb{R}_+$ that satisfies $|x(0)| < a$, but that (B.4) is violated.

Then, since the solution $x(t)$ is monotone, there exists $t_0 \in \mathbb{R}_+$ such that one of the following holds.

i) $x(t_0) = a$ and $x'(t_0) > 0$,

ii) $x(t_0) = -a$ and $x'(t_0) < 0$.

Let us treat case i). From (B.3), it follows that $|f(x(t_0))| = |f(a)| \leq ka$. Therefore, using (B.1),

$$\begin{aligned} x'(t_0) &= -\alpha x(t_0) + f(x(t_0)) \\ &= -\alpha a + f(a) \\ &\leq -\alpha a + ka \\ &\leq -(\alpha - k)a \\ &< 0 \end{aligned}$$

since $\alpha > k$. This is a contradiction with $x'(t_0) > 0$. Case ii) is treated similarly, and thus it follows that (B.4) holds for all $t \in \mathbb{R}_+$.

2 − a) If $|x(0)| < a$, then we have just proved that for all $t \in \mathbb{R}_+$, $|x(t)| \leq a$. From Part I, 1), this implies that $|x(t)| \leq |x(0)|e^{-(\alpha-k)t}$. Therefore, since $\alpha > k$, $\lim_{t\to\infty} x(t) = 0$.

2 − b) To show that $x \equiv 0$ is the only solution of (B.1) such that $x(0) = 0$, we first show that $x \equiv 0$ is a solution of (B.1). Condition (B.3) applied to 0 states that $|0| < a$ implies $|f(0)| \leq k|0| = 0$.

Uniqueness: let ϕ be a solution of (B.1) such that $\phi(0) = 0$. This implies that $|\phi(0) < a$, and as a consequence, it follows from Part I, 1) that for all $t \in \mathbb{R}_+$, $|\phi(t)| \leq |\phi(0)|e^{-(\alpha-k)t}$, hence for all $t \in \mathbb{R}_+$, $\phi(t) = 0$.

Part III. All solutions of the nonlinear equation $x' = -\alpha x + \ln(1 + x^2)$ tend to zero as $t \to \infty$, when $\alpha > 1$. Indeed, let $f(u) = \ln(1 + u^2)$. We have

$$|f'(u)| = \frac{2|u|}{1+u^2} \leq 1$$

since $(u-1)^2 = u^2 + 1 - 2u \geq 0$ (and hence $2u/(1 + u^2) \leq 1$). Then, $|f(u) - f(0)| \leq |u - 0|$ implies that $|f(u)| \leq |u|$, for all $u \in \mathbb{R}$. We thus have $k = 1 < \alpha$, the hypotheses of the exercise are satisfied and all solutions of the equation tend to zero. ◦

Solution – Exercise 2 – Using a variation of constants formula, we obtain

$$\phi(t) = e^{-at}c_1 + \int_0^t e^{-a(t-s)} f(s)ds, \quad t \geq 0$$

and

$$\psi(t) = e^{-at}c_2 + \int_0^t e^{-a(t-s)} f'(s)ds, \quad t \geq 0$$

Let us show this for the solution of (B.5), the solution of (B.6) is obtained using exactly the same method. The solution to a linear differential equation consists of the general solution to the homogeneous equation together with a particular solution to the nonhomogeneous equation. Here, the homogeneous equation is

$$x' = -ax$$

and basic integration yields the general solution $x(t) = c_1e^{-at}$. To obtain a particular solution to the nonhomogeneous equatiob (B.5), we use a variation of constants formula: assume that the constant in the solution $x(t) = ce^{-at}$ is a function of time, hence

$$x(t) = c(t)e^{-at}$$

Taking the derivative of this expression, we obtain

$$x' = c'e^{-at} - ace^{-at}$$

Substituting both this expression and $x = ce^{-at}$ into (B.5), we get

$$c'e^{-at} - ace^{-at} + ace^{-at} = f(t), \quad t \geq 0$$

and hence

$$c' = e^{at} f(t)$$

Integrating both sides of this expression gives

$$c(t) = \int_0^t e^{as} f(s)ds$$

so that a particular solution to (B.5) is given by

$$x(t) = c(t)e^{-at} = \left(\int_0^t e^{as} f(s)ds \right) e^{-at} = \int_0^t e^{-a(t-s)} f(s)ds$$

Summing the general solution to the homogeneous equation with this last expression gives the desired result.

Using the initial conditions yields

$$\begin{aligned} \phi(0) &= c_1 = 0 \\ \psi(0) &= c_2 = 0 \end{aligned}$$

Hence the system under consideration is

$$\begin{aligned} \phi(t) &= \int_0^t e^{-a(t-s)} f(s)ds, \quad t \geq 0 \\ \psi(t) &= \int_0^t e^{-a(t-s)} f'(s)ds, \quad t \geq 0 \end{aligned}$$

Recall that if $g(t) = \int_{t_0}^t h(s,t)ds$, then for all t_0,

$$g'(t) = h(t,t) + \int_{t_0}^t \frac{\partial h}{\partial t}(s,t)ds$$

This implies that

$$\phi'(t) = f(t) - a \int_0^t e^{-a(t-s)} f(s)ds$$

It follows that

$$\begin{aligned} \phi'(t) = \psi(t) &\Leftrightarrow f(t) - a \int_0^t e^{-a(t-s)} f(s)ds = \int_0^t e^{-a(t-s)} f'(s)ds \\ &\Leftrightarrow f(t) - a \int_0^t e^{-a(t-s)} f(s)ds = \left[e^{-a(t-s)} f(s) \right]_{s=0}^{s=t} - a \int_0^t e^{-a(t-s)} f(s)ds \\ &\Leftrightarrow f(t) = \left[e^{-a(t-s)} f(s) \right]_{s=0}^{s=t} \\ &\Leftrightarrow f(t) = f(t) - e^{-at} f(0) \\ &\Leftrightarrow f(0) = 0 \end{aligned}$$

$\circ$

Solution – Exercise 3 – 1 – a) For a vector-valued function, there is no mean value theorem with an equal sign. But the following holds (see, *e.g.*, [3, p. 44], [9, p. 209] or Rudin, p.113).

Theorem B.1.3. *Let $f : [a,b] \to F$ be a continuous mapping, with F a Banach space. Suppose that f admits a right derivative $f'_r(x)$ for all $x \in (a,b)$, and that $\|f'_r(x)\| \leq k$, where $k \geq 0$ is a constant. Then*

$$\|f(b) - f(a)\| \leq k(b-a)$$

and more generally, for all $x_1, x_2 \in [a,b]$,

$$\|f(x_2) - f(x_1)\| \leq k|x_2 - x_1|$$

Let us denote $M = \sup_{t\in[0,\alpha)} \|f(t,x)\|$, and let $n,p > N$, where N is sufficiently large that $\alpha - \frac{1}{n+p} \in [0,\alpha)$ and $\alpha - \frac{1}{n} \in [0,\alpha)$. Using Theorem B.1.3, we obtain that

$$\|x(\alpha - \frac{1}{n+p}) - x(\alpha - \frac{1}{n})\| \leq M|\frac{1}{n} - \frac{1}{n+p}|$$

$$\|z_{\alpha,n+p} - z_{\alpha,n}\| \leq M|\frac{1}{n} - \frac{1}{n+p}|$$

So the sequence $(z_{\alpha,n})_{n\in\mathbb{N}^*}$ is a Cauchy sequence.

1 − b) For all $t \in [0,\alpha)$,

$$\|x(t) - x(\alpha - \frac{1}{n})\| \leq \left| \int_{\alpha-\frac{1}{n}}^{t} \|f(x(s))\| ds \right|$$

$$\|x(t) - z_{\alpha,n}\| \leq M_\alpha |t - \alpha + \frac{1}{n}| \tag{B.8}$$

Since the sequence $(z_{\alpha,n})_{n\in\mathbb{N}^*}$ is a Cauchy sequence, there exists $x_\alpha = \lim_{n\to\infty}(z_{\alpha,n})$. Thus taking $n \to \infty$ in (B.8) gives

$$\|x(t) - x_\alpha\| \leq M_\alpha |t - \alpha|$$

1 − c) According to 1.b), we have

$$\lim_{t\to\alpha, t<\alpha} \|x(t) - x_\alpha\| \leq 0$$

Hence

$$\lim_{t\to\alpha, t<\alpha} x(t) = x_\alpha$$

2) Let

$$z(t) = \begin{cases} x(t) & \text{if } t \in [0,\alpha) \\ x_\alpha & \text{if } t = \alpha \end{cases}$$

Let us show that z is a solution of (B.7) on $[0,\alpha]$ if z is continuous on $[0,\alpha]$ and satisfies the integral equation

$$z(t) = z(t_0) + \int_{t_0}^{t} f(z(s)) ds$$

for all $t \in [0,\alpha]$, and with an arbitrary $t_0 \in [0,\alpha]$. We know by construction that z is continuous (since $\lim_{t\to\infty} x(t) = x_\alpha$).

Let $t \in [0, \alpha)$,

$$z(t) = x(t) = x(t_0) + \int_{t_0}^{t} f(x(s))ds$$

because x is a solution. For $t = \alpha$,

$$\begin{aligned} z(\alpha) &= x_\alpha \\ &= \lim_{t \to \alpha, t < \alpha} x(t) \\ &= \lim_{t \to \alpha, t < \alpha} x(t_0) + \int_{t_0}^{t} f(x(s))ds \\ &= \lim_{t \to \alpha, t < \alpha} z(t_0) + \int_{t_0}^{t} f(z(s))ds \end{aligned}$$

since for $t < \alpha$, we have $z(t) = x(t)$.

Furthermore, $t \mapsto f(z(t))$ is bounded on $[0, \alpha)$, which implies that

$$\int_{t_0}^{\alpha} f(z(s))ds = \lim_{t \to \alpha} \int_{t_0}^{t} f(z(s))ds$$

So

$$z(\alpha) = z(t_0) + \int_{t_0}^{\alpha} f(z(s))ds$$

So z is solution to (B.7) on $[0, \alpha]$. $\circ$

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Homework Sheet 2

Exercise 2.1 – Consider the system

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

where $a, b, c, d \in \mathbb{R}$ are constants such that $ad - bc = 0$. Discuss all possible behaviours of the solutions and sketch the corresponding phase plane trajectories. ◦

Exercise 2.2 – Let A be a constant $n \times n$ matrix.

i) Show that $|e^A| \leq e^{|A|}$.

ii) Show that $\det e^A = e^{\mathrm{tr} A}$.

iii) How should α be chosen so that $\lim_{t\to\infty} e^{-\alpha t} e^{At} = 0$.

◦

Exercise 2.3 – Let $X(t)$ be a fundamental matrix for the system $x' = A(t)x$, where $A(t)$ is an $n \times n$ matrix with continuous entries on $\mathbb{R}$. What conditions on $A(t)$ and C guarantee that $CX(t)$ is a fundamental matrix, where C is a constant matrix. ◦

Exercise 2.4 – Consider the system

$$x' = A(t)x \tag{B.9}$$

where $A(t)$ is a continuous $n \times n$ matrix on $\mathbb{R}$, and $A(t + \omega) = A(t)$, $\omega > 0$.

i) Show that $\mathcal{P}(\omega)$, the set of ω-periodic solutions of (B.9), is a vector space.

ii) Let f be a continuous $n \times 1$ matrix function on $\mathbb{R}$ with $f(t + \omega) = f(t)$. Show that, for the system

$$x' = A(t)x + f(t) \tag{B.10}$$

the following conditions are equivalent.

a) System (B.10) has a unique ω-periodic solution,

b) $[X^{-1}(\omega) - X^{-1}(0)]$ is nonsingular,

c) $\dim \mathcal{P}(\omega) = 0$.

◦

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Homework Sheet 2 – Solutions

Solution – Exercise 1 – The characteristic polynomial of the matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is $(a-\lambda)(d-\lambda)-bc = \lambda^2-(a+d)\lambda+ad-bc = \lambda^2-(a+d)\lambda$ since $ad-bc=0$. Thus A has the eigenvalues 0 and $a+d$. Hence solutions are of the form

$$\begin{aligned} x_1 &= c_1 \\ x_2 &= c_2 e^{(a+d)t} \end{aligned}$$

with $c_1, c_2 \in \mathbb{R}$, and there are three possibilities.

- If $a+d<0$, then all points on the x_1 axis are equilibria, and all trajectories in (x_1,x_2)-space go to them parallely to the x_2 axis.
- If $a+d=0$, then all points are equilibria.
- If $a+d>0$, then the x_1 axis is invariant and any solution that does not start on the x_1 axis diverges to $\pm\infty$ parallely to the x_2 axis.

$\circ$

Solution – Exercise 2 – **1)** We have $e^A = \sum_{k=0}^\infty A^k/k!$. Taking the norm,

$$\|e^A\| = \|\sum_{k=0}^\infty A^k/k!\|,$$

whence, by the triangle inequality, $\|e^A\| \leq \sum_{k=0}^\infty \|A^k/k!\| = e^{\|A\|}$.

2) Let J be the Jordan form of A, *i.e.*, there exists P nonsingular such that $P^{-1}AP = J$, where J has the form $\text{diag}(B_j)_{j=1,\dots,p}$, with B_j the Jordan block corresponding to the eigenvalue λ_j of multiplicity m_j. Then, since A and J are similar,

$$\begin{aligned} \det e^A &= \det e^J \\ &= \det(e^{B_1})^{m_1} \dots \det(e^{B_p})^{m_p} \\ &= e^{\lambda_1 m_1} \cdots e^{\lambda_p m_p} \\ &= \exp(\sum_{k=1}^p \lambda_k m_k) \\ &= \text{tr} A \end{aligned}$$

3) We can write

$$\lim_{t\to\infty} e^{-\alpha t}e^{At} = \lim_{t\to\infty} e^{(A-\alpha I)t}$$

Let Sp (A) be the spectrum of A, *i.e.*, the set of eigenvalues of A. Then, if $\lambda \in$ Sp (A), $\lambda - \alpha \in$ Sp $(A - \alpha I)$. We have $\lim_{t\to\infty} e^{(A-\alpha I)t} = 0$ if, and only if, $\Re(\mu) < 0$ for all $\mu \in$ Sp $(A - \alpha I)$, *i.e.*, $\Re(\mu + \alpha) < 0$ for all $\mu \in$ Sp (A), *i.e.*, $\Re(\mu) < \alpha$ for all $\mu \in$ Sp (A). Hence, choosing α greater than the eigenvalue of A with maximal real part ensures that $\lim_{t\to\infty} e^{(A-\alpha I)t} = 0$. ◦

Solution – Exercise 3 – We have

$$\begin{aligned}(CX(t))' &= CX'(t)\\ &= CA(t)X(t)\end{aligned}$$

For $CX(t)$ to be a fundamental matrix for $x' = A(t)x$ requires that $(CX(t))' = A(t)(CX(t))$. So C and $A(t)$ must commute. Also, a fundamental matrix must be nonsingular. As $X(t)$ is a fundamental matrix, it is nonsingular. Thus C must be nonsingular for $CX(t)$ to be a fundamental matrix. So, to conclude, if $X(t)$ is a fundamental matrix for the system $x' = A(t)x$, then $CX(t)$ is a fundamental matrix for $x' = A(t)x$ if C is nonsingular and commutes with $A(t)$. ◦

Solution – Exercise 4 – **1)** For $x \in \mathbb{R}^n$, $x \in \mathcal{P}(\omega)$ if it satisfies (B.9). Let $x_1, x_2 \in \mathbb{R}^n$ and $\alpha_1, \alpha_2 \in \mathbb{R}$. Then, as $\mathbb{R}^n$ is a vector space, $\alpha_1 x_1 + \alpha_2 x_2 \in \mathbb{R}^n$. Now assume that, moreover, $x_1, x_2 \in \mathcal{P}(\omega)$. Then,

$$\begin{aligned}(\alpha_1 x_1 + \alpha_2 x_2)' &= \alpha_1 x_1' + \alpha_2 x_2'\\ &= \alpha_1 A(t)x_1 + \alpha_2 A(t)x_2\\ &= A(t)(\alpha_1 x_1 + \alpha_2 x_2)\end{aligned}$$

and therefore, $\alpha_1 x_1 + \alpha_2 x_2 \in \mathcal{P}(\omega)$, and $\mathcal{P}(\omega)$ is a vector space.

2) There were of course several approaches to this problem. The simplest one required the use of Theorem B.2.4, stated and proved later.

c) ⇒ b) Let V be the nonsingular matrix such that $X(t+\omega) = X(t)V$, that we know to exist from Theorem 2.4.2. Then $X^{-1}(t+\omega) = V^{-1}X^{-1}(t)$, and $X^{-1}(t+\omega) - X^{-1}(t) = (V^{-1} - I)X^{-1}(t)$.

Suppose that $\dim \mathcal{P}(\omega) = 0$. Then 1 is not an eigenvalue of V. This implies that 1 is neither an eigenvalue of V^{-1}; in turn, 0 is not an eigenvalue of $V^{-1} - I$. This means that $V^{-1} - I$ is nonsingular, and since $X(t)$ is nonsingular, $(V^{-1} - I)X^{-1}(t)$ is nonsingular. Thus we conclude that if $\dim \mathcal{P}(\omega) = 0$, then $(X^{-1}(t+\omega) - X^{-1}(t))$ is nonsingular.

b) ⇒ c) The previous argument works the other way as well.

c) ⇒ a) Suppose $\dim \mathcal{P}(\omega) = 0$ and that x_1, x_2 are two solutions to (B.10). Then $x_1' = A(t)x_1 + f(t)$ and $x_2' = A(t)x_2 + f(t)$. Therefore,

$$(x_1 - x_2)' = A(t)x_1 + f(t) - A(t)x_2 - f(t) = A(t)(x_1 - x_2)$$

which implies that $x_1 - x_2$ is a solution to (B.9), and therefore, $\dim \mathcal{P}(\omega) \neq 0$, a contradiction. Thus the solution to (B.10) is unique.

a) ⇒ c) Let x be the unique ω-periodic solution of (B.10), and assume that $\dim \mathcal{P}(\omega) \neq 0$, *i.e.*, there exists y, non trivial ω-periodic solution to (B.9). Then

$$\begin{aligned}(x+y)' &= A(t)x + f(t) + A(t)y \\ &= A(t)(x+y) + f(t)\end{aligned}$$

and so $x+y$ is a solution to (B.10), which is a contradiction since x is the unique solution to (B.10). ∘

Theorem B.2.4. *Consider the system*

$$x' = A(t)x$$

where $A(t)$ has continuous entries on $\mathbb{R}$ and is such that $A(t+\omega) = A(t)$ for some $\omega \in \mathbb{R}$. Then 1 is an eigenvalue of $A(t)$.

Proof. For some constant vector $c \neq 0$, we have $x(t) = X(t)c$. Also, because of periodicity, $x(t+\omega) = X(t+\omega)c$. As x is periodic of period ω, $x(t) = x(t+\omega)$, so that, using the previously obtained forms,

$$\begin{aligned}X(t)c &= X(t+\omega)c \\ X(t)c &= X(t)Vc \\ c &= Vc\end{aligned}$$

Hence c is an eigenvector of V with corresponding eigenvalue (Floquet multiplier) $\lambda = 1$. □

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Homework Sheet 3

Exercise 3.1 – Compute the solution of the differential equation

$$\begin{aligned} x'(t) &= x(t) - y(t) - t^2 \\ y'(t) &= x(t) + 3y(t) + 2t \end{aligned} \tag{B.11}$$

∘

Exercise 3.2 – Consider the initial value problem

$$\begin{aligned} x'(t) &= A(t)x(t) \\ x(t_0) &= x_0 \end{aligned} \tag{B.12}$$

We have seen that the solution of this initial value problem is given by

$$x(t) = R(t, t_0)x_0$$

where $R(t, t_0)$ is the resolvent matrix of the system. Suppose that we are in the case where the following condition holds

$$\forall t, s \in \mathcal{I}, \quad A(t)A(s) = A(s)A(t) \tag{B.13}$$

with $\mathcal{I} \subset \mathbb{R}$.

i) Show that $M(t) = \exp\left(\int_{t_0}^t A(s)ds\right)$ is a solution of the matrix initial value problem

$$\begin{aligned} M'(t) &= A(t)M(t) \\ M(t_0) &= I_n \end{aligned}$$

where I_n is the $n \times n$ identity matrix. [Hint: Use the formal definition of a derivative, *i.e.*, $M'(t) = \lim_{h\to 0}(M(t+h) - M(t))/h$]

ii) Deduce that, when (B.13) holds,

$$R(t, t_0) = \exp\left(\int_{t_0}^t A(s)ds\right)$$

iii) Deduce the following theorem.

Theorem B.3.5. *Let U, V be constant matrices that commute, and suppose that $A(t) = f(t)U + g(t)V$ for scalar functions f, g. Then*

$$R(t, t_0) = \exp\left(\int_{t_0}^t f(s)dsU\right)\exp\left(\int_{t_0}^t g(s)dsV\right) \tag{B.14}$$

∘

Exercise 3.3 – Find the resolvent matrix associated to the matrix

$$A(t) = \begin{pmatrix} a(t) & -b(t) \\ b(t) & a(t) \end{pmatrix} \tag{B.15}$$

where a, b are continuous functions on $\mathbb{R}$. ∘

Exercise 3.4 – Consider the linear system

$$\begin{aligned} x' &= \frac{1}{t}x + ty \\ y' &= y \end{aligned} \tag{B.16}$$

with initial condition $x(t_0) = x_0$, $y(t_0) = y_0$.

i) Solve the initial value problem (B.16).

ii) Deduce the formula for the principal solution matrix $R(t, t_0)$.

iii) Show that in this case,

$$R(t, t_0) \neq \exp\left(\int_{t_0}^{t} A(s)ds\right)$$

with

$$A(t) = \begin{pmatrix} \frac{1}{t} & t \\ 0 & 1 \end{pmatrix}$$

∘

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Solutions – Homework Sheet 3

Solution – Exercise 3.1 – Let $\xi(t) = (x(t), y(t))^T$,

$$A = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix}$$

and $B(t) = (-t^2, 2t)^T$. The system (B.11) can then be written as

$$\xi' = A\xi + B(t)$$

This is a nonhomogeneous linear system, so we use the variation of constants,

$$\xi(t) = e^{(t-t_0)A}\xi_0 + \int_{t_0}^{t} e^{(t-s)A}B(s)ds$$

where $\xi_0 = \xi(t_0)$. Let us assume for simplicity that $t_0 = 0$.

Eigenvalues of A are $(\lambda - 2)^2$, with associated subspace

$$\begin{aligned} \ker(A - 2I) &= \left\{ \begin{pmatrix} x \\ y \end{pmatrix}; \quad \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\} \\ &= \left\{ \begin{pmatrix} x \\ y \end{pmatrix}; \quad x + y = 0 \right\} \end{aligned}$$

Thus $\dim\ker(A - 2I) = 1 \neq 2$, and A is not diagonalisable. Let us compute the Jordan canonical form of A. There exists P nonsingular such that

$$P^{-1}AP = \begin{pmatrix} 2 & \alpha \\ 0 & 2 \end{pmatrix}$$

where α is a constant that has to be determined.

$$P^{-1}AP = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$$

$$P = \begin{pmatrix} 1 & -1 \\ -1 & 0 \end{pmatrix}, \quad P^{-1} = \begin{pmatrix} 0 & -1 \\ -1 & -1 \end{pmatrix}$$

Then

$$e^{At} = Pe^{(P^{-1}AP)t}P^{-1}$$

with

$$e^{(P^{-1}AP)t} = e^{P^{-1}(At)P}$$

We have $P^{-1}AP = 2I + N$, where

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

Therefore, $e^{(P^{-1}AP)t} = e^{(2It+Nt)} = e^{2t}e^{Nt}$. Now, N is nilpotent ($N^2 = 0$), so $e^{Nt} = \sum_{n=0}^{\infty} \frac{t^n}{n!} N^n = I + Nt$. As a consequence,

$$\begin{aligned} e^{(P^{-1}AP)t} &= e^{2t}(I + Nt) \\ &= e^{2t}\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right) \\ &= \begin{pmatrix} e^{2t} & te^{2t} \\ 0 & e^{2t} \end{pmatrix} \end{aligned}$$

Thus

$$\begin{aligned} e^{At} &= P \begin{pmatrix} e^{2t} & te^{2t} \\ 0 & e^{2t} \end{pmatrix} P^{-1} \\ &= \begin{pmatrix} (1-t)e^{2t} & -te^{2t} \\ te^{2t} & (1+t)e^{2t} \end{pmatrix} \\ &= e^{2t} \begin{pmatrix} 1-t & -t \\ t & 1+t \end{pmatrix} \end{aligned}$$

We still have to compute $\int_0^t e^{-As}B(s)ds$. We have

$$e^{-As}B(s) = e^{-2s} \begin{pmatrix} 1+s & s \\ -s & 1-s \end{pmatrix} \begin{pmatrix} -s^2 \\ 2s \end{pmatrix} = e^{-2s} \begin{pmatrix} -s^3 - s^2 + 2s^2 \\ s^3 + 2s - 2s^2 \end{pmatrix} = e^{-2s} \begin{pmatrix} s^2 - s^3 \\ s^3 - 2s^2 + 2s \end{pmatrix}$$

Let $I_1(t) = \int_0^t e^{-2s} s ds$, $I_2(t) = \int_0^t e^{-2s} s^2 ds$ and $I_3(t) = \int_0^t e^{-2s} s^3 ds$. Then

$$\int_0^t e^{-As}B(s)ds = \begin{pmatrix} I_1(t) - I_3(t) \\ I_3(t) - 2I_2(t) + 2I_1(t) \end{pmatrix}$$

Evaluating the integrals, we obtain

$$\int_0^t e^{-As}B(s)ds = \begin{pmatrix} e^{-2t}\left(\frac{1}{4}t^2 + \frac{1}{4}t + \frac{1}{8} + \frac{1}{2}t^3\right) - \frac{1}{8} \\ e^{-2t}\left(\frac{1}{4}t^2 - \frac{3}{4}t - \frac{3}{8} - \frac{1}{2}t^3\right) + \frac{3}{8} \end{pmatrix}$$

As a conclusion,

$$\begin{aligned} \xi(t) &= e^{2t} \begin{pmatrix} 1-t & -t \\ t & 1+t \end{pmatrix} \xi_0 + \begin{pmatrix} 1-t & -t \\ t & 1+t \end{pmatrix} \begin{pmatrix} e^{-2t}\left(\frac{1}{4}t^2 + \frac{1}{4}t + \frac{1}{8} + \frac{1}{2}t^3\right) - \frac{1}{8} \\ e^{-2t}\left(\frac{1}{4}t^2 - \frac{3}{4}t - \frac{3}{8} - \frac{1}{2}t^3\right) + \frac{3}{8} \end{pmatrix} \\ &= \begin{pmatrix} e^{-2t}x_0 - e^{-2t}tx_0 - e^{-2t}ty_0 + \frac{1}{2}e^{-2t}t + \frac{1}{8}e^{-2t} - \frac{1}{8} - \frac{t}{4} + \frac{3}{4}e^{-2t}t^2 \\ e^{-2t}tx_0 + e^{-2t}y_0 + e^{-2t}ty_0 - \frac{1}{4}e^{-2t}t^2 - e^{-2t}t + \frac{t}{4} - \frac{3}{8}e^{-2t} + \frac{3}{8} \end{pmatrix} \end{aligned}$$

is the solution of (B.11) going through (x_0, y_0) at time 0. ◦

Solution – Exercise 3.2 – **1)** We have

$$\frac{1}{h}\left(M(t+h)-M(t)\right)=\frac{1}{h}\left(\exp\left(\int_{t_0}^{t+h}A(s)ds\right)-\exp\left(\int_{t_0}^{t}A(s)ds\right)\right)$$

Since, for all $s_1,s_2\in\mathcal{I}$, $A(s_1)A(s_2)=A(s_2)A(s_1)$, we have

$$\left(\int_{t_0}^{s_1}A(u)du\right)\left(\int_{t_0}^{s_2}A(u)du\right)=\left(\int_{t_0}^{s_2}A(u)du\right)\left(\int_{t_0}^{s_1}A(u)du\right)$$

It follows that

$$\begin{aligned}\frac{1}{h}\left(M(t+h)-M(t)\right)&=\frac{1}{h}\left(\exp\left(\int_{t}^{t+h}A(s)ds\right)\exp\left(\int_{t_0}^{t}A(s)ds\right)-\exp\left(\int_{t_0}^{t}A(s)ds\right)\right)\\&=\frac{1}{h}M(t)\left(\exp\left(\int_{t}^{t+h}A(s)ds\right)-I\right)\\&=\left(\frac{1}{h}\exp\left(\int_{t}^{t+h}A(s)ds\right)-\frac{1}{h}I\right)M(t)\end{aligned}$$

The second term in this equality tends to zero as $h\to 0$, and thus

$$\lim_{h\to 0}\frac{1}{h}(M(t+h)-M(t))=A(t)M(t)$$

therefore $M'(t)=A(t)M(t)$, hence the result.

2) The implication is trivial.

3) If U and V commute and that $A(t)=Uf(t)+Vg(t)$, then

$$\begin{aligned}A(s)A(t)&=(Uf(s)+Vg(s))(Uf(t)+Vg(t))\\&=U^2f(s)f(t)+UVf(s)g(t)+VUg(s)f(t)+V^2g(s)g(t)\\&=U^2f(t)f(s)+VUg(t)f(s)+UVf(t)g(s)+V^2g(t)g(s)\\&=(Uf(t)+Vg(t))Uf(s)+(Uf(t)+Vg(t))Vg(s)\\&=(Uf(t)+Vg(t))(Uf(s)+Vg(s))\end{aligned}$$

that is, $A(s)$ and $A(t)$ commute for all t,s. Then

$$\begin{aligned}R(t,t_0)=\exp\left(\int_{t_0}^{t}A(s)ds\right)&=\exp\left(\int_{t_0}^{t}Uf(s)+Vg(s)ds\right)\\&=\exp\left(\int_{t_0}^{t}f(s)dsU+\int_{t_0}^{t}g(s)dsV\right)\\&=\exp\left(\int_{t_0}^{t}f(s)dsU\right)\exp\left(\int_{t_0}^{t}g(s)dsV\right)\end{aligned}$$

○

Solution – Exercise 3.3 – Writing

$$A(t) = a(t)\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + b(t)\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix} = a(t)I + b(t)V,$$

it is obvious the Theorem B.3.5 can be used here, since I commutes with all matrices. Thus,

$$R(t,t_0) = \exp\left(\int_{t_0}^t a(s)ds I\right)\exp\left(\int_{t_0}^t b(s)dsV\right)$$

Let $\alpha(t) = \int_{t_0}^t a(s)ds$ and $\beta(t) = \int_{t_0}^t b(s)ds$. Then $R(t,t_0) = e^{\alpha(t)I}e^{\beta(t)V} = e^{\alpha(t)}Ie^{\beta(t)V}$. Now notice that $V^2 = I$, $V^3 = -V$, etc., so that we can write that

$$V^n = \begin{cases} (-1)^p I \text{ if } n = 2p,\\ (-1)^p V \text{ if } n = 2p+1.\end{cases}$$

This implies that

$$\begin{aligned} e^{\beta(t)V} &= \sum_{n=0}^\infty \frac{1}{n!}\beta(t)^n V^n\\ &= \left(\sum_{p=0}^\infty \frac{(-1)^p}{(2p)!}\beta(t)^{2p}\right) I + \left(\sum_{p=0}^\infty \frac{(-1)^p}{(2p+1)!}\beta(t)^{2p+1}\right) V\\ &= \cos(\beta(t))I + \sin(\beta(t))V \end{aligned}$$

Thus

$$\begin{aligned} R(t,t_0) &= e^{\alpha(t)}(\cos(\beta(t))I + \sin(\beta(t))V)\\ &= \begin{pmatrix} e^{\alpha(t)}\cos(\beta(t)) & -e^{\alpha(t)}\sin(\beta(t))\\ e^{\alpha(t)}\sin(\beta(t)) & e^{\alpha(t)}\cos(\beta(t))\end{pmatrix} \end{aligned}$$

∘

Solution – Exercise 3.4 – **1)** Notice that the y' equation in (B.16) does not involve x. Therefore, we can solve it directly, giving $y(t) = Ce^t$, with $C \in \mathbb{R}$. Substituting this into the equation for x', we have

$$x' = \frac{1}{t}x(t) + tCe^t$$

To solve this nonhomogeneous first-order scalar equation, we start by solving the homogeneous part, $x' = x/t$. This equation is separable, giving the solution $x(t) = Kt$, for $K \in \mathbb{R}$. Now we use a variation of constants approach to find a particular solution to the nonhomogeneous problem. We use the ansatz $x(t) = K(t)t$, which, when differentiated and substituted into the nonhomogeneous equation, gives $K'(t) = Ce^t$, and hence, $K(t) = Ce^t$ is a particular solution, giving the general solution $x(t) = Kt + Ce^t$.

Let $t_0 \neq 0$ (to avoid problems with $1/t$), and suppose $x(t_0) = x_0$, $y(t_0) = y_0$. Then $x_0 = Kt_0 + Ct_0e^{t_0}$ and $y_0 = Ce^{t_0}$. It follows that $K = x_0/t_0 - y_0$ and $C = e^{-t_0}y_0$, and the solution to the equation going through the point (x_0, y_0) at time t_0 is given by

$$\xi(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} (\frac{x_0}{t_0} - y_0)t + e^{-t_0}y_0te^t \\ e^{-t_0}y_0e^t \end{pmatrix} = \begin{pmatrix} \frac{x_0}{t_0}t + y_0t(e^{t-t_0} - 1) \\ y_0e^{t-t_0} \end{pmatrix}.$$

2) The solution to the IVP

$$\begin{aligned} \xi' &= A(t)\xi \\ \xi(t_0) &= \xi_0 \end{aligned}$$

is given by $\xi(t) = R(t, t_0)\xi_0$. Thus, the resolvent matrix (or principal solution matrix) for (B.16) is given by

$$R(t, t_0) = \begin{pmatrix} \frac{t}{t_0} & -t + te^{t-t_0} \\ 0 & e^{t-t_0} \end{pmatrix}$$

3) First of all, notice that $A(t)$ and $A(s)$ do not commute. Let us compute $B(t) = \int_{t_0}^t A(s)ds$.

$$\begin{aligned} B(t) &= \begin{pmatrix} \int_{t_0}^t \frac{1}{s}ds & \int_{t_0}^t sds \\ 0 & \int_{t_0}^t ds \end{pmatrix} \\ &= \begin{pmatrix} \ln\frac{t}{t_0} & \frac{1}{2}(t^2 - t_0^2) \\ 0 & t - t_0 \end{pmatrix} \end{aligned}$$

which, for convenience, we denote

$$B(t) = \begin{pmatrix} \alpha & \beta \\ 0 & \gamma \end{pmatrix}$$

Eigenvalues of $B(t)$ are α and γ. As $R(t_0, t_0) = B(t_0) = I$, we are concerned with finding a $t \neq t_0$ such that $B(t)$ is diagonalizable. If a t exists such that $\alpha \neq \gamma$, then $B(t)$ is diagonalizable, *i.e.*, there exists P nonsingular such that

$$P^{-1}B(t)P = D = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$$

Then

$$e^{B(t)} = P\begin{pmatrix} e^\alpha & 0 \\ 0 & e^\beta \end{pmatrix}P^{-1}$$

We find

$$P = \begin{pmatrix} 1 & \beta \\ 0 & \gamma - \alpha \end{pmatrix}, \quad P^{-1} = \frac{1}{\gamma - \alpha}\begin{pmatrix} \gamma - \alpha & -\beta \\ 0 & 1 \end{pmatrix}$$

Thus, after a few computations,

$$e^{B(t)} = \begin{pmatrix} e^{\alpha} & \frac{\beta}{\gamma-\alpha}(e^{\gamma}-e^{\alpha}) \\ 0 & e^{\gamma} \end{pmatrix} = \begin{pmatrix} \frac{t}{t_0} & \Delta \\ 0 & e^{t-t_0} \end{pmatrix}$$

The element Δ in this matrix is the only one different from the elements in $R(t,t_0)$. We have

$$\Delta = \frac{t^2-t_0^2}{2(t-t_0-\ln\frac{t}{t_0})}\left(e^{t-t_0}-\frac{t}{t_0}\right) \neq t(e^{t-t_0}-1)$$

$\circ$

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Homework Sheet 4

Exercise 3.1 – A differential equation of the form

$$x' = f(t, x(t), x(t-\omega)) \tag{B.17}$$

for $\omega > 0$, is called a *delay differential equation* (or also a differential difference equation, or an equation with deviating argument), and ω is called the *delay.* The basic initial value problem for (B.17) takes the form

$$\begin{aligned} x' &= f(t, x(t), x(t-\omega)) \\ x(t) &= \phi_0(t), \quad t_0 - \omega \le t \le t_0 \end{aligned} \tag{B.18}$$

i) Use the *method of steps* to construct the solution to (B.18) on the interval $[t_0, t_0+\omega]$, that is, find how to construct the solution to the non delayed problem

$$\begin{aligned} x' &= f(t, x(t), \phi_0(t-\omega)) \\ x(t_0) &= \phi_0(t_0), \quad t_0 \le t \le t_0 + \omega \end{aligned} \tag{B.19}$$

ii) Discuss existence and uniqueness of the solution on the interval $[t_0, t_0+\omega]$, depending on the nature of ϕ_0 and f.

iii) Suppose that $\phi_0 \in C^0([t_0-\omega, t_0])$. Discuss the regularity of the solution to (B.18) on the interval $[t_0 + k\omega, t_0 + (k+1)\omega]$, $k \in \mathbb{N}$.

∘

Exercise 3.2 – Consider the delay initial value problem

$$\begin{aligned} x'(t) &= ax(t-\omega) \\ x(t) &= C, \quad t \in [t_0-\omega, t_0] \end{aligned} \tag{B.20}$$

with $a, C \in \mathbb{R}$, $\omega \in \mathbb{R}_+^*$. Using the ideas of the previous exercise, find the solution to (B.20) on the interval $[t_0 + k\omega, t_0 + (k+1)\omega]$, $k \in \mathbb{N}$. ∘

Exercise 3.3 – Compute A_i^n and e^{tA_i} for the following matrices.

$$A_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 0 & 1 & -\sin(\theta) \\ -1 & 0 & \cos(\theta) \\ -\sin(\theta) & \cos(\theta) & 0 \end{pmatrix}.$$

∘

Exercise 3.4 – Compute e^{tA} for the matrix

$$A=\begin{pmatrix}1&1&0\\-1&0&-1\\0&-1&1\end{pmatrix}.$$

○

Exercise 3.5 – Let $A\in\mathcal{M}_n(\mathbb{R})$ be a matrix (independent of t), $\|\cdot\|$ be a norm on $\mathbb{R}^n$ and $|||\cdot|||$ the associated operator norm on $\mathcal{M}_n(\mathbb{R})$.

i) a) Show that for all $t\in\mathbb{R}$ and all $k\in\mathbb{N}^*$, there exists $C_k(t)\geq 0$ such that,

$$\left|\left|\left|e^{\frac{t}{k}A}-\left(I+\frac{t}{k}A\right)\right|\right|\right|\leq\frac{1}{k^2}C_k(t).$$

with $\lim_{k\to\infty}C_k(t)=\frac{t^2}{2}|||A^2|||$.

b) Show that for all $t\in\mathbb{R}$ and all $k\in\mathbb{N}^*$,

$$\left|\left|\left|I+\frac{t}{k}A\right|\right|\right|\leq e^{\frac{|t|}{k}|||A|||}.$$

c) Deduce that

$$e^{tA}=\lim_{k\to\infty}\left(I+\frac{t}{k}A\right)^k.$$

ii) Suppose now that A is symmetric and that its eigenvalues are $>-\alpha$, with $\alpha>0$.

a) Show by induction that, for $k\geq 0$,

$$(\alpha I+A)^{-(k+1)}=\int_0^\infty e^{-t(\alpha I+A)}\frac{t^k}{k!}dt.$$

b) Deduce that for all $u>0$,

$$|||(I+uA)^{-k}|||\leq M \text{ if, and only if, } |||e^{-tA}|||\leq M,\quad\forall t>0.$$

c) Show that

$$\left(\forall t>0,\quad e^{-tA}\geq 0\right)\Leftrightarrow\left(\exists\lambda_0,\quad\forall\lambda>\lambda_0,\quad(\lambda I+A)^{-1}\geq 0\right)$$

where by convention, for $B=(b_{ij})\in\mathcal{M}_n(\mathbb{R})$, writing that $B\geq 0$ means that $b_{ij}\geq 0$ for all $i,j=1,\dots,n$.

iii) Do the results of part **2)** hold true if A is a nonsymmetric matrix?

○

McMaster University – Math4G03/6G03
Fall 2003
Homework Sheet 4 – Solutions

Solution – Exercise 1 – **1)** The proposed method consists in considering (B.18) as a nondelayed IVP on the interval $[t_0, t_0+\omega]$. Indeed, on this interval, we can consider (B.19). That the latter is a nondelayed problem is obvious if we rewrite the differential equation as

$$x'(t) = g(t, x(t)) \tag{B.21}$$

with $g(t, x(t)) = f(t, x(t), \phi_0(t-\omega))$, which is well defined on the interval $[t_0, t_0+\omega]$ since for $t \in [t_0, t_0+\omega]$, $t-\omega \in [-\omega, 0]$, on which the function ϕ_0 is defined.

We can then use the integral form to construct the solution on the interval $[t_0, t_0+\omega]$,

$$\begin{aligned} x(t) &= x(t_0) + \int_{t_0}^t g(s, x(s))ds \\ &= \phi_0(t_0) + \int_{t_0}^t f(s, x(s), \phi_0(s-\omega))ds \end{aligned}$$

2) Obviously, the discussion to make is on the nature of the function f. As problem (B.19) is an ordinary differential equations initial value problem, existence and uniqueness of solutions on the interval $[t_0, t_0+\omega]$ follow the usual scheme. To discuss the required properties on f and ϕ_0, the best is to use (B.21). Recall that a vector field has to be continuous both in t and in x for solutions to exist. Thus to have existence of solutions to the equation (B.21), g must be continuous in t and x. This implies that $f(t, x, \phi_0(t-\omega))$ must be continuous in t, x. Thus ϕ_0 has to be continuous on $[t_0-\omega, t_0]$.

Now, for uniqueness of solutions to (B.21), we need g to be Lipschitz in x, *i.e.*, we require the same property from f. Note that this does not imply either ϕ_0 or the way f depends on ϕ_0.

3) Every successive integration raises the regularity of the solution: x is C^1 on $[t_0, t_0+\omega]$, C^2 on $[t_0+\omega, t_0+2\omega]$, etc. Hence, x is C^n on $[t_0+(n-1)\omega, t_0+n\omega]$. ∘

Solution – Exercise 2 – We proceed as previously explained. We assume for simplicity that $t_0 = 0$. To find the solution on the interval $[0, \omega]$, we consider the nondelayed IVP

$$\begin{aligned} x_1' &= a x_0(t) \\ x_1(0) &= C \end{aligned}$$

where $x_0(t) = C$ for $t \in [0, \omega]$. The solution to this IVP is straightforward, $x_1(t) = C + aCt = C(1+at)$, defined on the interval $[0, \omega]$. To integrate on the second interval, we consider the IVP

$$\begin{aligned} x_2' &= a[C(1+at)] \\ x_2(\omega) &= x_1(\omega) = C + aC\omega \end{aligned}$$

Hence we find the solution to the differential equation to be, on the interval $[\omega, 2\omega]$,

$$x_2(t) = C\left(1 + at + \frac{1}{2}a^2t^2 - \frac{1}{2}a^2\omega^2\right)$$

Iterating this process one more time with the IVP

$$\begin{aligned} x_3' &= a\left[C\left(1 + at + \frac{1}{2}a^2t^2 - \frac{1}{2}a^2\omega^2\right)\right] \\ x_3(2\omega) = x_2(2\omega) &= \frac{3}{2}a^2C\omega^2 + 2aC\omega + C \end{aligned}$$

we find, on the interval $[2\omega, 3\omega]$, the solution

$$x_3(t) = C\left(1 + at + \frac{1}{2}a^2t^2 + \frac{1}{6}a^3t^3 - \frac{1}{2}ta^3\omega^2 - \frac{1}{3}a^3\omega^3 - \frac{1}{2}a^2\omega^2\right)$$

We develop the intuition that the solution at step n (*i.e.*, on the interval $[(n-1)\omega, n\omega]$) must take the form

$$x_n(t) = C\sum_{k=0}^{n} a^k \frac{(t-(k-1)\omega)^k}{k!} \tag{B.22}$$

This can be proved by induction (we will not do it here). ◦

Solution – Exercise 3 – For matrix A_1, we have $A_1^2 = I$, $A_1^3 = A_1$, etc. Hence, $A_1^{2n} = I$ and $A_1^{2n+1} = A_1$, which implies

$$\begin{aligned} eAt &= \sum_{n=0}^{\infty} \frac{t^{2n}}{(2n)!}A_1^{2n} + \sum_{n=0}^{\infty} \frac{t^{2n+1}}{(2n+1)!}A_1^{2n+1} \\ &= \left(\sum_{n=0}^{\infty} \frac{t^{2n}}{(2n)!}\right) I + \left(\sum_{n=0}^{\infty} \frac{t^{2n+1}}{(2n+1)!}\right) A_1 \\ &= \cosh tI + \sinh tA \\ &= \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix} \end{aligned}$$

For matrix A_2, remark that it can be written as $A_2 = I + N$, where

$$N = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

is a nilpotent matrix. Thus $A_2^2 = (I+N)^2 = I+2N+N^2 = I+2N$, $A_2^3 = (I+2N)(I+N) = I + 2N + N + 2N^2 = I + 3N$, and it is easily shown by induction that $(I+N)^n = I + nN$.

It follows that

$$\begin{aligned}
e^{A_2 t} &= \left(\sum_{n=0}^{\infty} \frac{t^n}{n!}\right) I + \left(\sum_{n=0}^{\infty} \frac{nt^n}{n!}\right) N \\
&= e^t I + t\left(\sum_{n=0}^{\infty} \frac{t^{n-1}}{(n-1)!}\right) N \\
&= e^t I + te^t N \\
&= e^t \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}
\end{aligned}$$

Finally, for matrix A_3, we have that

$$A_3^2 = \begin{pmatrix} -\cos^2\theta & -\frac{1}{2}\sin 2\theta & \cos\theta \\ -\frac{1}{2}\sin 2\theta & -\sin^2\theta & \sin\theta \\ -\cos\theta & -\sin\theta & 1 \end{pmatrix}$$

and $A_3^3 = 0$, *i.e.*, A_3 is nilpotent for the index 3. Therefore, $e^{A_3 t} = I + tA_3 + \frac{t^2}{2}A_3^2$. ◦

Solution – Exercise 4 – The Jordan canonical form of A is

$$J = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

Now, to compute $e^J t$, remark that J has the form

$$\left(\begin{array}{c:cc} 0 & 0 & 0 \\ \hdashline 0 & \multicolumn{2}{c}{\multirow{2}{*}{A_1}} \\ 0 & & \end{array}\right)$$

where $e^{A_1 t}$ has been computed in Exercise 3. Hence,

$$\left(\begin{array}{c:cc} e^{0t} & 0 & 0 \\ \hdashline 0 & \multicolumn{2}{c}{\multirow{2}{*}{$e^{A_1 t}$}} \\ 0 & & \end{array}\right) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^t & te^t \\ 0 & 0 & e^t \end{pmatrix}$$

We have $J = P^{-1}AP$, where

$$P = \begin{pmatrix} -1 & -1 & 2 \\ 1 & 0 & -1 \\ 1 & 1 & -1 \end{pmatrix}, \quad P^{-1} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$

are the matrices of change of basis that transform A to its Jordan canonical form. Then $A = PJP^{-1}$, and $e^{At} = Pe^{Jt}P^{-1}$, *i.e.*,

$$e^{At} = \begin{pmatrix} (2-t)e^t - 1 & e^t - 1 & (1-t)e^t - 1 \\ 1 - e^t & 1 & 1 - e^t \\ 1 + (t-1)e^t & 1 - e^t 1 + te^t & \end{pmatrix}$$

∘

Solution – Exercise 5 – This exercise was far from trivial.

1–a) Consider the map

$$\begin{aligned} t &\mapsto e^{At} \\ \mathbb{R} &\to \mathcal{M}_n(\mathbb{R}) \end{aligned}$$

We have $(e^{At})' = Ae^{At}$ and $(e^{At})^{(k)} = A^k e^{At}$, where $u^{(k)}$ denotes the kth derivative of u.

$$e^{\frac{t}{k}A} = I + \frac{t}{k}A + \frac{t^2}{2k^2}A^2 + \frac{t^2}{k^2}\varepsilon(\frac{t}{k})$$

Thus

$$e^{\frac{t}{k}A} - I - \frac{t}{k}A = \frac{t^2}{2k^2}A^2 + \frac{t^2}{k^2}\varepsilon(\frac{t}{k})$$

Therefore, taking the norm $|||\cdot|||$ of this expression,

$$\left|\left|\left| e^{\frac{t}{k}A} - (I + \frac{t}{k}A)\right|\right|\right| \leq \frac{1}{k^2}\left\{\frac{t^2}{2}|||A^2||| + t^2\left|\left|\left|\varepsilon(\frac{t}{k})\right|\right|\right|\right\} = \frac{1}{k^2}C_k(t)$$

We have

$$\lim_{k\to\infty} C_k(t) = \frac{t^2}{2}|||A^2|||$$

Let $S_k = \sum_{j=3}^{\infty}\frac{t^j}{k^{j-2}j!}|||A|||^j$. This series is uniformly convergent, which implies that we can change the order in the following limit,

$$\lim_{k\to\infty} S_k = \sum_{j=3}^{\infty}\lim_{k\to\infty}\left(\frac{t^j}{k^{j-2}j!}|||A|||^j\right) = 0$$

1–b) We have already seen (Exercise 2, Assignment 2) that $|||e^A||| \leq e^{|||A|||}$. Therefore,

$$\left|\left|\left| e^{\frac{t}{k}A}\right|\right|\right| \leq e^{\frac{t}{k}|||A|||}$$

But

$$\begin{aligned} e^{\frac{|t|}{k}|||A|||} &= \sum_{k=0}^{\infty}\frac{1}{k!}\frac{|t|^k|}{k^k}|||A|||^k \\ &= 1 + \frac{|t|}{k}|||A||| + \frac{|t|^2|}{2k^2}|||A|||^2 + \cdots \end{aligned}$$

which, since $\frac{|t|^2}{2k^2}|||A|||^2 + \cdots \geq 0$, implies that

$$\begin{aligned} e^{\frac{|t|}{k}|||A|||} &\geq 1 + \frac{|t|}{k}|||A||| = |||I||| + \left|\left|\left|\frac{t}{k}A\right|\right|\right| \\ &\geq \left|\left|\left| I + \frac{t}{k}A\right|\right|\right| \end{aligned}$$

(since $\|I\| = \sup_{\|v\|\leq 1}\|IV\| = \sup_{\|v\|\leq 1}\|v\| = 1$).

1–c) We skip the scalar case, and consider the case $n \geq 2$. We have

$$\begin{aligned}
e^{At} - (I + \frac{t}{k}A)^k &= (e^{\frac{t}{k}A})^k - (I + \frac{t}{k}A)^k \\
&= \left(e^{\frac{t}{k}A} - (I + \frac{t}{k}A)\right)\sum_{j=0}^{k-1}(e^{\frac{t}{k}A})^{k-1-j}(I + \frac{t}{n}A)^j
\end{aligned}$$

since for two matrices $E, F \in \mathcal{M}_n(\mathbb{R})$ (or $\mathcal{M}_n(\mathbb{C})$) that commute, $E^n - F^n = (E - F)\sum_{j=0}^{k-1} E^{k-1-j}F^j$.

Therefore,

$$\left\|e^{At} - (I + \frac{t}{k}A)^k\right\| \leq \left\|e^{\frac{t}{k}A-(I+\frac{t}{k}A)}\right\| \sum_{j=0}^{k-1}\left[\left\|e^{\frac{t(k-1-j)}{k}A}\right\| \left\|(I + \frac{t}{k}A)^j\right\|\right] \tag{B.23}$$

Now, we have $\left\|e^{\frac{t(k-1-j)}{k}A}\right\| \leq e^{\frac{|t|(k-1-j)}{k}\|A\|}$. Also,

$$\left\|(I + \frac{t}{k}A)^j\right\| \leq \left\|I + \frac{t}{k}A\right\|^j \leq \left(e^{\frac{|t|}{k}\|A\|}\right)^j = e^{\frac{j|t|}{k}\|A\|}$$

where the last inequality results from question **1–b)**. Therefore,

$$\begin{aligned}
\left\|e^{At} - (I + \frac{t}{k}A)^k\right\| &\leq \frac{1}{k^2}C_k(t)\sum_{j=0}^{k-1} e^{|t|\frac{k-1-j}{k}\|A\|}e^{\frac{j|t|}{k}\|A\|} \\
&= \frac{1}{k^2}C_k(t)\sum_{j=0}^{k-1} e^{|t|\frac{k-1}{k}\|A\|} \\
&= \frac{1}{k^2}C_k(t)ke^{|t|\frac{k-1}{k}\|A\|} \\
&= \frac{1}{k}C_k(t)e^{|t|\frac{k-1}{k}\|A\|}
\end{aligned}$$

We thus have

$$\left\|e^{At} - (I + \frac{t}{k}A)^k\right\| \leq \frac{1}{k}C_k(t)e^{|t|\frac{k-1}{k}\|A\|} = \frac{1}{k}D_k(t)$$

As $k \to \infty$, $C_k(t) \to \frac{t^2}{2}\|A^2\|$ and $e^{|t|\frac{k-1}{k}\|A\|} \to e^{|t|\|A\|}$. Therefore, $\lim_{k\to\infty} D_k(t) = \frac{t^2}{2}\|A^2\| e^{|t|\|A\|}$, which in turn implies that $\lim_{k\to\infty}(I + \frac{t}{k}A)^k = e^{At}$.

2–a) We now suppose that A is a symmetric matrix. Recall that any symmetric matrix is diagonalizable, with real eigenvalues. Furthermore, there exists a matrix P such that $P^{-1} = P^T$ and that $P^TAP = \text{diag}(\lambda_i)$, with $\lambda_i \in \text{Sp}\,(A)$.

We assume that the eigenvalues λ_i, $i = 1, \ldots, m$, are such that $\lambda_i > -\alpha$, for $\alpha > 0$. We want to show by induction that the following holds.

$$\forall k \geq 0, \quad (\alpha I + A)^{-(k+1)} = \int_0^\infty e^{-(\alpha I + A)t}\frac{t^k}{k!}dt \tag{B.24}$$

Suppose that $k=0$. Equation (B.24) reads

$$(\alpha I + A)^{-1} = \int_0^\infty e^{-(\alpha I + A)t} dt$$

The matrix $(\alpha I + A)^{-1}$ is nonsingular. Indeed, suppose that $\det(\alpha I + A) = 0$. This is equivalent to $\det(-\alpha I - A) = 0$, which implies that $-\alpha$ is an eigenvalue of A, a contradiction with the hypothesis on the localization of the spectrum.

Now remark that if B is a nonsingular matrix, the equality $\frac{d}{dt}e^{Bt} = Be^{Bt}$ implies that $B\frac{d}{dt}e^{Bt} = e^{Bt}$. Since $(\alpha I + A)^{-1}$ is nonsingular, we thus have

$$\frac{d}{dt}e^{-(\alpha I + A)t} = -(\alpha I + A)e^{-(\alpha I + A)t}$$

and therefore

$$e^{-(\alpha I + A)t} = -(\alpha I + A)^{-1}\frac{d}{dt}\left(e^{-(\alpha I + A)t}\right)$$

and so, integrating,

$$\begin{aligned}
\int_0^\infty e^{-(\alpha I + A)t} ds &= \int_0^\infty -(\alpha I + A)^{-1}\frac{d}{dt}\left(e^{-(\alpha I + A)t}\right)dt \\
&= -(\alpha I + A)^{-1}\int_0^i nfty\frac{d}{dt}\left(e^{-(\alpha I + A)t}\right)dt \\
&= -(\alpha I + A)^{-1}\left[e^{-(\alpha I + A)t}\right]_0^\infty \\
&= -(\alpha I + A)^{-1}\left[\lim_{t\to\infty}\left(e^{-(\alpha I + A)t}\right) - I\right] \qquad \text{(B.25)}
\end{aligned}$$

Now,

$$e^{-(\alpha I + A)t} = P\begin{pmatrix} e^{-(\alpha+\lambda_1)t} & & 0 \\ & \ddots & \\ 0 & & e^{-(\alpha+\lambda_n)t}\end{pmatrix}P^{-1}$$

Since for all $i = 1, \ldots, n$, $\lambda_i > -\alpha$, it follows that $\lim_{t\to\infty} e^{-(\alpha+\lambda_i)t} = 0$, which in turn implies that $\lim_{t\to\infty} e^{-(\alpha I + A)t} = 0$. Using this in (B.25) gives (B.24) for $k=0$.

Now assume (B.24) holds for $k = j$, *i.e.*,

$$(\alpha I + A)^{-j} = \int_0^\infty e^{-(\alpha I + A)t}\frac{t^{j-1}}{(j-1)!}dt$$

Then

$$\begin{aligned}
\int_0^\infty e^{-(\alpha I + A)t}\frac{t^j}{j!}dt &= \int_0^\infty -(\alpha I + A)^{-1}\frac{d}{dt}\left(e^{-(\alpha I + A)t}\right)\frac{t^j}{j!}dt \\
&= -(\alpha I + A)^{-1}\int_0^\infty \frac{d}{dt}\left(e^{-(\alpha I + A)t}\right)\frac{t^j}{j!}dt \\
&= -(\alpha I + A)^{-1}\left\{\left[e^{-(\alpha I + A)t}\frac{t^j}{j!}\right]_0^\infty - \int_0^\infty e^{-(\alpha I + A)t}\frac{t^{j-1}}{(j-1)!}dt\right\}
\end{aligned}$$

As we did in the $k = 0$ case, we now use the bound on the eigenvalues to get rid of the term

$$\frac{t^j}{j!}e^{-(\alpha I+A)t} = P\begin{pmatrix} \frac{t^j}{j!}e^{-(\alpha+\lambda_1)t} & & 0 \\ & \ddots & \\ 0 & & \frac{t^j}{j!}e^{-(\alpha+\lambda_n)t}\end{pmatrix} P^{-1} \longrightarrow 0 \quad \text{as } t \to \infty$$

Therefore,

$$\begin{aligned}\int_0^\infty e^{-(\alpha I+A)t}\frac{t^j}{j!}dt &= -(\alpha I+A)^{-1}\int_0^\infty e^{-(\alpha I+A)t}\frac{t^{j-1}}{(j-1)!}dt \\ &= -(\alpha I+A)^{-1}(\alpha I+A)^{-j} \\ &= -(\alpha I+A)^{-(j+1)}\end{aligned}$$

from which we deduce that (B.24) holds for all $k \geq 0$.

2–b) Let us begin with the implication $\underline{(\forall u > 0, |||(I+uA)^{-k}||| \leq M) \Rightarrow (\forall t > 0, |||e^{-At}||| \leq M)}$. We know from **1–c)** that $e^{At} = \lim_{k\to\infty}(I + \frac{t}{k}A)^k$. Thus

$$e^{-At} = \lim_{k\to\infty}\left(\left(I+\frac{t}{k}A\right)^k\right)^{-1} = \lim_{k\to\infty}\left(I+\frac{t}{k}\right)^{-k} \tag{B.26}$$

Let $u = t/k$ with $k \in \mathbb{N}^*$ and $t > 0$. Then

$$\begin{aligned} &\forall t > 0,\ \forall k \in \mathbb{N}^*,\ \left|\left|\left|(I+\frac{t}{k}A)^{-k}\right|\right|\right| \leq M \\ &\Rightarrow \forall t > 0,\ \lim_{k\to\infty}\left|\left|\left|(I+\frac{t}{k}A)^{-k}\right|\right|\right| \leq M \\ &\Rightarrow \forall t > 0,\ \left|\left|\left|\lim_{k\to\infty}(I+\frac{t}{k}A)^{-k}\right|\right|\right| \leq M\end{aligned}$$

which, using (B.26), implies that

$$\forall t > 0,\ |||e^{-At}||| \leq M$$

We now treat the reverse implication, $\underline{(\forall t > 0, |||e^{-At}||| \leq M) \Rightarrow (\forall u > 0, |||(I+uA)^{-k}||| \leq M)}$. We have

$$(I+uA)^{-k} = [u(\frac{1}{u}I+A)]^{-k} = u^{-k}[\frac{1}{u}I+A]^{-k}$$

Suppose that $-\alpha > -1/u$, *i.e.*, $0 < u < 1/\alpha$. Then, from **2–a)**, it follows that

$$(I+uA)^{-k} = u^{-k}\int_0^\infty e^{-(\frac{1}{u}I+A)t}\frac{t^{j-1}}{(j-1)!}dt$$

which, when taking the norm, gives

$$\begin{aligned}\left|\!\left|\!\left|(I+uA)^{-k}\right|\!\right|\!\right| &\leq u^{-k}\int_0^\infty \left|\!\left|\!\left|e^{-At}\right|\!\right|\!\right| e^{-\frac{t}{u}}\frac{t^{j-1}}{(j-1)!}dt \\ &\leq \frac{Mu^{-k}}{(k-1)!}\int_0^\infty e^{-\frac{t}{u}}t^{k-1}dt\end{aligned}$$

Let $\chi = t/u$, then $dt = ud\chi$, and

$$\begin{aligned}\left|\!\left|\!\left|(I+uA)^{-k}\right|\!\right|\!\right| &\leq \frac{Mu^{-k}}{(k-1)!}\int_0^\infty e^{-\chi}u^{k-1}\chi^{k-1}d\chi \\ &\leq \frac{M}{(k-1)!}\int_0^\infty e^{-\chi}\chi^{k-1}d\chi\end{aligned}$$

The latter integral is $\Gamma(k)$, the Gamma Function. It is well known[1] that for $k \in \mathbb{N}$, $\Gamma(k) = (k-1)!$. Thus,

$$\left|\!\left|\!\left|(I+uA)^{-k}\right|\!\right|\!\right| \leq M$$

2–c) Let us begin with the forward implication ($\Rightarrow$). To apply **2–b)** with $k = 0$, it suffices that the eigenvalues of A be greater than $-\alpha$. Take $\lambda_0 = \alpha$. Then $\lambda > \alpha = \lambda_0$, and so

$$\begin{aligned}(\lambda I + A)^{-1} &= \int_0^\infty e^{-(\lambda I + A)t}dt \\ &= \int_0^\infty e^{-\lambda t}e^{-At}dt & \geq 0\end{aligned}$$

since the eigenvalues $\lambda \in \mathbb{R}$ and by hypothesis on e^{-At}.

Now for the reverse implication ($\Leftarrow$). That there exists $\lambda_0 \in \mathbb{R}$ such that for all $k \in \mathbb{N}^*$, $(\lambda I + A)^{-1} \geq 0$ implies that

$$\forall \lambda > \lambda_0, \quad \forall k \in \mathbb{N}^*, \quad (\lambda I + A)^{-k} \geq 0$$

for λ sufficiently large. Take $\lambda = k/t$, the previous expression can be written

$$\forall t > 0, \quad \forall k \geq k_0, \quad (\frac{k}{t}I + A)^{-k} \geq 0$$

where k_0 is sufficiently large. This implies that

$$\forall t > 0, \quad \forall k \geq k_0, \quad \left(\frac{k}{t}\right)^{-k}(I + \frac{t}{k}A)^{-k} \geq 0$$

As $(k/t)^{-k} > 0$,

$$\forall t > 0, \quad \forall k \geq k_0, \quad (I + \frac{t}{k}A)^{-k} \geq 0$$

[1] See, *e.g.*, M. Abramowitz and I.E. Stegun, *Handbook of Mathematical Functions.* Dover, 1965.

so

$$\forall t > 0, \quad \lim_{k \to \infty} (I + \frac{t}{k} A)^{-k} \geq 0$$

So that this finally implies that

$$\forall t > 0, \quad e^{-At} \geq 0$$

3) The results of the previous part hold. However, in the case of a nonsymmetric matrix, we need to ask for the real part of the eigenvalues to be greater than $-\alpha$. $\circ$

MATH 4G03/6G03

JULIEN ARINO

DURATION OF EXAMINATION: 72 Hours
MCMASTER UNIVERSITY FINAL EXAMINATION DECEMBER 2003

THIS EXAMINATION PAPER INCLUDES 4 PAGES AND 3 QUESTIONS. YOU ARE RESPONSIBLE FOR ENSURING THAT YOUR COPY OF THE PAPER IS COMPLETE.

Detailed Instructions

You have 72 hours, from the time you pick up and sign for this examination sheet, to complete this examination. You are to work on this examination by yourself. Any hint of collaborative work will be considered as evidence of academic dishonesty. You are not to have any outside contacts concerning this subject, except myself.

You can use any document that you find useful. If using theorems from outside sources, give the bibliographic reference, and show clearly how your problem fits in with the conditions of application of the theorem. When citing theorems from the lecture notes, refer to them by the number they have on the last version of the notes, as posted on the website on the first day of the examination period.

Pay attention to the form of your answers: as this is a take-home examination, you are expected to hand back a very legible document, in terms of the presentation of your answers. Show your calculations, but try to be concise.

This examination consists of 1 independent question and 2 problems. In questions or problems that have multiple parts, you are always allowed to consider as proved the results of a previous part, even if you have not actually done that part.

Foreword to the correction. This examination was long, but established the results in a very guided way. Exercise 1 was almost trivial. Both of the Problems dealt with Sturm theory. This comes as an illustration of the richness of behaviors that can be observed in differential equations: simple equations such as (B.29) can have very complex behaviors. Concerning the difficulty of the problems, it was not excessive. Problem 2 is a shortened and simplified version of a review problem for the CAPES, a French competition to hire high school teachers. The original problem comprised 23 questions, and was written by candidates in 5 hours. Problem 3 introduced the Wronskian, which we did not have time to cover during class. It also established further results of Sturm type.

Exercise 5.1 – Consider the mapping $A : t \mapsto A(t)$, continuous from $\mathbb{R}$ to $\mathcal{M}_n(\mathbb{R})$, periodic of period ω and such that $A(t)A(s) = A(s)A(t)$ for all $t, s \in \mathbb{R}$. Consider the equation

$$x'(t) = A(t)x \tag{B.27}$$

Let $R(t, t_0)$ be the resolvent of (B.27).

i) Show that $R(t + \omega, t_0 + \omega) = R(t, t_0)$ for all $t, t_0 \in \mathbb{R}$.

ii) Let u be an eigenvector of $R(\omega, 0)$, associated to the eigenvalue λ. Show that the solution of (B.27) taking the value u for $t_0 = 0$ is such that

$$x(t + \omega) = \lambda x(t), \quad \forall t \in \mathbb{R}. \tag{B.28}$$

iii) Conversely, show that if x is a nontrivial solution of (B.27) such that (B.28) holds, then λ is an eigenvalue of $R(\omega, 0)$.

○

Problem 5 2 – The aim of this problem is to study some properties of the solutions of the differential equation

$$x'' + q(t)x = 0, \tag{B.29}$$

where q is a continuous function from $\mathbb{R}$ to $\mathbb{R}$.

i) Show that for $t_0, x_0, y_0 \in \mathbb{R}$, there exists a unique solution of (B.29) such that

$$x(t_0) = x_0, \quad x'(t_0) = y_0$$

Preliminary results : convex functions. A function $f : \mathcal{I} \subset \mathbb{R} \to \mathbb{R}$ is *convex* if, for all $x, y \in \mathcal{I}$ and all $\lambda \in [0, 1]$,

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y).$$

Before proceeding with the study of the solutions of (B.29), we establish a few useful results on convex functions.

ii) Let f be a function defined on $\mathbb{R}$, convex and nonnegative. Suppose that f has two zeros t_1, t_2 and that $t_1 < t_2$. Show that f is zero on the interval $[t_1, t_2]$.

Let $c \in \mathbb{R}$ and f be a convex function that is bounded from above on the interval $[c, +\infty)$. It can then be shown that f is decreasing on $[c, +\infty)$. Using this fact, show the following.

iii) Every convex function that is bounded from above on $\mathbb{R}$ is constant.

Part I.

iv) Let $a, b \in \mathbb{R}$, $a < b$. We assume that (B.29) has a solution x, zero at a and at b and positive on (a, b). Show that

$$\int_a^b |q(t)|dt > \frac{4}{b-a}.$$

v) We suppose that $\int_0^\infty |q(t)|dt$ converges. Let x be a bounded solution of (B.29). Determine the behaviour of x' as $t \to \infty$.

vi) We suppose that $q \in C^1$ and is positive and increasing on $\mathbb{R}_+$. Show that all solutions of (B.29) are bounded on $\mathbb{R}_+$.

Part II. We suppose in this part that q is nonpositive and is not the zero function.

vii) Let x be a solution of (B.29). Show that x^2 is a convex function.

viii) Show that if x is a solution of (B.29) that has two distinct zeros, then $x \equiv 0$.

ix) Show that if x is a bounded solution of (B.29), then $x \equiv 0$.

Part III.

x) Let x, y be two solutions of (B.29). Show that the function $xy' - x'y$ is constant.

xi) Let x_1 and x_2 be the solutions of (B.29) that satisfy

$$\begin{aligned} x_1(0) = 1, &\quad x_1'(0) = 0, \\ x_2(0) = 0, &\quad x_2'(0) = 1. \end{aligned}$$

Show that (x_1, x_2) is a basis of the vector space $\mathcal{S}$ on $\mathbb{R}$ of the solutions of (B.29). What is the value of $x_1 x_2' - x_1' x_2$? Can the functions x_1 and x_2 have a common zero? Justify your answer.

xii) Discuss the results of question **11)** in the context of linear systems, *i.e.*, transform (B.29) into a system of first-order differential equations and express question **11)** and its answer in this context.

xiii) Show that if q is an even function, then the function x_1 is even and the function x_2 is odd.

Problem 5 3 – The aim of this problem is to show some elementary properties of the *Wronskian* of a system of solutions, and to use them to study a second-order differential equation.

Consider the nth order ordinary differential equation

$$x^{(n)} = a_0(t)x + a_1(t)x' + \cdots + a_{n-1}(t)x^{(n-1)}(t) \tag{B.30}$$

where $x^{(k)}$ denotes the kth derivative of x, $\frac{d^k x}{dt^k}$.

i) Find the matrix $A(t)$ such that this system can be written as the first-order linear system $y' = A(t)y$.

Part I : Wronskian Let $f_1, \dots, f_n$ be n functions from $\mathbb{R}$ into $\mathbb{R}$ that are $n-1$ times differentiable. We define $W(f_1, \dots, f_n)$, the *Wronskian* of $f_1, \dots, f_n$, by

$$W(f_1, \dots, f_n)(t) = \det \begin{pmatrix} f_1(t) & \cdots & f_n(t) \\ f_1'(t) & \cdots & f_n'(t) \\ \vdots & & \vdots \\ f_1^{(n-1)}(t) & \cdots & f_n^{(n-1)}(t) \end{pmatrix}.$$

If $f_1, \dots, f_n$ are linearly dependent, then $W(f_1, \dots, f_n) = 0$. The converse is false. Remember that the set of solutions of (B.30) forms a vector space S of dimension n.

ii) Using the equivalent linear system $y' = A(t)y$, show that every system of n solutions of (B.30) whose Wronskian is nonzero at a time τ constitutes a basis of S.

iii) Using the linear system $x' = A(t)x$, show that for every set of n solutions,

$$W(t) = W(s) \exp\left(\int_s^t a_{n-1}(u) du \right).$$

Part II : a theorem of Sturm Let us now consider the second-order differential equation

$$a_2(t)x'' + a_1(t)x' + a_0(t)x = 0 \tag{B.31}$$

The objective here is to show the following theorem of Sturm.

Theorem B.5.6 (Sturm)**.** *Let f_1, f_2 be two independent solutions of (B.31). Between two consecutive zeros of f_1, there is exactly one zero of f_2.*

We suppose f_1, f_2 are two independent solutions of (B.31).

iv) Let u and v be two consecutive zeros of f_1. Using the Wronskian $W(f_1, f_2)$, show that u and v cannot be zeros of f_2.

v) Deduce the theorem. [Hint: consider the function f_1/f_2.]

Part III : another theorem of Sturm. Let us assume that we confine ourselves to segments of $\mathbb{R}$ where $a_0(t) \neq 0$.

vi) Let

$$x(t) = u(t) \exp\left(\int_0^t \frac{a_1(s)}{a_2(s)} ds \right)$$

Show that (B.31) becomes $u'' + q(t)u = 0$.

We want to show the following theorem of Sturm.

Theorem B.5.7 (Sturm)**.** *Let*

$$x'' + q(t)x = 0, \quad q(t) \leq 0 \tag{B.32}$$

in an interval (t_1, t_2)*. Every solution of (B.32) that is not identically zero has at most one zero in the interval* $[t_1, t_2]$*.*

vi) Let ϕ be a solution of (B.32) on (t_1, t_2), and v be a zero of ϕ in this interval. Discuss the properties of ϕ. [Hint: Use of Problem 2, Part II is possible, but not strictly necessary.]

vii) Let $u < v$ be another zero of ϕ in the interval (t_1, t_2). Discuss properties of ϕ. Conclude.

Solution – Exercise 1 – **1)** We have

$$
\begin{aligned}
R(t+\omega, t_0+\omega) &= \exp\left(\int_{t_0+\omega}^{t+\omega} A(s)ds\right) \\
&= \exp\left(\int_{t_0}^{t} A(s+\omega)ds\right) \\
&= \exp\left(\int_{t_0}^{t} A(s)ds\right) \\
&= R(t, t_0)
\end{aligned}
$$

2) Let u be an eigenvector associated to the eigenvalue λ, *i.e.*,

$$R(\omega, 0)u = \lambda u$$

Let x be the solution of (B.27) such that $x(t_0) = x(0) = u$. As x is a solution of (B.27), we have that, for all t,

$$x(t) = R(t, t_0)u = R(t, 0)u$$

Therefore,

$$
\begin{aligned}
x(t+\omega) &= R(t+\omega, 0)u \\
&= R(t+\omega, \omega)R(\omega, 0)u \\
&= R(t+\omega, \omega)\lambda u \\
&= R(t, 0)\lambda u \\
&= \lambda R(t, 0)u \\
&= \lambda x(t)
\end{aligned}
$$

and hence (B.28).

3) Let x be a nonzero solution of (B.27) such that, for all $t \in \mathbb{R}$,

$$x(t+\omega) = \lambda x(t)$$

We assume that, for all $t \in \mathbb{R}$, $x(t+\omega) = \lambda x(t)$. This is true in particular for $t = 0$, and hence $x(\omega) = \lambda x(0)$. As $x \not\equiv 0$, there exists $v \in \mathbb{R} - \{0\}$ such that $x(0) = v$. Therefore,

$$x(t) = \lambda v$$

and as a consequence,

$$R(\omega, 0)v = \lambda v$$

and λ is an eigenvalue of $R(\omega, 0)$. ◦

Solution – Problem 2 – This problem concerns what are called Sturm theory type results, that is, results dealing with the behavior of second order differential equations.

1) This is a standard application of the existence-uniqueness result of Cauchy-Lipschitz. To see that, transform the second order equation into a system of first order equations, by setting $y = x'$. Then, differentiating y, we obtain

$$y' + qx = 0$$

Therefore, (B.27) is equivalent to the system of first order equations

$$\begin{aligned} x' &= y \\ y' &= -qx \end{aligned}$$

This is a linear system, hence satisfies a Lipschitz condition, and we can apply the Cauchy-Lipschitz theorem.

2) Since f is nonnegative and convex, we have that, for all $\lambda \in [0,1]$,

$$0 \leq f\left((1-\lambda)t_1 + \lambda t_2\right) \leq (1-\lambda)f(t_1) + \lambda f(t_2)$$

But we have supposed that $f(t_1) = f(t_2) = 0$. Hence we have that for all $\lambda \in [0,1]$,

$$f\left((1-\lambda)t_1 + \lambda t_2\right) = 0$$

and so f is zero on $[t_1, t_2]$.

3)
4)
5)
6)
7)
8)
9)
10)
11)
12)
13) ◦

Solution – Problem 3 – This problem was also about Sturm results. But it also introduced the notion of Wronskian, which is a very general tool intimately linked to the notion of resolvent matrix.

1) We let $y_1 = x, y_2 = x', \ldots, y_n = x^{(n-1)}$. As a consequence, $y_1' = y_2, y_2' = y_3, \ldots, y_{n-1}' = y_n$, and $y_n' = a_0(t)y_1 + a_1(t)y_2 + \cdots + a_{n-1}(t)y_{n-1}$. Written in matrix form, this is equivalent to $y' = A(t)y$, with $y = (y_1, \ldots, y_n)^T$ and

$$A(t) = \begin{pmatrix} 0 & 1 & 0 & & \ldots & 0 \\ 0 & 0 & 1 & 0 & \ldots & 0 \\ \vdots & & & & & \vdots \\ \vdots & & & & 1 & 0 \\ 0 & & & & & 1 \\ a_0(t) & a_1(t) & & & \ldots & a_{n-1}(t) \end{pmatrix} \tag{B.33}$$

2) We know that the system is equivalent to $y' = A(t)y$, with $A(t)$ given by (B.33). To every basis $(\phi_1, \ldots, \phi_n)$ of the vector space of solutions of

$$y' = A(t)y \tag{B.34}$$

there corresponds a basis $(\varphi_1, \ldots, \varphi_n)$ of (B.30), where φ_i is the first coordinate of the vector ϕ_i for every i. The converse is also true.

We know that a system $(\phi_1, \ldots, \phi_n)$ of solutions of (B.34) is a basis if $\det(\phi_1, \ldots, \phi_n) \neq 0$, and it suffices for this that $\det(\phi_1, \ldots, \phi_n)$ be nonzero at one point.

Since we have $\det(\phi_1, \ldots, \phi_n) = W(\varphi_1, \ldots, \varphi_n)$, the result follows.

3) This is a direct application of Liouville's theorem, which states that if $R(t, s)$ is the resolvent of $A(t)$, then

$$\det R(t, s) = \exp\left(\int_s^t \mathrm{tr} A(u) du\right)$$

And if a system of coordinates is fixed, for every fundamental matrix $\hat{\Phi}$,

$$\det \hat{\Phi}(t) = \det \hat{\Phi}(s) \exp\left(\int_s^t \mathrm{tr} A(u) du\right)$$

From Liouville's theorem,

$$\det(\Phi(t)\Phi^{-1}(s)) = \exp\left(\int_s^t \mathrm{tr} A(u) du\right)$$

which implies that

$$\det \Phi(t) = \det \Phi(s) \exp\left(\int_s^t \mathrm{tr} A(u) du\right)$$

Now, note that $\det \Phi(t) = W(t)$ and $\det \Phi(s) = W(s)$. This implies the result.

Note that for a system of solutions of (B.30), $W(\varphi) \neq 0$ iff φ are linearly independent (*i.e.*, we have the converse implication).

4)

5)

6)

7) $\circ$

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Homework Sheet 1

Periodic solutions of differential equations

In this problem, we will study the solutions of some differential equations, and in particular, their periodic solutions.

Let $T > 0$ be a real number, P be the vector space of real valued, continuous and T-periodic functions defined on $\mathbb{R}$, and let $a \in P$. Define

$$A = \int_0^T a(t)dt, \qquad g(t) = \exp\left(\int_0^t a(u)du\right),$$

and endow P with the norm

$$\|x\| = \sup_{t\in\mathbb{R}} |x(t)|.$$

First part

1. For what value(s) of A does the differential equation

$$x'(t) = a(t)x(t) \tag{E1}$$

admit non trivial T-periodic solutions?

We now let $b \in P$, and consider the differential equation

$$x'(t) = a(t)x(t) + b(t). \tag{E2}$$

2.a. Describe the set of maximal solutions to (E2) and the intervals of definition of these solutions.

2.b. Describe the set of maximal solutions to (E2) that are T-periodic, first assuming $A \neq 0$, then $A = 0$.

Second part

In this part, we let H be a real valued C^1 function defined on $\mathbb{R}^2$, and consider the differential equation

$$x'(t) = a(t)x(t) + H(x(t), t). \tag{E3}$$

3. Check that a function x is solution to (E3) if and only if it satisfies the condition

$$x(t) = g(t)\left(x(0) + \int_0^t g(s)^{-1} H(x(s), s)ds\right).$$

4. Suppose that H is T-periodic with respect to its second argument, and that $A \neq 0$. Show that, for all functions $x \in P$, the formula

$$U(_H x)(t) = \frac{e^A}{1-e^A} g(t) \int_t^{t+T} g(s)^{-1} H(x(s), s)ds,$$

defines a function $U_H x \in P$, and that x est solution to (E3) if and only if $U_H x = x$.

In the rest of the problem, we let F be a real-valued C^1 function defined on $\mathbb{R}^2$, T-periodic with respect to its second argument; for all $\varepsilon > 0$, define $H_\varepsilon = \varepsilon F$ and $U_\varepsilon = U_{H_\varepsilon}$, so that the differential equation (E3) is written

$$x'(t) = a(t)x(t) + \varepsilon F(x(t), t). \tag{E4}$$

Assume that $A \neq 0$. For all $r > 0$, we denote B_r the closed ball with centre 0 and radius r in the normed space P. We want to show the following assertion: for all $r > 0$, there exists $\varepsilon_1 > 0$ such that, for all $\varepsilon \leq \varepsilon_1$, the differential equation (E4) has a unique solution $x \in B_r$, that we will denote x_ε.

We denote α_r (resp. β_r) the upper bound of the set $|F(v,s)|$ (resp. $|\frac{\partial F}{\partial v}(v,s)|$), where $v \in [-r, r]$ and $s \in [0, T]$.

5.a. Find a real $\varepsilon_0 > 0$ such that, for all $\varepsilon \leq \varepsilon_0$, $U_\varepsilon(B_r) \subset B_r$.

5.b. Find a real $\varepsilon_1 \leq \varepsilon_0$ such that, for all $\varepsilon \leq \varepsilon_1$, the restriction of U_ε to B_r be a contraction of B_r.

5.c. Conclude.

6. Study the behavior of the function x_ε when $\varepsilon \to 0$, the number r being fixed.

7. We now suppose that the function a is a constant $k \neq 0$ et that the function F takes the form $F(v,s) = f(v)$. Determine the solution x_ε of (E4).

8. We now consider $T = 1$, $k = -1$ and $f(v) = v^2$, and thus (E4) takes the form

$$x'(t) = -x(t) + \varepsilon x(t)^2. \tag{E5}$$

8.a. Give possible values of ε_0 and ε_1.

8.b. Determine the x_ε of (E5).

8.c. Let $\alpha \in \mathbb{R}$. Show that there exists a unique maximal solution φ_α of (E5) such that $\varphi_\alpha(0) = \alpha$. Determine precisely this solution, and graph several of these solutions.

Third part

Here, we consider the differential equation

$$x'(t) = kx(t) + \varepsilon f(x(t)), \tag{E6}$$

where $k < 0$, f is C^1 and zero at zero. We let

$$\lambda = \sup_{u \in [-1,1]} |f'(u)|,$$

and assume that $\varepsilon\lambda < -k$.

We propose to show the following result: if x is a maximal solution of (E6) such that $|x(0)| < 1$, then it is defined on $[0, \infty)$ and, for all $t \geq 0$,

$$|x(t)| \leq |x(0)| e^{(k+\varepsilon\lambda)t}.$$

9. In this question, we suppose that the set of t such that $|x(t)| > 1$ is non-empty, and we denote its lower bound by θ. Show that, for all $t \in [0, \theta]$,

$$|x(t)| \leq |x(0)| e^{(k+\varepsilon\lambda)t}.$$

10. Conclude.

N.B. This result expresses the *stability* and the *asymptotic stability* of the trivial solution of (E6).

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Homework Sheet 1 – Solutions

1. The equation (E1) is a separable equation, so we write

$$\begin{aligned} x'(t) = a(t)x(t) &\Leftrightarrow \frac{x'(t)}{x(t)} = a(t) \\ &\Leftrightarrow \ln|x(t)| = \int_0^t a(s)ds + C \\ &\Leftrightarrow |x(t)| = \exp\left(\int_0^t a(s)ds + C\right) \\ &\Leftrightarrow x(t) = K\exp\left(\int_0^t a(s)ds\right), \end{aligned}$$

where it was assumed that integration starts at $t_0 = 0$, and where the sign of $|x(t)|$ is absorbed into $K \in \mathbb{R}$. Since $x(0) = K$, the general solution to (E1) is thus

$$x(t) = x(0)\exp\left(\int_0^t a(s)ds\right). \tag{B.35}$$

A nontrivial solution (B.35) is T-periodic if it satisfies $x(t+T) = x(t)$ for all $t \geq 0$. In particular (for simplicity), there must hold that $x(T) = x(0)$. This leads to

$$\begin{aligned} x(T) = x(0) &\Leftrightarrow x(0)\exp\left(\int_0^T a(s)ds\right) = x(0) \\ &\Leftrightarrow \exp\left(\int_0^T a(s)ds\right) = 1 \\ &\Leftrightarrow \left(\int_0^T a(s)ds\right) = 0 \\ &\Leftrightarrow A = 0. \end{aligned}$$

2.a. We know that the general solution to the homogeneous equation (E1) associated to (E2) is given by (B.35). To find the general solution to (E2), we need a particular solution to (E2), or to use integrating factors or a variation of constants approach. We do the latter, since we already have the solution (B.35) to (E1). Returning to the solution with undetermined value for K, we consider the ansatz

$$\phi(t) = K(t)\exp\left(\int_0^t a(s)ds\right) = K(t)g(t),$$

where the second equality uses the definition of $g(t)$. We have

$$\begin{aligned}\phi'(t) &= K'(t)g(t) + K(t)g'(t)\\ &= K'(t)g(t) + K(t)a(t)g(t).\end{aligned}$$

The function ϕ is solution to (E2) if and only if it satisfies (E2); therefore, ϕ is solution if and only if

$$\begin{aligned}\phi'(t) = a(t)\phi(t) + b(t) &\Leftrightarrow K'(t)g(t) + K(t)a(t)g(t). = a(t)K(t)g(t) + b(t)\\ &\Leftrightarrow K'(t)g(t) = b(t)\\ &\Leftrightarrow K'(t) = \frac{b(t)}{g(t)}, \text{ for } g(t) \neq 0\\ &\Leftrightarrow K(t) = \int_0^t \frac{b(s)}{g(s)}ds + C.\end{aligned}$$

Note that the remark that $g(t) \neq 0$ is made “for form”: as it is defined, $g(t) > 0$ for all $t \geq 0$. We conclude that the general solution to (E2) is given by

$$x(t) = \left(\int_0^t \frac{b(s)}{g(s)}ds + C\right)\exp\left(\int_0^t a(s)ds\right).$$

Since it will be useful to have information in terms of $x(0)$ (as in question **1.**), we note that $C = x(0)$. Thus, the solution to (E2) through $x(0) = 0$ is given by

$$x(t) = \left(\int_0^t \frac{b(s)}{g(s)}ds + x(0)\right)\exp\left(\int_0^t a(s)ds\right). \tag{B.36}$$

With integrating factors, we would have done as follows: write the equation (E2) as

$$x'(t) - a(t)x(t) = b(t).$$

The integrating factor is then

$$\mu(t) = \exp\left(-\int a(t)dt\right),$$

and the general solution to (E2) is given by

$$x(t) = \frac{1}{\mu(t)}\left(\int_0^t \mu(s)b(s)ds + C\right)$$

Maximal solutions are solutions that are the restriction of no other solution.

2.b. Solutions to (E2) are T-periodic if $x(T) = x(0)$; therefore, a T-periodic solution satisfies

$$\begin{aligned} x(T) = x(0) &\Leftrightarrow \left(\int_0^T \frac{b(s)}{g(s)} ds + x(0) \right) \exp\left(\int_0^T a(s)ds \right) = x(0) \\ &\Leftrightarrow \left(\int_0^T \frac{b(s)}{g(s)} ds + x(0) \right) = x(0)e^{-A} \\ &\Leftrightarrow \int_0^T \frac{b(s)}{g(s)} ds = \left(e^{-A} - 1\right) x(0) \end{aligned}$$

Second part

3. Before proceeding, note that

$$\begin{aligned} g'(t) &= \frac{d}{dt} \exp\left(\int_0^t a(s)ds \right) \\ &= a(t) \exp\left(\int_0^t a(s)ds \right) \\ &= a(t)g(t). \end{aligned}$$

We differentiate $x(t) = g(t) \left(x(0) + \int_0^t g(s)^{-1} H(x(s), s)ds \right)$. This gives

$$\begin{aligned} x'(t) &= g'(t) \left(x(0) + \int_0^t g(s)^{-1} H(x(s), s)ds \right) + g(t) \frac{H(x(t), t)}{g(t)} \\ &= g'(t) \left(x(0) + \int_0^t g(s)^{-1} H(x(s), s)ds \right) + H(x(t), t) \\ &= a(t)g(t) \left(x(0) + \int_0^t g(s)^{-1} H(x(s), s)ds \right) + H(x(t), t) \\ &= a(t)x(t) + H(x(t), t), \end{aligned}$$

and thus $x(t) = g(t) \left(x(0) + \int_0^t g(s)^{-1} H(x(s), s)ds \right)$ is solution to (E3).

4. Let $x \in P$. Then $U_H x \in P$ if and only if $U_H x$ is T-periodic. We have

$$(U_H x)(t + T) = \frac{e^A}{1 - e^A} g(t + T) \int_{t+T}^{t+2T} g(s)^{-1} H(x(s), s)ds.$$

Remark that

$$\begin{aligned}
g(t+T) &= \exp\left(\int_0^{t+T} a(s)ds\right) \\
&= \exp\left(\int_0^{t} a(s)ds + \int_t^{t+T} a(s)ds\right) \\
&= g(t)\exp\left(\int_t^{t+T} a(s)ds\right) \\
&= e^A g(t),
\end{aligned}$$

since $a(t)$ is T-periodic. Therefore,

$$\begin{aligned}
(U_H x)(t+T) &= \frac{e^A}{1-e^A} e^A g(t) \int_{t+T}^{t+2T} g(s)^{-1} H(x(s),s)ds \\
&= \frac{e^A}{1-e^A} e^A g(t) \int_{t}^{t+T} g(s-T)^{-1} H(x(s-T),s-T)ds.
\end{aligned}$$

Now

$$\begin{aligned}
g(s-T) &= \exp\left(\int_0^{s-T} a(u)du\right) \\
&= \exp\left(\int_0^{s} a(u)du + \int_s^{s-T} a(u)du\right) \\
&= g(s)\exp\left(-\int_{s-T}^{s} a(u)du\right) \\
&= e^{-A} g(s).
\end{aligned}$$

So, finally,

$$\begin{aligned}
(U_H x)(t+T) &= \frac{e^A}{1-e^A} e^{2A} g(t) \int_{t}^{t+T} g(s)^{-1} H(x(s),s)ds \\
&= e^{2A}(U_H x)(t),
\end{aligned}$$

since H is T-periodic in its second argument and $x \in P$. Therefore, $U_H x \in P$ for $x \in P$.

Suppose that $x(t) = (U_H x)(t)$. Then,

$$\begin{aligned}
x'(t) &= \frac{e^A}{1-e^A}\left(g'(t)\int_t^{t+T} \frac{H(x(s),s)}{g(s)}ds + g(t)\left(\frac{H(x(t+T),t+T)}{g(t+T)} - \frac{H(x(t),t)}{g(t)}\right)\right) \\
&= \frac{e^A}{1-e^A}\left(a(t)g(t)\int_t^{t+T} \frac{H(x(s),s)}{g(s)}ds + g(t)\left(\frac{H(x(t),t)}{e^A g(t)} - \frac{H(x(t),t)}{g(t)}\right)\right) \\
&= a(t)\frac{e^A}{1-e^A} g(t) g(t)\int_t^{t+T} \frac{H(x(s),s)}{g(s)}ds + \frac{e^A}{1-e^A} g(t)\frac{(1-e^A)H(x(t),t)}{e^A g(t)} \\
&= a(t)x(t) + H(x(t),t).
\end{aligned}$$

5.a. We seek $\varepsilon_0 > 0$ such that for all $\varepsilon \leq \varepsilon_0$, $\|x\| \leq r \Rightarrow \|U_\varepsilon x\| \leq r$. Therefore, we compute $\|U_\varepsilon x\|$. We have, letting $H(x,s) = \varepsilon F(x,s)$,

$$\begin{aligned}
\|U_\varepsilon x\| &= \sup_{t\in\mathbb{R}} |(U_\varepsilon x)(t)| \\
&= \sup_{t\in\mathbb{R}} \left| \frac{e^A}{1-e^A} g(t) \int_t^{t+T} g(s)^{-1} \varepsilon F(x(s),s) ds \right| \\
&= \varepsilon \frac{e^A}{|1-e^A|} \sup_{t\in\mathbb{R}} |g(t)| \left| \int_t^{t+T} \frac{F(x(s),s)}{g(s)} ds \right| \\
&\leq \varepsilon \frac{e^A}{|1-e^A|} \sup_{t\in\mathbb{R}} |g(t)| \int_t^{t+T} \left| \frac{F(x(s),s)}{g(s)} \right| ds.
\end{aligned}$$

Note that we keep the absolute value of $|1-e^A|$, since A could be negative, leading to a negative value for $1-e^A$. Let $\|g^{-1}\| = \sup_{t\in\mathbb{R}} |g^{-1}(t)|$. We then have

$$\begin{aligned}
\|U_\varepsilon x\| &\leq \varepsilon \frac{e^A}{|1-e^A|} \|g\| \|g^{-1}\| \sup_{t\in\mathbb{R}} \int_t^{t+T} |F(x(s),s)| \, ds \\
&\leq \varepsilon \frac{e^A}{|1-e^A|} \|g\| \|g^{-1}\| \int_t^{t+T} \alpha_r ds,
\end{aligned}$$

since $x(s) \in [-r, r]$, and so

$$\|U_\varepsilon x\| \leq \varepsilon \frac{e^A}{|1-e^A|} \|g\| \|g^{-1}\| \alpha_r T.$$

Letting

$$\varepsilon_0 = \left(\frac{e^A}{|1-e^A|} \|g\| \|g^{-1}\| \alpha_r T \right)^{-1} r,$$

we see that if $\varepsilon \leq \varepsilon_0$, then $\|U_\varepsilon x\| \leq r$.

5.b. For the restriction of U_ε to be a contraction, we must have the inequality obtained above, as well as, for $x, y \in B_r$, $d(U_\varepsilon x, U_\varepsilon y) < d(x,y)$. In terms of the induced norm, this means that $\|U_\varepsilon x - U_\varepsilon y\| < \|x - y\|$. Therefore, letting $x, y \in P$ be such that $\|x\| \leq r$ and

$\|y\| \leq r$, we compute

$$\begin{aligned}
\|U_\varepsilon x - U_\varepsilon\| &= \sup_{t\in\mathbb{R}} |(U_\varepsilon x)(t) - (U_\varepsilon y)(t)| \\
&= \sup_{t\in\mathbb{R}} \left| \frac{e^A}{1-e^A} g(t) \int_t^{t+T} \frac{\varepsilon F(x(s),s) - \varepsilon F(y(s),s)}{g(s)} ds \right| \\
&= \varepsilon \left| \frac{e^A}{1-e^A} \right| \sup_{t\in\mathbb{R}} |g(t)| \left| \int_t^{t+T} \frac{F(x(s),s) - F(y(s),s)}{g(s)} ds \right| \\
&\leq \varepsilon \left| \frac{e^A}{1-e^A} \right| \sup_{t\in\mathbb{R}} |g(t)| \int_t^{t+T} \left| \frac{F(x(s),s) - F(y(s),s)}{g(s)} \right| ds.
\end{aligned}$$

For $s \in [0,T]$ and $x(s) \in [-r,r]$, we have, picking a $y(s) \in [-r,r]$,

$$\begin{aligned}
|F(x(s),s)| &= |F(x(s),s) - F(y(s),s) + F(y(s),s)| \\
&\leq |F(x(s),s) - F(y(s),s)| + |F(y(s),s)| \\
&\leq \beta_r |x(s) - y(s)| + \alpha_r,
\end{aligned}$$

from the mean value theorem, and thus

$$|F(x(s),s)| \leq 2\beta_r r + \alpha_r.$$

5.c. We used the contraction mapping principle (Theorem **??**):

- for $\varepsilon \leq \varepsilon_1$, U_ε is a contraction of B_r,
- P is complete (it is closed in $\mathcal{C}(\mathbb{R},\mathbb{R}) \cap \mathcal{B}(\mathbb{R})$ endowed with the supremum norm) and B_r is closed in P.

Therefore, we conclude that for a given $r > 0$, for all $\varepsilon \leq \varepsilon_1$, there exists a unique solution x_ε of (E4) in B_r.

6. This is the contraction mapping theorem with a parameter:

$$\|x_\varepsilon - x_{\varepsilon'}\| = \|U_\varepsilon x_\varepsilon - U_{\varepsilon'} x_{\varepsilon'}\| \leq \underbrace{\|U_\varepsilon x_\varepsilon - U_\varepsilon x_{\varepsilon'}\|}_{\leq K\|x_\varepsilon - x_{\varepsilon'}\|} + \|U_\varepsilon x_{\varepsilon'} - U_{\varepsilon'} x_{\varepsilon'}\|.$$

But we have

$$\begin{aligned}\|U_\varepsilon x_\varepsilon - U_{\varepsilon'}x_{\varepsilon'}\|(t) &\leq |\varepsilon-\varepsilon'|\frac{e^A}{|1-e^A|}\int_t^{t+T}|g(t)g(s)^{-1}F(x(s),s)|ds\\ &\leq |\varepsilon-\varepsilon'|\underbrace{\frac{e^A}{|1-e^A|}e^{TA}T\alpha_r}_{=K'}.\end{aligned}$$

Thus, we have $\|x_\varepsilon - x_{\varepsilon'}\| \leq \dfrac{|\varepsilon-\varepsilon'|K'}{1-K}$ and therefore $\varepsilon\in\mathbb{R}\mapsto x_\varepsilon\in P$ is continuous; it follows that $\lim_{\varepsilon\to 0}x_\varepsilon = x_0$. But the only periodic solution of (E1) when $A\neq 0$ is the zero solution. Therefore, $x_\varepsilon\to 0$ when $\varepsilon\to 0$.

7. Let $x_0(t)=c_0$, then $g(t)=e^{kt}$ and $A=kT$.

$$\begin{aligned}U_\varepsilon x_0(t) &= \frac{\varepsilon e^{kT}}{1-e^{kT}}e^{kt}f(t_0)\int_t^{t+T}e^{-ks}ds = \frac{\varepsilon e^{kT}}{1-e^{kT}}f(c_0)e^{kt}\left[-\frac{1}{k}e^{-ks}\right]_t^{t+T}\\ &= \frac{\varepsilon e^{kT}}{1-e^{kT}}f(c_0)\frac{e^{kt}}{k}\left(e^{-kt}-e^{-kt}e^{-kT}\right) = \frac{\varepsilon e^{kT}}{1-e^{kT}}\frac{f(c_0)}{k}\left(1-e^{-kT}\right) = -\frac{\varepsilon}{k}f(c_0)\end{aligned}$$

The constant function $x_\varepsilon(t)=c_0$ is solution (where c_0 is the unique solution of the equation $\varepsilon f(x)+kx=0$ for ε sufficiently small).

Note : letting $g(x)=-\dfrac{\varepsilon}{k}f(x)$, it follows that $g'(x)=-\dfrac{\varepsilon}{k}f'(x)$. Thus, for $r>0$ given, there exists $\varepsilon_0>0$ such that $\varepsilon\leq\varepsilon_0$ implies $g([-r,r])\subset[-r,r]$, and there exists $\varepsilon_1\leq\varepsilon_0$ such that $\sup_{x\in[-r,r]}|g'(x)|<1$, the fixed point theorem can be applied easily.

8.a. Using the formula obtained in **5.a.** with $\alpha_r=r^2$, $\beta_r=2r$, $g(t)=e^{-t}$, $A=-T$ then

$$\varepsilon_0=\frac{r(1-e^{-T})e^Te^{-T}}{r^2T}=\frac{1-e^{-T}}{rT},\quad \varepsilon_1=\frac{1}{2}\frac{1-e^{-T}}{2rT}=\frac{\varepsilon_0}{4}.$$

8.b. The zero function is clearly a 1-periodic solution of (E5). By uniqueness of solutions, $x_\varepsilon=0$ is the only solution of (E5).

8.c. The vector field $-x+\varepsilon x^2$ is C^1, and therefore existence and uniqueness of a maximal solution φ_α is a direct consequence of the theorem of Cauchy-Lipschitz.

We solve the equation $x' = -x+\varepsilon x^2$ without constraint of periodicity. There are two constant solutions , $x(t)=0$ and $x(t)=1/\varepsilon$. By uniqueness, any other solution never takes the values 0 and $1/\varepsilon$. We have:

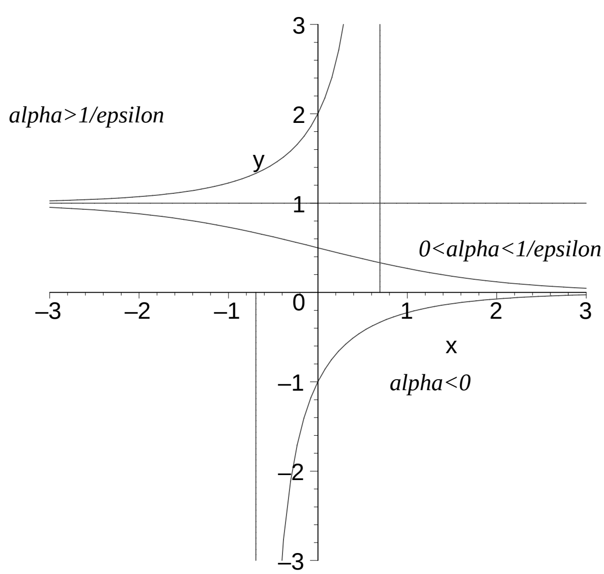
$$\frac{x'}{x(1-\varepsilon x)} = \frac{d}{dt}\left(\ln\left|\frac{x}{1-\varepsilon x}\right|\right) = -1 \quad \Rightarrow \quad \frac{x}{1-\varepsilon x} = \lambda e^{-t} \quad \Rightarrow \quad x(t) = \frac{\lambda e^{-t}}{1+\varepsilon\lambda e^{-t}}, \quad \lambda \in \mathbb{R}^*.$$

The condition $x(0)=\alpha$ gives $\lambda = \dfrac{\alpha}{1-\varepsilon\alpha}$ if $\alpha \neq 0$ and $\alpha \neq 1/\varepsilon$, and as a consequence, letting $\beta = \dfrac{1}{\alpha} - \varepsilon$ we obtain

$$\varphi_\alpha(t) = \frac{1}{\beta e^t + \varepsilon}.$$

- If $\beta \geq 0$ then φ_α is defined on $\mathbb{R}$.
- If $\beta < 0$, we let $t_0 = \ln\left(-\frac{\varepsilon}{\beta}\right)$.
  - If $\alpha > 0$ then $-\dfrac{\varepsilon}{\beta} > 1$, that is, $t_0 > 0$, φ_α is defined on $]-\infty, t_0[$.
  - If $\alpha < 0$ then $-\dfrac{\varepsilon}{\beta} < 1$, that is, $t_0 < 0$, φ_α is defined on $]t_0, +\infty[$.

Here are a few representative solutions obtained when setting $\varepsilon = 1$.



Third part

9. $\theta > 0$ since $|x(0)| < 1$, and we have $x(s) \in [-1,1]$ for all $s \in [0,\theta]$, by definition of θ. Since $f(0) = 0$ and $|f'|$ is bounded by λ on $[-1,1]$, we have, from the inequality of finite variations, $|f(u) - \underbrace{f(0)}_{=0}| \leq \lambda|u|$ for all $u \in [-1,1]$, that is, $|f(x(s)) - \underbrace{f(0)}_{=0}| \leq \lambda|x(s)|$.

Let $t \in [0,\theta]$. From **4.**,

$$|e^{-kt}x(t)| = \left|x(0) + \varepsilon \int_{s=0}^{t} e^{-ks} f(x(s))\, ds\right| \leq |x(0)| + \varepsilon\lambda \int_{s=0}^{t} e^{-ks}|x(s)|\, ds,$$

from which $|e^{-kt}x(t)| \leq |x(0)|e^{\varepsilon\lambda t}$ (using Gronwall's lemma with $\varphi(t) = e^{-kt}|x(t)|$, $\eta = |x(0)|$, $\zeta = \varepsilon\lambda$), giving the inequality.

10. Since $\varepsilon\lambda < 0$, it follows that $|x(0)|e^{(k+\varepsilon\lambda)t} \leq |x(0)| < 1$. Letting $\mathcal{E} = \{t > 0 \mid |x(t)| > 1\}$, which is assumed non empty, then $\theta = \inf \mathcal{E} > 0$ (by continuity, since $|x(0)| < 1$ there exists $\eta > 0$ such that $|x(t)| < 1$ on $[0,\eta]$, and thus $\theta \geq \eta > 0$). Since $\lim_{t\to\theta^-} |x(t)| \leq 1$ and $\lim_{t\to\theta^+} |x(t)| \geq 1$, it follows $|x(\theta)| = 1$.

On $[0,\theta]$, $|x(t)| \leq |x(0)|e^{(k+\varepsilon\lambda)t}$ and, taking the limit, $|x(\theta)| < 1$, which is impossible.

First conclusion : $\mathcal{E} = \emptyset$ and, if J is the interval of definition of x, then $\forall t \in J \cap [0,+\infty[$, $|x(t)| \leq 1$.

If J admits an upper bound $b \in \mathbb{R}$, then x' is bounded in a neighborhood of b. Thus x admits a limit in b. The same is therefore true for x'. We then know that x can be extended beyond b, contradicting the maximality of J.

Final conclusion : $J \cap [0,+\infty[= [0,+\infty[$, x is defined on $[0,+\infty[$ and the proof of question **9.** holds true for all $t \geq 0$, i.e.,

$$\forall t \in [0,+\infty[,\ |x(t)| \leq |x(0)|e^{(k+\varepsilon\lambda)t}.$$

N.B. This result expresses the *stability* and the *asymptotic stability* of the trivial solution of (E6).

This subject was the *Première composition de mathématiques* for the contest determining admission to *École Polytechnique* in France, for MP (Math-Physics) track students, in 2004. Students, in their second year of university, have 4 hours to write this *première composition*.

The original subject comprised another question, originally question 3, which was suppressed in this homework sheet. To be complete, this question is included here:

2'. We suppose here that $T = 2\pi$ and that the function a is a constant k.

2'.a. Assuming $k \neq 0$, express the Fourier coefficients $\hat{x}(n)$, $n \in \mathbb{Z}$, of a solution x of (E2) belonging to P, as a function of k and the Fourier coefficients of b. What is the mode of convergence of the Fourier series of x?

2'.b. What happens when $k = 0$?

2'.a. If $k \neq 0$ then $A = 2\pi k \neq 0$, and from **2.**, there exists a unique 2π-periodic solution. Since the mapping $x \mapsto \widehat{x}(n)$ is linear, and from the relation $\widehat{x'}(n) = in\widehat{x}(n)$, we have

$$\widehat{x'}(n) = k\widehat{x}(n) + \widehat{b}(n) \Rightarrow \widehat{x}(n) = \frac{\widehat{b}(n)}{in - k}.$$

Since x is $\mathcal{C}^1$, we know that the Fourier series of x is normally convergent. Since $\widehat{b}(n) \to 0$, we ca also say that $\widehat{x}(n) = o\left(\frac{1}{n}\right)$.

2'.b. Applying here again the result of **2.b.**,

- If $\widehat{b}(0) = 0$ then all solutions are 2π-periodic. In this case, solutions satisfy $\hat{x}(n) = \hat{b}(n)/in$ for $n \in \mathbb{Z}$ non zero and $\hat{x}(0)$ varies with the solutions under consideration.

- If $\widehat{b}(0) \neq 0$ then no solution is periodic.