Three different outcomes to the same problem in population dynamics

Consider a population of rabbits and suppose that the rate at which new rabbits are born depends on the current number of rabbits. Also, suppose that rabbits eat only carrots, for which they compete with each other: if there are too many rabbits, some rabbits starve to death; this is known as *intraspecific competition*, because it takes place within a species (*interspecific* competition occurs when rabbits compete for carrots with, say, the Green Giant).

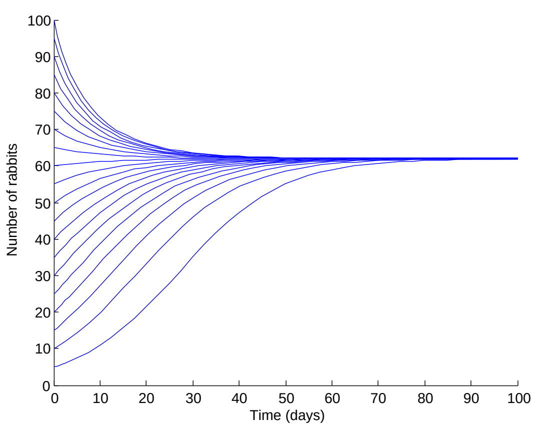
Let t be a real number representing days and $N(t)$ be the number of rabbits on day t (t can take any positive value). The *logistic* equation,

$$N'(t) = rN(t)\left(1 - \frac{N(t)}{K}\right),$$

describes the evolution of the rabbit population. Remark that it involves a *derivative*, $N'(t)$; it is called an *ordinary differential equation* (ODE), and the unknown, $N(t)$, is a function rather than a number.

Solving ODEs requires advanced mathematics, but the logistic equation is not too difficult (although well beyond the scope of this presentation). The constant r is the *intrinsic growth rate*, that is, how many new rabbits would be born every day if there were an unlimited supply of carrots, and K is the *carrying capacity* of the environment, the number of rabbits that can survive with the limited number of carrots that is effectively there.

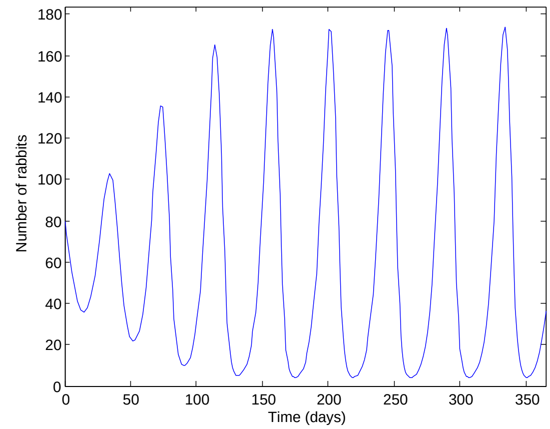
In the following figure, several solutions to the ODE logistic are represented, corresponding to different initial numbers of rabbits; the horizontal axis shows time and the vertical axis shows the number of rabbits, so following a curve from left to right indicates the evolution of the population through time. All the solutions tend to the same value, which turns out to be K, the carrying capacity of the environment.



Now suppose that it takes τ days between the instant rabbits compete for a carrot and the outcome of this competition, that is, the death of a rabbit. In this case, we use a *delay differential equation* (DDE),

$$N'(t) = rN(t)\left(1 - \frac{N(t-\tau)}{K}\right).$$

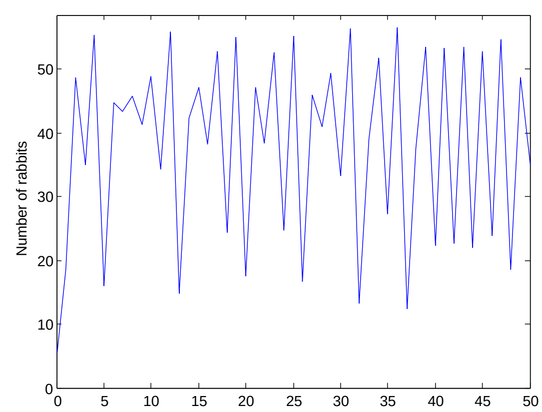
The constant τ is called the *time delay*. DDEs are much more complicated than ODEs, and have been studied only since the 1950s. In the next figure, a solution to the DDE logistic is represented. The big difference with the ODE logistic is that the population of rabbits oscillates.



The last version of the logistic equation uses *discrete* time, where time is an integer quantity instead of varying continuously as in the previous two equations. The solution is a *sequence* where the number of rabbits on day $t+1$ is given as a function of the number of rabbits on day t by the following equation,

$$N(t+1) = rN(t)\left(1 - \frac{N(t)}{K}\right).$$

Because the solution is not continuous, this is also more difficult than the ODE logistic. The next figure shows a solution to this equation. Such a solution is *chaotic*: it oscillates, but in a very irregular way, and two solutions that start very close to each other will be very different after a small time. The discrete logistic equation was among the first to be discovered that show this type of behavior.



In conclusion, three different *modeling paradigms* produce three very different types of behaviors, even though the phenomenon being described is the same. This is a situation that mathematical modelers often encounter, and good modeling requires an understanding of both the phenomenon being modeled, and of the tools used.