MATH 3610 – 04
A few epidemic models
Analysis of nonlinear systems of ODEs

Julien Arino
University of Manitoba
julien.arino@umanitoba.ca

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The University of Manitoba campuses are located on original lands of Anishinaabeg, Ininew, Anisininew, Dakota and Dene peoples, and on the National Homeland of the Red River Métis.
We respect the Treaties that were made on these territories, we acknowledge the harms and mistakes of the past, and we dedicate ourselves to move forward in partnership with Indigenous communities in a spirit of Reconciliation and collaboration.

Outline

SIS model without vital dynamics

SIR model of Kermack and McKendrick

SIRS model with demography

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A SIS model

Consider a disease that confers no immunity. In this case, individuals are either

- *susceptible* to the disease, with the number of such individuals at time t denoted by $S(t)$,
- or *infected* by the disease (and are also *infective* in the sense that they propagate the disease), with the number of such individuals at time t denoted by $I(t)$.

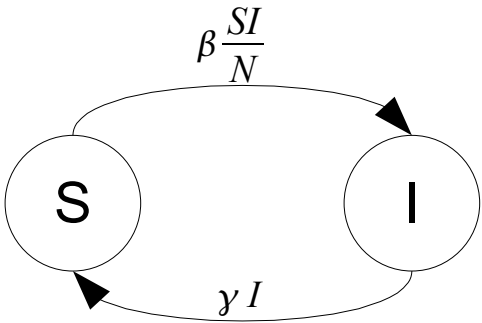
We want to model the evolution with time of S and I (t is omitted unless necessary).

Hypotheses

- Individuals recover from the disease at the *per capita* rate γ.
- The disease does not confer immunity.
- There is no birth or death.
- Infection is of *standard incidence* type, $\beta = SI/N$.

(for details, see slides on *residence time*)

Flow diagram of the model



The evolution of $I(t)$ is described by the following equation (see slides on *residence time*):

$$I' = \beta \frac{(N - I)I}{N} - \gamma I.$$

Develop and reorder the terms, giving

$$I' = (\beta - \gamma)I - \frac{\beta}{N} I^2 \tag{1}$$

This is a logistic-type equation. It can be solved as a Bernoulli equation or as a separable equation, giving, for an initial number of infectives $I(0) = I_0$,

$$I(t) = \frac{(\beta - \gamma) N I_0}{(\beta - \gamma) N e^{-(\beta - \gamma)t} + \beta I_0 \left(1 - e^{-(\beta - \gamma)t}\right)}$$

From $S = N - I$, we deduce that the solution $(S(t), I(t))$ for the complete system, with initial condition $S(0) + I(0) = S_0 + I_0 = N$ is, for $t \geq 0$,

$$S(t) = N - \frac{(\beta - \gamma) N I_0}{(\beta - \gamma) N e^{-(\beta-\gamma)t} + \beta I_0 \left(1 - e^{-(\beta-\gamma)t}\right)}$$

and

$$I(t) = \frac{(\beta - \gamma) N I_0}{(\beta - \gamma) N e^{-(\beta-\gamma)t} + \beta I_0 \left(1 - e^{-(\beta-\gamma)t}\right)}$$

Behavior of the solutions

Consider only I for the moment.

$$I(t) = \frac{(\beta - \gamma) N I_0}{(\beta - \gamma) N e^{-(\beta-\gamma)t} + \beta I_0 \left(1 - e^{-(\beta-\gamma)t}\right)}$$

So

- If $\beta - \gamma > 0$, then $e^{-(\beta-\gamma)t} \to 0$ as $t \to \infty$, and therefore

$$\lim_{t\to\infty} I(t) = \frac{(\beta - \gamma) N I_0}{\beta I_0} = \frac{\beta - \gamma}{\beta} N = \left(1 - \frac{\gamma}{\beta}\right) N.$$

- If $\beta - \gamma < 0$, then $e^{-(\beta-\gamma)t} \to \infty$ at $t \to \infty$. This implies that the denominator in $I(t)$ tends to $-\infty$ as $t \to \infty$, and so

$$\lim_{t\to\infty} I(t) = 0, \text{ with } I(t) > 0 \text{ for all } t.$$

- If $\beta = \gamma$, then $I(t) = 0$ for all t.

The basic reproduction number

Define the *basic reproduction number* (the average number of people that an infectious individual will infect, when introduced in a population of susceptibles) as

$$\mathcal{R}_0 = \frac{\beta}{\gamma}$$

We have

$$(\mathcal{R}_0 < 1 \Leftrightarrow (\beta - \gamma) < 0) \text{ and } (\mathcal{R}_0 > 1 \Leftrightarrow (\beta - \gamma) > 0).$$

Therefore, previous cases can be rewritten

- If $\mathcal{R}_0 < 1$, then $\lim_{t\to\infty} I(t) = 0$.
- If $\mathcal{R}_0 > 1$, then

$$lim_{t\to\infty} I(t) = \left(1 - \frac{1}{\mathcal{R}_0}\right) N.$$

(the case $\mathcal{R}_0 = 1$ is usually omitted)

Plotting this in Maple

```
> f:=R->piecewise(R<1,0,R>1,(1-1/R)*1000);
> plot(f(R),R=0..10);
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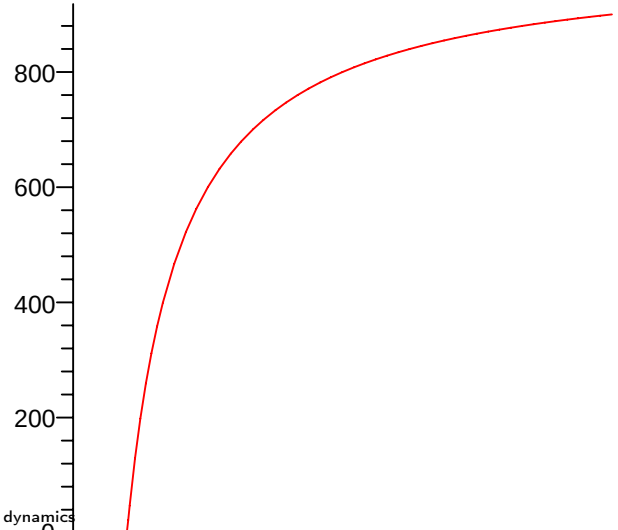

SIS model without vital dynamics

SIR model of Kermack and McKendrick

SIRS model with demography

A Contribution to the Mathematical Theory of Epidemics.

By W. O. KERMACK and A. G. MCKENDRICK.

(Communicated by Sir Gilbert Walker, F.R.S.—Received May 13, 1927.)

(From the Laboratory of the Royal College of Physicians, Edinburgh.)

Introduction.

(1) One of the most striking features in the study of epidemics is the difficulty of finding a causal factor which appears to be adequate to account for the magnitude of the frequent epidemics of disease which visit almost every population. It was with a view to obtaining more insight regarding the effects of the various factors which govern the spread of contagious epidemics that the present investigation was undertaken. Reference may here be made to the work of Ross and Hudson (1915–17) in which the same problem is attacked. The problem is here carried to a further stage, and it is considered from a point of view which is in one sense more general. The problem may be summarised as follows: One (or more) infected person is introduced into a community of individuals, more or less susceptible to the disease in question. The disease spreads from

In this case the equations are

$$\left.\begin{aligned} \frac{dx}{dt} &= -\kappa xy \\ \frac{dy}{dt} &= \kappa xy - ly \\ \frac{dz}{dt} &= ly \end{aligned}\right\}$$

and as before $x + y + z = \mathrm{N}$.

Kermack and McKendrick

In 1927, Kermack and McKendrick started publishing a series of papers on epidemic models. In the first of their papers, they have this model as a particular case:

$$\begin{aligned} S' &= -\beta SI \\ I' &= \beta SI - \gamma I \\ R' &= \gamma I \end{aligned} \tag{2}$$

In this case the equations are

$$\left.\begin{aligned} \frac{dx}{dt} &= -\kappa xy \\ \frac{dy}{dt} &= \kappa xy - ly \\ \frac{dz}{dt} &= ly \end{aligned}\right\} \tag{29}$$

and as before $x + y + z = \mathrm{N}$.

Thus

$$\frac{dz}{dt} = l(\mathrm{N} - x - z),$$

and $\frac{dx}{dz} = -\frac{\kappa}{l}x$, whence $\log \frac{x_0}{x} = \frac{\kappa}{l}z$, since we assume that z_0 is zero.

Thus

$$\frac{dz}{dt} = l\left(\mathrm{N} - x_0 e^{-\frac{\kappa}{l}z} - z\right).$$

Analyzing the system

First, note (as KMK) that the total population in the system is constant. This is deduced from the fact that

$$N' = (S + I + R)' = -\beta SI + \beta SI - \gamma I + \gamma I = 0.$$

Since this is true for all values of t, we have N constant.

Let us ignore the R equation for now. We can compute

$$\frac{dI}{dS} = \frac{dI}{dt}\frac{dt}{dS} = \frac{I'}{S'} = \frac{\gamma}{\beta S} - 1$$

This gives

$$I(S) = S - \frac{\gamma}{\beta}\ln S + K,$$

which, considering the initial condition (S_0, I_0), is,

$$I(S) = S - \frac{\gamma}{\beta}\ln S + I_0 - (S_0 - \frac{\gamma}{\beta}\ln S_0).$$

This gives a curve in the (S, I) plane.

$$I(S) = S - \frac{\gamma}{\beta} \ln S + I_0 - (S_0 - \frac{\gamma}{\beta} \ln S_0).$$

Typically, assume $S \approx N$ and $I > 0$ small. Let us denote $S_\infty = \lim_{t \to \infty} S(t)$. We want to find the value of S when $I \to 0$. Then

$$I_0 - \frac{\gamma}{\beta} \ln S_0 = S_\infty - \frac{\gamma}{\beta} \ln S_\infty$$

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SIRS model with demography

The SIRS model – Assumptions (1/2)

- Like KMK, individuals are S, I or R.
- Infection is βSI (mass action) or $\beta SI/N$ (proportional incidence).
- Different interpretation of the R class: R stands for "recovered", individuals who are immune to the disease following recovery.
- Recovery from the disease (movement from I class to R class) occurs at the per capita rate γ.
  (Time spent in I before recovery is exponentially distributed.)
- Immunity can be lost: after some time, R individuals revert back to S individuals.
- Time spent in R class before loss of immunity is exponentially distributed, with mean $1/\nu$.

The SIRS model – Assumptions (2/2)

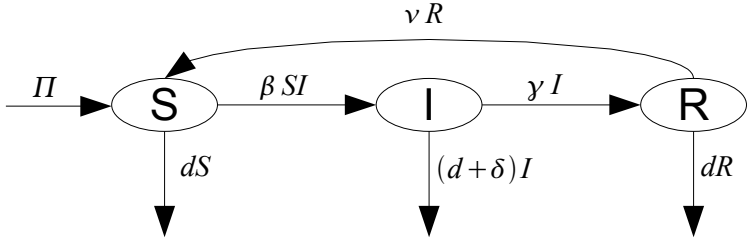
- There is birth and death of individuals:
  - No vertical transmission of the disease (mother to child) or of immunity, so all birth is into the S class.
    Birth occurs at the rate Π.
  - Individuals in all classes die of at the per capita rate d, i.e., the average life duration is exponentially distributed with mean $1/d$.
  - The disease is lethal: infected individuals are subject to additional mortality at the per capita rate δ.

Note that birth and death can have different interpretations:

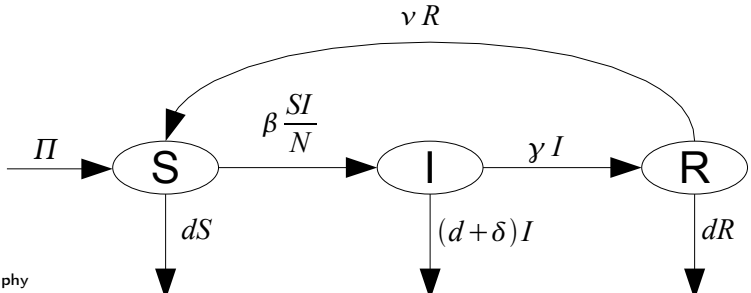
- birth and death in the classical sense,
- but also, entering the susceptible population and leaving it.

Flow diagrams for the models

Mass action



Standard incidence



SIRS models

Mass action

$$S' = \Pi + \nu R - \beta SI - dS \tag{3a}$$
$$I' = \beta SI - (d + \delta + \gamma)I \tag{3b}$$
$$R' = \gamma I - (d + \nu)R \tag{3c}$$

Proportional incidence

$$S' = \Pi + \nu R - \beta SI - dS \tag{4a}$$
$$I' = \beta SI - (d + \delta + \gamma)I \tag{4b}$$
$$R' = \gamma I - (d + \nu)R, \tag{4c}$$

where $N = S + I + R$.

SIRS model with mass action incidence

Consider (3):

$$
\begin{aligned}
S' &= \Pi + \nu R - \beta SI - dS \\
I' &= \beta SI - (d + \delta + \gamma) I \\
R' &= \gamma I - (d + \nu) R
\end{aligned}
$$

Steps of the analysis

1. Assess well-posedness of the system:
   1.1 Determine whether solutions exist and are unique.
   1.2 Determine whether solutions remain in a realistic region and are bounded.
2. Find the equilibria of the system.
3. Determine the local stability properties of the equilibria.
4. Determine the global stability properties of the equilibria (**much harder**, often not possible).

Existence and uniqueness of solutions

Theorem 1 (Cauchy-Lipschitz)

Consider the equation $x' = f(x)$, with $x \in \mathbb{R}^n$, and suppose that $f \in C^1$. Then there exists a unique solution of $x' = f(x)$ such that $x(t_0) = x_0$, where $t_0 \in \mathbb{R}$ and $x_0 \in \mathbb{R}^n$, defined on the largest interval $J \ni t_0$ on which $f \in C^1$.

Equilibria

Definition 2 (Equilibrium point)

Consider a differential equation

$$x' = f(x), \tag{5}$$

with $x \in \mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}^n$. Then x^* is an equilibrium (solution) of (5) if $f(x^*) = 0$.

Linearization

Consider x^* an equilibrium of (5). For simplicity, assume here that $x^* = 0$ (it is always possible to do this, by considering $y = x - x^*$).

Taylor's theorem:

$$f(x) = Df(0)x + \frac{1}{2}D^2f(0)(x,x) + \cdots,$$

where $Df(0)$ is the Jacobian matrix of f evaluated at 0.

Stability of equilibria

Definition 3 (Stable and unstable EP)

Let ϕ_t be the flow of (5), assumed to be defined for all $t \in \mathbb{R}$. An equilibrium x^* of (5) is (locally) *stable* if for all $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x \in \mathcal{N}_\delta(x^*)$ and $t \geq 0$, there holds

$$\phi_t(x) \in \mathcal{N}_\varepsilon(x^*).$$

The equilibrium point is *unstable* if it is not stable.

Definition 4 (Asymptotically stable EP)

Let ϕ_t be the flow of (5) is (locally) *asymptotically stable* if there exists $\delta > 0$ such that for all $x \in \mathcal{N}_\delta(x^*)$ and $t \geq 0$, there holds

$$\lim_{t \to \infty} \phi_t(x) = x^*.$$

Clearly, Asymtotically Stable $\Rightarrow$ Stable.

Hyperbolic EPs, sinks, sources

Definition 5 (Sink)

An equilibrium point x^* of (5) is *hyperbolic* if none of the eigenvalues of the matrix $Df(x^*)$ (Jacobian matrix of f evaluated at x^*) have zero real parts.

Definition 6 (Sink)

An equilibrium point x^* of (5) is a *sink* if all the eigenvalues of the matrix $Df(x^*)$ have negative real parts.

Definition 7 (Source)

An equilibrium point x^* of (5) is a *source* if all the eigenvalues of the matrix $Df(x^*)$ have positive real parts.

Theorem 8

If x^* is a sink of (5) and for all the eigenvalues λ_j of the matrix $Df(x^*)$

$$Re\,(\lambda_j) < -\alpha < 0,$$

where $Re\,(\lambda)$ denotes the real part of λ, then for a given $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x \in \mathcal{N}_\delta(x^*)$, the flow $\phi_t(x)$ of (5) satisfies

$$\|\phi_t(x) - x^*\| \leq \varepsilon e^{-\alpha t}$$

for all $t \geq 0$.

Theorem 9

If x^* is a stable equilibrium point of (5), no eigenvalue of $Df(x^*)$ has positive real part.