Shallow water

Partial differential equations

Julien Arino
University of Manitoba
julien.arino@umanitoba.ca

Fall 2024

The University of Manitoba campuses are located on original lands of Anishinaabeg, Ininew, Anisininew, Dakota and Dene peoples, and on the National Homeland of the Red River Métis.
We respect the Treaties that were made on these territories, we acknowledge the harms and mistakes of the past, and we dedicate ourselves to move forward in partnership with Indigenous communities in a spirit of Reconciliation and collaboration.

Outline

Model formulation

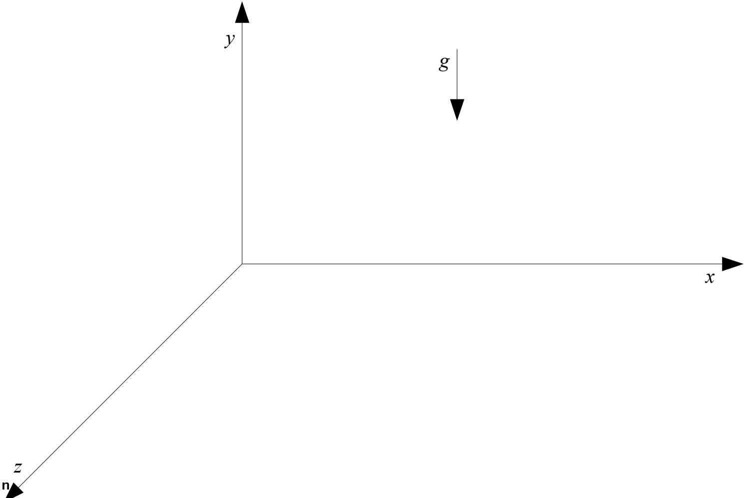
Case of smooth solutions

Linearization

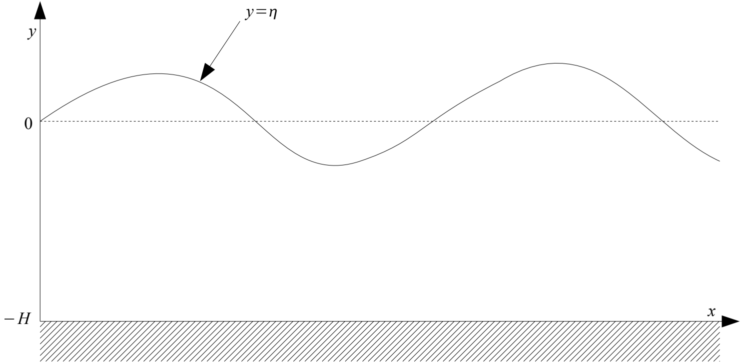
Traveling wave solutions

Spatial domain

We consider the motion of a body of water that is infinite in the z direction, with or without boundary in the x direction, and the vertical direction of gravity taken as the y direction.



- Water depth at rest, H, small compared to distance L_0 over which significant changes can occur in the x direction.
- Undisturbed water surface, $y = 0$.
- Moving upper free surface $y = \eta$, measured from $y = 0$.
- Sea floor $y = -H$.

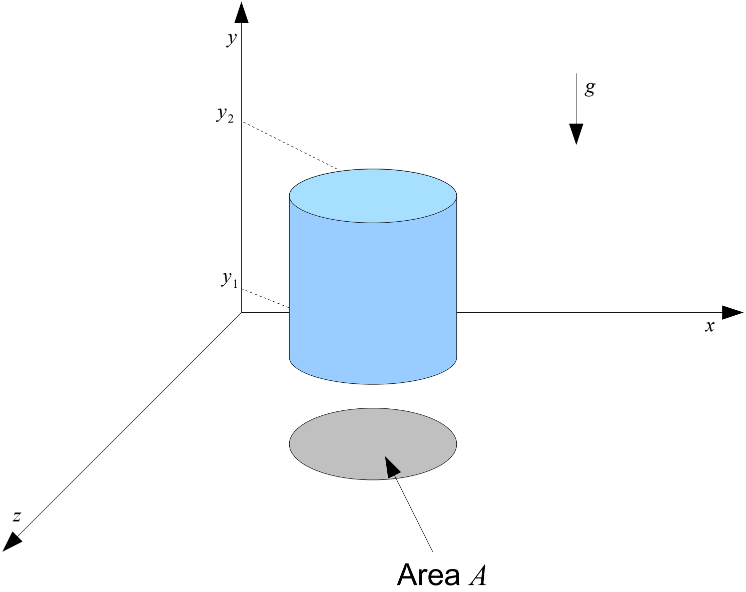


- u velocity in the x direction. Assume independent of depth y.
- ρ mass density of water.
- $p(x, y, t)$ pressure in fluid at point (x, y) at time t. In water, magnitude at any (x, y) is same in all directions.

Fluid motion independent of z, so

- $u = u(x, t)$
- $\eta = \eta(x, t)$.

Take a cylindrical water column, with base area A, between y_1 and $y_2 > y_1$.



Weight of water column:

$$\iint_A \int_{y_1}^{y_2} (-\rho g)\ dydxdz$$

Pressure differential:

$$\iint_A (p(x, y_2, t) - p(x, y_1, t))\ dxdz$$

So we must have

$$\iint_A \int_{y_1}^{y_2} (-\rho g)\ dydxdz = \iint_A (p(x, y_2, t) - p(x, y_1, t))\ dxdz$$

$$\iint_A \int_{y_1}^{y_2} (-\rho g)\ dydxdz = \iint_A \left(p(x, y_2, t) - p(x, y_1, t)\right)\ dxdz$$

is equivalent to

$$\iint_A \int_{y_1}^{y_2} \left(\frac{\partial p}{\partial y} + \rho g\right)\ dydxdz = 0$$

This must be true for any water column, i.e., any A, y_1, y_2. Therefore,

$$\frac{\partial p}{\partial y} + \rho g = 0$$

(otherwise, we would be able to find a water column where the integrand is positive, leading to a positive value of the integral on that column).

Water is incompressible

If you force a body of water to deform, the volume of that body of water remains constant, i.e., water is an *incompressible fluid*.

$\Rightarrow \rho$, the density, is a constant, and from

$$\frac{\partial p}{\partial y} + \rho g = 0$$

we get

$$p = -\rho g y + C,$$

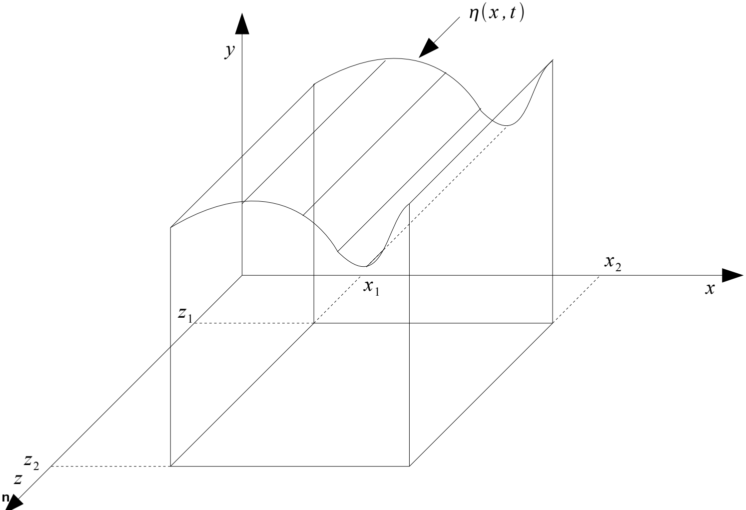
so if p is measured relative to the pressure above the free upper surface $y = \eta$,

$$p = \rho g(\eta - y)$$

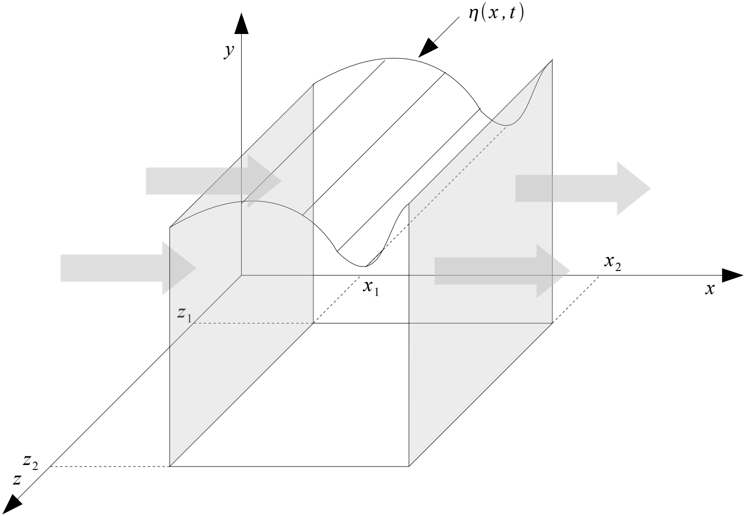
Water accumulation

Consider a fixed volume V,

$$V = \{z_1 \leq z \leq z_2, x_1 \leq x \leq x_2, -H \leq y \leq \eta\}$$



Water enters V through x_1 face and leaves V through x_2 face.

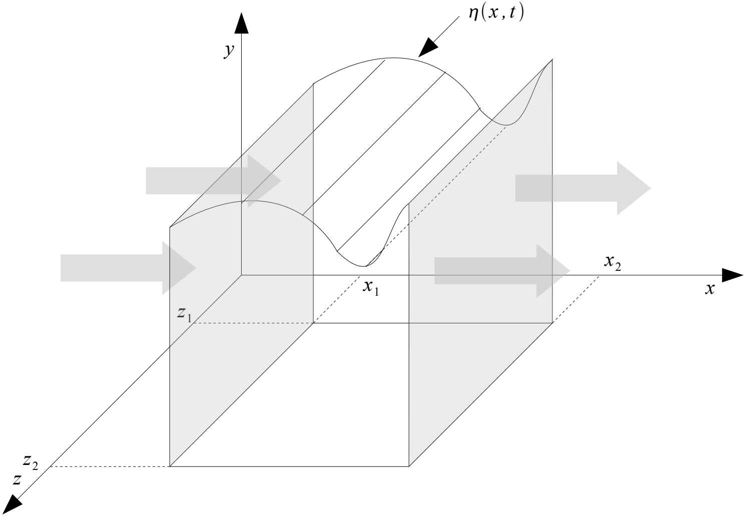


Rate of water accumulation in V is

$$\frac{d}{dt}\int^{z_2}\int^{x_2}\int^{\eta}\rho\ dydxdz = \Delta z\frac{d}{dt}\int^{x_2}\rho h\ dx,$$

Water flux

Net flux of water entering V through its faces $x = x_1$ and $x = x_2$ is



$$\left[\int^{z_2}\int^{\eta} u\, dydz\right] - \left[\int^{z_2}\int^{\eta} u\, dydz\right] = -\Delta z\, [\rho u h]^{x_2}$$

Conservation of mass

Of course, the mass must conserve in V, so the two expressions must be equal, i.e.,

$$\frac{d}{dt}\int_{x_1}^{x_2} \rho h \; dx + [\rho u h]_{x_1}^{x_2} = 0$$

Newton's second law for deformable media (Euler): rate of increase of horizontal momentum (in the x direction) in V must equal the sum of the net influx of momentum into the volume and the net horizontal force acting on the column.

(Momentum: product of mass and velocity of an object).

Rate of increase of momentum

$$\frac{d}{dt}\int_{z_1}^{z_2}\int_{x_1}^{x_2}\int_{-H}^{\eta} \rho u \; dydxdz = \Delta z \frac{d}{dt}\int_{x_1}^{x_2} \rho u h dx$$

Momentum flux

Net influx of momentum through faces $x = x_1$ and $x = x_2$ is

$$\left[\int_{z_1}^{z_2}\int_{-H}^{\eta} (\rho u)u \; dydz\right]_{x=x_1} - \left[\int_{z_1}^{z_2}\int_{-H}^{\eta} (\rho u)u \; dydz\right]_{x=x_2}$$

$$= -\Delta z \left[\rho u^2 h\right]_{x_1}^{x_2}$$

There is no flux through $y = -H$ and $y = \eta$, and no net flux through $z = z_1$ and $z = z_2$.

Forces acting on V

Ignore friction at $y = -H$. Then only contributions to horizontal forces come from pressure at $x = x_1$ and $x = x_2$, so net horizontal forces acting on V is

$$\left[\int_{z_1}^{z_2}\int_{-H}^{\eta} p \; dydz\right]_{x_1}^{x_2} = -\left[\Delta z \int_{-H}^{\eta} \rho g(\eta - y)\; dy\right]_{x_1}^{x_2}$$

$$= \left[-\Delta z \rho g(\eta y - \frac{1}{2}y^2)\Big|_{-H}^{\eta}\right]_{x_1}^{x_2}$$

$$= \left[-\frac{1}{2}\Delta z \rho g h^2\right]_{x_1}^{x_2}$$

Conclusion from Newton's second law

$$\frac{d}{dt}\int_{x_1}^{x_2} \rho u h \, dx + \left[\rho u^2 h + \frac{1}{2}\rho g h^2\right]_{x_1}^{x_2} = 0$$

The general model

Pressure magnitude:

$$p = \rho g(\eta - y) \tag{1}$$

Horizontal velocity:

$$\frac{d}{dt}\int_{x_1}^{x_2} \rho h \; dx + [\rho u h]_{x_1}^{x_2} = 0 \tag{2}$$

Free surface height:

$$\frac{d}{dt}\int_{x_1}^{x_2} \rho u h \; dx + \left[\rho u^2 h + \frac{1}{2}\rho g h^2\right]_{x_1}^{x_2} = 0 \tag{3}$$

Model formulation

Case of smooth solutions

Linearization

Traveling wave solutions

Suppose u and h are smooth (with continuous first order partial derivatives), then (2) and (3) take a much simpler form,

$$\int_{x_1}^{x_2} \left(\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(uh) \right) \; dx = 0$$

and

$$\int_{x_1}^{x_2} \left(\frac{\partial}{\partial t}(uh) + \frac{\partial}{\partial x}(u^2 h + \frac{1}{2} g h^2) \right) \; dx = 0$$

Since the intervals of integration $[x_1, x_2]$ are arbitrary, and that the integrands are continuous, we have

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(uh) = 0$$

and

$$\frac{\partial}{\partial t}(uh) + \frac{\partial}{\partial x}(u^2 h + \frac{1}{2} g h^2) = 0$$

We write

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(uh) = 0$$

and

$$\frac{\partial}{\partial t}(uh) + \frac{\partial}{\partial x}(u^2 h + \frac{1}{2}gh^2) = 0$$

as

$$h_t + (uh)_x = 0 \tag{4}$$

and

$$(uh)_t + (u^2 h + \frac{1}{2}gh^2)_x = 0 \tag{5}$$

From (4),

$$h_t = -(uh)_x = -(u_x h + u h_x)$$

Equation (5) can be rewritten as

$$\begin{aligned}
(5) &\Leftrightarrow u_t h + u h_t + (u^2 h + \frac{1}{2} g h^2)_x = 0 \\
&\Leftrightarrow u_t h - u(u_x h + u h_x) + 2 u u_x h + u^2 h_x + g h h_x = 0 \\
&\Leftrightarrow u_t h - u u_x h - \cancel{u^2 h_x} + 2 u u_x h + \cancel{u^2 h_x} + g h h_x = 0 \\
&\Leftrightarrow u_t h + u u_x h + g h h_x = 0
\end{aligned}$$

Therefore, provided $h \neq 0$, we get

$$h_t + (uh)_x = 0 \tag{6a}$$

$$u_t + u u_x + g h_x = 0 \tag{6b}$$

which describes the evolution of u and h.

The model for smooth solutions

$$h_t + (uh)_x = 0 \tag{6a}$$

$$u_t + uu_x + gh_x = 0 \tag{6b}$$

If $-\infty < x < \infty$, then all we need is an initial condition, i.e., functions describing the initial state of u and h:

$$u(x,0) = u_0(x), \quad h(x,0) = h_0(x), \qquad -\infty < x < \infty.$$

If x has a boundary, then we need boundary conditions.

Suppose the bottom is flat (H is constant), and that the deviation from the undisturbed depth H is small compared to H itself, then

$$h = (H + \zeta) = H(1 + \frac{\zeta}{H}) \simeq H, \qquad h_t = \zeta_t, \qquad h_x = \zeta_x.$$

If $|u|$ is also small, then uu_x can be neglected. Then we can linearize

$$h_t + (uh)_x = 0 \tag{6a}$$
$$u_t + uu_x + gh_x = 0, \tag{6b}$$

getting

$$\zeta_t + Hu_x = 0 \tag{7a}$$
$$u_t + g\zeta_x = 0 \tag{7b}$$

Differentiate (7b) with respect to x:

$$u_{tx} + g\zeta_{xx} = 0$$

and therefore,

$$u_{tx} = -g\zeta_{xx} \tag{8}$$

Differentiate (7a) with respect to t:

$$\zeta_{tt} + Hu_{xt} = 0 \tag{9}$$

If u has continuous second-order partial derivatives, then from Clairaut's theorem, $u_{tx} = u_{xt}$. Therefore, substituting (8) into (9),

$$\zeta_{tt} - HG\zeta_{xx} = 0$$

that is

$$\zeta_{tt} = c^2\zeta_{xx}, \qquad c^2 = Hg$$

The one-dimensional wave equation (1)

The partial differential equation

$$\zeta_{tt} = c^2 \zeta_{xx} \tag{10}$$

with $c^2 = Hg$, is the one-dimensional wave equation. Initial conditions are given by

$$\zeta(x,0) = h_0(x) - H \equiv \zeta_0(x)$$
$$\zeta_t(x,0) = -Hu_x(x,0) = -H[u_0(x)]_x \equiv \nu_0(x)$$

The one-dimensional wave equation (2)

Things can also be expressed in terms of u. Using the same type of simplification used before for ζ, we get

$$u_{tt} = c^2 u_{xx} \tag{11}$$

with $c^2 = Hg$. Initial conditions are given by

$$\begin{aligned} u(x,0) &= u_0(x) \\ u_t(x,0) &= -g\zeta_x(x,0) = -g[h_0(x)]_x \equiv v_0(x) \end{aligned}$$

Traveling wave solutions

This was obtained by d'Alembert. Consider

$$u_{tt} = c^2 u_{xx} \tag{11}$$

Note that this can be written as

$$\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)u = 0$$

This implies that for any F, G, the sum

$$u(x,t) = F(x-ct) + G(x+ct)$$

satisfies (11).

Derivation of the solution

Introduce the new variables

$$a = x - ct \qquad \text{and} \qquad b = x + ct$$

We have

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial a} + \frac{\partial u}{\partial b} \qquad \frac{\partial u}{\partial t} = -c\frac{\partial u}{\partial a} + c\frac{\partial u}{\partial b}$$

$$\frac{\partial^2}{\partial x^2}u = \left(\frac{\partial}{\partial a} + \frac{\partial}{\partial b}\right)^2 u = \frac{\partial^2 u}{\partial a^2} + 2\frac{\partial^2 u}{\partial a \partial b} + \frac{\partial^2 u}{\partial b^2}$$

$$\frac{\partial^2}{\partial t^2}u = \left(-c\frac{\partial}{\partial a} + c\frac{\partial}{\partial b}\right)^2 u = c^2\left(\frac{\partial^2 u}{\partial a^2} - 2\frac{\partial^2 u}{\partial a \partial b} + \frac{\partial^2 u}{\partial b^2}\right)$$

So the equation

$$u_{tt} = c^2 u_{xx} \tag{11}$$

is written

$$4\frac{\partial^2 u}{\partial a \partial b} = 0$$

Integrate with respect to b:

$$\frac{\partial u}{\partial a} = \xi(a)$$

and thus

$$\begin{aligned} u(x,t) = u(a,b) &= \int \xi(a) da + G(b) \\ &= F(a) + G(b) \\ &= F(x - ct) + G(x + ct) \end{aligned}$$

Set

$$u(x,0) = f(x) \qquad u_t(x,0) = g(x)$$

Then d'Alembert's formula gives

$$u(x,t) = \frac{f(x-ct)+f(x+ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)ds$$

Case of a Dirac delta initial condition

Suppose $u_0(x) = 0$ and $v_0(x) = \delta(x)$, for $-\infty < x < \infty$, with δ the Dirac delta,

$$\delta(x) = \begin{cases} \infty & \text{if } x = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Therefore,

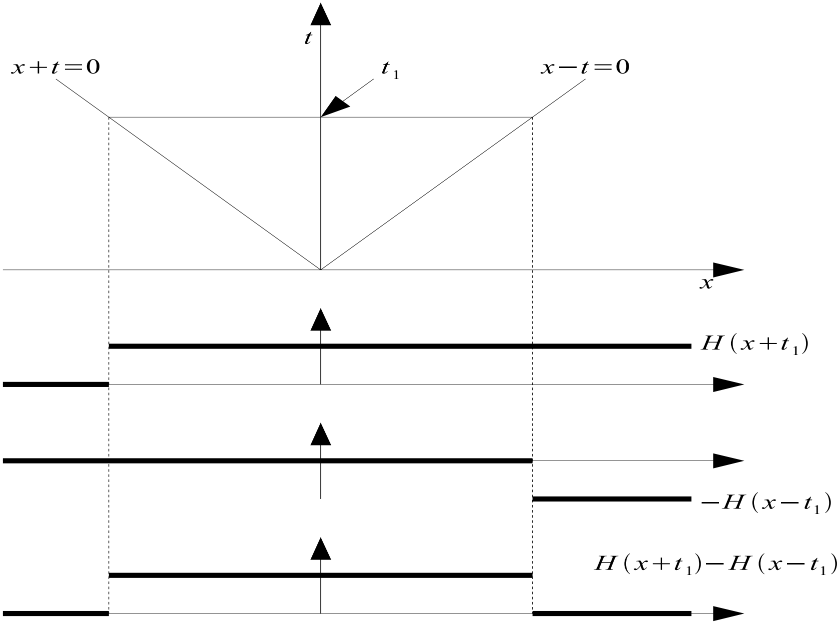
$$u(x,t) = \frac{1}{2c} \int_{x-ct}^{x+ct} \delta(z) dz = \frac{1}{2c} \left\{ H(x+ct) - H(x-ct) \right\},$$

with H the Heaviside function,

$$H(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x > 0. \end{cases}$$

For simplicity, take $c = 1$. This gives

$$u(x,t) = \frac{1}{2}\left\{H(x+t) - H(x-t)\right\},$$


t
x+t=0
t1
x−t=0
x
H(x+t1)
−H(x−t1)
H(x+t1)−H(x−t1)

As t increases, we move further up in the top graph in (x,t)-space, resulting in a wider and wider square pulse.