MATH 4370/7370 – Linear Algebra and Matrix Analysis

Eigenvalues, eigenvectors, similarity and Geršgorin disks

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Outline

Definition 3.1

Let $A \in \mathcal{M}_n(\mathbb{F})$. If $\lambda \in \mathbb{C}$ and $\mathbf{v} \neq \mathbf{0} \in \mathbb{F}^n$ are such that $A\mathbf{v} = \lambda\mathbf{v}$, then λ is an **eigenvalue** of A associated to the **eigenvector** $\mathbf{v}$. We also say that $(\lambda, \mathbf{v})$ form an **eigenpair**.

The eigenpair equation takes the form $A\mathbf{v} = \lambda\mathbf{v}$, for $\mathbf{v} \neq \mathbf{0}$. Rewriting this,

$$A\mathbf{v} = \lambda\mathbf{v} \iff A\mathbf{v} - \lambda\mathbf{v} = \mathbf{0} \iff A\mathbf{v} - \lambda\mathbb{I}\mathbf{v} = \mathbf{0} \iff (A - \lambda\mathbb{I})\mathbf{v} = \mathbf{0}$$

(We could also have obtained $(\lambda\mathbb{I} - A)\mathbf{v} = \mathbf{0}$)

Hence, since we seek $\mathbf{v} \neq \mathbf{0}$, the homogeneous system $(A - \lambda\mathbb{I})\mathbf{v} = \mathbf{0}$ must have non-trivial solutions; this implies that $A - \lambda\mathbb{I}$ must be *singular*. So, if λ is an eigenvalue, there must hold that $\det(A - \lambda\mathbb{I}) = 0$

Remark 3.2

It is essential to remember that one seeks a nonzero vector $\mathbf{v}$. *Clearly, if* $\mathbf{v} = \mathbf{0}$, *then* $A\mathbf{v} = \lambda\mathbf{v}$ *for any* λ, *since this just means that* $\mathbf{0} = \mathbf{0}$

Often, we use normalised eigenvectors, $\tilde{\mathbf{v}} = \mathbf{v}/||\mathbf{v}||$, so that $\|\tilde{\mathbf{v}}\| = 1$

Also, for eigenvectors $\mathbf{v}$ that have all their components nonpositive, we typically use $-\mathbf{v}$, so that all components are nonnegative.

Definition 3.3 (Spectrum of a matrix)

The **spectrum** of $A \in \mathcal{M}_n$ is the set of all its eigenvalues and its denoted $\sigma(A)$

Theorem 3.4

$0 \in \sigma(A) \iff A$ *is singular*

Theorem 3.5

$A \in \mathcal{M}_n(\mathbb{F})$, $\lambda, \mu \in \mathbb{C}$ *given. Then* $\lambda \in \sigma(A)$ *if and only if* $\lambda + \mu \in \sigma(A + \mu\mathbb{I})$

Definition 3.6 (Characteristic polynomial/equation)

The **characteristic polynomial** of $A \in \mathcal{M}_n$ is

$$p_A(z) = \det(A - z\mathbb{I}).$$

The **characteristic equation** of A is $p_A(z) = 0$

By the Fundamental Theorem of Algebra, if $p_A(z)$ has degree n, then $p_A(z)$ has n complex roots including multiplicity (or at most n roots if not counting multiplicity)

These roots are the eigenvalues of A and thus $\sigma(A)$ has at most n elements in $\mathbb{C}$

Theorem 3.7

Let $A \in \mathcal{M}_n$. Then

$$\operatorname{tr}(A) = \sum_{i=1}^{n} \lambda_i \quad and \quad \det(A) = \prod_{i=1}^{n} \lambda_i$$

Theorem 3.8

Let $p(T)$ be a k-degree polynomial. If $(\lambda, \mathbf{v})$ eigenpair of $A \in \mathcal{M}_n$, then $(p(\lambda), \mathbf{v})$ is an eigenpair for $p(A)$

Definition 3.9 (Algebraic multiplicity of an eigenvalue)

Let $A \in \mathcal{M}_n$. The (**algebraic**) **multiplicity** of $\lambda \in \sigma(A)$ is its multiplicity as a zero of the characteristic polynomial $p_A(\lambda)$

Definition 3.10 (Spectral radius of a matrix)

The **spectral radius** of $A \in \mathcal{M}_n$ is

$$\rho(A) = \max\{|\lambda|, | \lambda \in \sigma(A)\}$$

Proposition 3.11

For all $\lambda \in \sigma(A)$, $A \in \mathcal{M}_n$, λ lies in the closed bounded disk in $\mathbb{C}$,

$$\{z \in \mathbb{C} : z| \leq \rho(A)\}$$

Theorem 3.12 (Every square matrix is close to nonsingular matrices)

Let $A \in \mathcal{M}_n$, then there exists $\delta > 0$ such that $A + \varepsilon\mathbb{I}$ is non-singular for $0 < |\varepsilon| < \delta$

Theorem 3.13

Let $A \in \mathcal{M}_n$. Suppose that $\lambda \in \sigma(A)$ has algebraic multiplicity k. Then

$$\operatorname{rank}(A - \lambda\mathbb{I}) \geq n - k$$

with equality when $k = 1$

Definition 3.14 (Similarity/permutation similarity)

Let $A, B \in \mathcal{M}_n$. We say that B is **similar** to A if there exists a nonsingular $S \in \mathcal{M}_n$ such that

$$B = S^{-1}AS$$

The transformation $A \mapsto S^{-1}AS$ is a **similarity transformation** with similarity matrix S. If $S = P$ with P a **permutation matrix** and that $B = P^TAP$, A and B are **permutation similar**. In both cases, we denote "A similar to B" as $A \sim B$

Theorem 3.15

Similarity is an equivalence relation, i.e., it is reflexive, symmetric, and transitive

Theorem 3.16

Let A, $B \in \mathcal{M}_n$. If A is similar to B, then they have the same characteristic polynomial, i.e.,

$$p_A(t) = p_B(t)$$

Corollary 3.17

Let A, $B \in \mathcal{M}_n$. If $A \sim B$, then

1. *A and B have the same eigenvalues*
2. *If B is a diagonal matrix, then the main diagonal entries are the eigenvalues of A*
3. $B = 0 \iff A = 0$
4. $B = \mathbb{I} \iff A = \mathbb{I}$

Definition 3.18

If $A \in \mathcal{M}_n$. If A is similar to a diagonal matrix, then A is **diagonalisable**

Theorem 3.19

Let $A \in \mathcal{M}_n$*.*

1. $$A \sim \begin{pmatrix} \Lambda & C \\ 0 & D \end{pmatrix} \qquad (1)$$

   with $\Lambda = diag(\lambda_1, \ldots, \lambda_k)$*,* $D \in \mathcal{M}_{n-k}$*,* $1 \leq k \leq n \iff k$ *linear independent vectors in* $\mathbb{C}^n$*, each of which is an eigenvector of* A

2. A *diagonalisable* $\iff$ *there are* n *linearly independent eigenvectors of* A

Theorem 3.19 (continued)

3. *If* $x^{(1)}, \dots, x^{(n)}$ *are linear independent eigenvectors of* A*, define*

$$S = [x^{(1)} \dots x^{(n)}].$$

Then $S^{-1}AS$ *is diagonal.*

4. *If*

$$A \sim \begin{pmatrix} \Lambda & C \\ 0 & D \end{pmatrix},$$

then the diagonal entries of Λ *are eigenvalues of* A*, if* $A \sim \Lambda = \mathrm{diag}(\lambda_1, \dots, \lambda_n)$*, then* $\lambda_1, \dots, \lambda_n$ *are the eigenvalues of* A

Lemma 3.20

Let $\lambda_1, \ldots, \lambda_k$, $k \geq 2$ *be* k *distinct eigenvalue of* A. *Let* $x^{(i)}$ *be an eigenvector associated to* λ_i, $i = 1, \ldots, k$. *Then* $x^{(1)}, \ldots, x^{(k)}$ *are linear independent*

Theorem 3.21

If $A \in \mathcal{M}_n$ *has* n *distinct eigenvalues, then it is diagonalisable*

Lemma 3.22

Let $B = \bigoplus_{i=1}^{d} B_{ii}$. *Then* B *is diagonalisable if and only if each of the* B_{ii} *is diagonalisable*

Definition 3.23

Two matrices A and B in $\mathcal{M}_n$ are **simultaneously diagonalisable** if there exists a matrix $S \in \mathcal{M}_n$ non-singular such that $S^{-1}AS$ and $S^{-1}BS$ are diagonal

Theorem 3.24

Let $A, B \in \mathcal{M}_n$ be diagonalisable. Then A and B commute if and only if A and B are simultaneously diagonalisable

Remark 3.25

See Definition 1.3.16 and following for commuting families and simultaneously diagonalisable families

Theorem 3.26

Let $A \in \mathcal{M}_n$, then

1. $\sigma(A) = \sigma(A^T)$
2. $\sigma(A^*) = \overline{\sigma(A)}$

Definition 3.27

Take $A \in \mathcal{M}_n$, for a given $\lambda \in \sigma(A)$, the set of $\mathbf{x} \in \mathbb{C}^n$ such that $A\mathbf{x} = \lambda\mathbf{x}$ is the eigenspace associated to λ. Every non-zero vector in the eigenspace associated to $\lambda \in \sigma(A)$ is an eigenvector of A associated to λ

Definition 3.28

Let $A \in \mathcal{M}_n$ and $\lambda \in \sigma(A)$. The dimension of the eigensapce associated to λ is the **geometric multiplicity** of λ. We say that λ is **simple** if its algebraic multiplicity is one, it is **semisimple** if its algebraic and geometric multiplicities are equal

Proposition 3.29

Let λ be an eigenvalue of A. We have that the algebraic multiplicity is greater of equal to the geometric multiplicity. Furthermore, if the algebraic multiplicity is one the the geometric multiplicity is one as well

Definition 3.30

Let $A \in \mathcal{M}_n$. We say that A is

- **defective** if the geometric multiplicity is less then the algebraic multiplicity for *some* eigenvalue
- **non-defective** if *for all* eigenvalues, the geometric multiplicity equals the algebraic multiplicity
- **non-derogatory** if *for all* eigenvalues, the geometric multiplicity is one
- **derogatory** otherwise

Theorem 3.31

Let $A \in \mathcal{M}_n$

1. *A is diagonalisable if and only if it is nondefective*
2. *A has distinct eigenvalues if and only if A is nonderogatory and non-defective*

Remark 3.32

$\sigma(A) = \sigma(A^T)$, *however they might have different spaces associated to each eigenvalue*

Definition 3.33 (Left wigenvector)

Let $\mathbf{0} \neq \mathbf{y} \in \mathbb{C}^n$, then we say that $\mathbf{y}$ is a **left eigenvector** of $A \in \mathcal{M}_n$ associated to $\lambda \in \sigma(A)$ if $\mathbf{y}^* A = \lambda \mathbf{y}^*$

Theorem 3.34

Let $\mathbf{0} \neq \mathbf{x} \in \mathbb{C}^n$, $A \in \mathcal{M}_n$. Assume that $A\mathbf{x} = \lambda \mathbf{x}$ for some λ. If $\mathbf{x}^* A = \mu \mathbf{x}^*$, then $\lambda = \mu$

Remark 3.35

y *is a left eigenvector associated to* λ *is also a right eigenvector of* A^* *associated to* $\bar{\lambda}$.
$\bar{\mathbf{y}}$ *eigenvector of* A^T *associated to* λ

Let $A \in \mathcal{M}_n$ diagonalisable, S non-singular matrix, $S^{-1}AS = \Lambda$. Partition $S = [x_1, \ldots, x_n]$ and $S^{-*} = [y_1, \ldots, y_n]$, where x_i and y_i are the right and left eigenvectors associated to λ_i, respectively.

Theorem 3.36

Let $A \in \mathcal{M}_n$, $\mathbf{x}, \mathbf{y} \in \mathbb{C}^n$, λ, $\mu \in \mathbb{C}$. *Assume* $A\mathbf{x} = \lambda\mathbf{x}$ *and* $\mathbf{y}^* A = \mu\mathbf{y}^*$

1. *If* $\lambda \neq \mu$, *then* $\mathbf{y}^*\mathbf{x} = 0$, *then* $\mathbf{x} \perp \mathbf{y}$
2. *If* $\lambda = \mu$ *and* $\mathbf{y}^*\mathbf{x} \neq 0$, *then there exists* S *non-singular of the form* $S = [\mathbf{x}S_1]$ *such that* $S^{-*} = [\mathbf{y}/(\mathbf{x}^*\mathbf{y})Z_1]$ *and* $A = S \begin{pmatrix} \lambda & 0 \\ 0 & B \end{pmatrix} S^{-1}$

Conversely, if A *is similar to a block matrix of the form*

$$\begin{pmatrix} \lambda & 0 \\ 0 & B \end{pmatrix}, B \in \mathcal{M}_{n-1}$$

then it has a non-orthogonal pair of left and right eigenvectors associated to λ

Theorem 3.37

Let A, $B \in \mathcal{M}_n$, with $A \sim B$ with similarity matrix S. If $(\lambda, \mathbf{x})$ is an eigenpair of B, then $(\lambda, S\mathbf{x})$ is an eigenpair of A. If $(\lambda, \mathbf{y})$ is a left eigenpair of B, then $(\lambda, S^{-*}\mathbf{y})$ is a left eigenpair of A

Theorem 3.38

Let $A \in \mathcal{M}_n$, $\lambda \in \mathbb{C}$, $\mathbf{x}$, $\mathbf{y} \in \mathbb{C}^n$ non-zero. Suppose that $\lambda \in \sigma(A)$ and $A\mathbf{x} = \lambda\mathbf{x}$ and $\mathbf{y}^*A = \lambda\mathbf{y}^*$

1. *If λ has algebraic multiplicity 1, then $\mathbf{y}^*\mathbf{x} \neq 0$*
2. *If λ has geometric multiplicity 1, then it has algebraic multiplicity 1 if and only if $\mathbf{y}^*\mathbf{x} \neq 0$.*

This section is based mostly on Varga's book *Geršgorin and His Circles* [**?**], which is highly recommended reading if you enjoy matrix theory.
Let $A \in \mathcal{M}_n(\mathbb{C})$. Denote $N = \{1, \dots, n\}$. For $i \in N$, define

$$r_i(A) = \sum_{j \in N \setminus \{i\}} |a_{ij}|$$

to be the ith deleted row sums of A. Assume that $r_i(A) = 0$ if $n = 1$. Let

$$\Gamma_i(A) = \{z \in \mathbb{C} \mid |z - a_{ii}| \leq r_i(A)\} \qquad i \in N$$

be the ith **Gershgorin disk** of A and

$$\Gamma(A) = \bigcup_{i \in N} \Gamma_i(A)$$

be the **Gershgorin set** of A. Γ_i and Γ are closed and bounded in $\mathbb{C}$. $\Gamma_i(A)$ is a disk centred at a_{ii} and with radius $r_i(A)$, $i \in N$.

Theorem 3.39 (Gershgorin, 1931)

For all $A \in \mathcal{M}_n(\mathbb{C})$ *and for all* $\lambda \in \sigma(A)$*, there exists* $k \in \mathbb{N}$ *such that*

$$|\lambda - a_{kk}| \leq r_k(A)$$

i.e., $\lambda \in \Gamma_k(A)$ *and thus* $\lambda \in \Gamma(A)$*. Since this is true for all* λ*, we have*

$$\sigma(A) \subseteq \Gamma(A)$$

Remark 3.40

This also works with deleted column sums; indeed, just consider A^T *in this case. However, this typically gives different disks*

Corollary 3.41

Let $A \in \mathcal{M}_n(\mathbb{C})$, *then*

$$\rho(A) = \max\{|\lambda|,\ \lambda \in \sigma(A)\} \leq \max_{i \in \mathbb{N}} \sum_{j \in \mathbb{N}} |a_{ij}|$$

Definition 3.42 (Strictly diagonally dominant matrix)

$A \in \mathcal{M}_n(\mathbb{C})$ is **strictly diagonally dominant** (SDD) if

$$\forall i \in N, |a_{ii}| > r_i(A)$$

Theorem 3.43

Let $A \in \mathcal{M}_n(\mathbb{C})$. *If A SDD then A is nonsingular*

Let $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{x} > \mathbf{0}$, i.e., $\mathbf{x} = (x_1, \dots, x_n)$ is such that $x_i > 0$ for all i. Let $X = \mathrm{diag}(\mathbf{x}) = \mathrm{diag}(x_1, \dots, x_n)$ such that X is invertible. Let $A \in \mathcal{M}_n(\mathbb{C})$, then $X^{-1}AX = \left[\dfrac{a_{ij}x_j}{x_i}\right]_{i,j \in N}$. Also $X^{-1}AX$ similar to A, so $\sigma(X^{-1}AX) = \sigma(A)$.b

Let $r_i^{x_i}(A) = r_i(X^{-1}AX) = \displaystyle\sum_{j \in N \setminus \{i\}} \frac{|a_{ij}|x_j}{x_i}$ be the ith weighted rows sums of A. Let

$$\Gamma_i^{r^x} = \{z \in \mathbb{C},\ |z - a_{ii}| \leq r_i^x(A)\}$$

and

$$\Gamma^{r^x} = \bigcup_{i \in N} \Gamma_i^{r^x}$$

be the ith **weighted Gershgorin disk** and the **weighted Gershgorin set**of A, respectively

Corollary 3.44

For any matrix $A \in \mathcal{M}_n(\mathbb{C})$ *and* $x \in \mathbb{R}^n$, $x > 0$,

$$\sigma(A) \subset \Gamma^{r^x}(A)$$

Question: How many eigenvalues are contained in each "component"?
Assume $n \geq 2$. Let S be a proper subset of N, i.e., $\emptyset \neq S \subsetneq N$, with $|S|$ its cardinality.

Let $A \in \mathcal{M}_n(\mathbb{C})$, $\mathbf{x} > \mathbf{0}$ in $\mathbb{R}^n$ and

$$F_S^{r^{\mathbf{x}}} = \bigcup_{i \in S} \Gamma_i^{r^{\mathbf{x}}}(A)$$

Then

$$\Gamma_S^{r^{\mathbf{x}}}(A) \cap \Gamma_{N \setminus S}^{r^{\mathbf{x}}}(A) = \emptyset$$

Theorem 3.45

For all $A \in \mathcal{M}_m(\mathbb{C})$, *for all* $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{x} > \mathbf{0}$ *for which*

$$\Gamma_S^{r^{\mathbf{x}}}(A) \cap \Gamma_{N \setminus S}^{r^{\mathbf{x}}}(A) = \emptyset$$

for some proper subset S *of* N, *then* $\Gamma_S^{r^{\mathbf{x}}}(A)$ *contains exactly* $|S|$ *eigenvalues of* A

We have seen that a matrix A that is SDD is nonsingular. Can we weaken this? What if diagonal dominance is not strict, i.e., $|a_{ii}| = r_i(A)$ for some $i \in N$, $|a_{ii}| \geq r_i(A)$ for all $i \in N$. This is not sufficient for nonsingularity. If we take the matrix

$$\begin{pmatrix} 2 & 1 \\ 2 & 2 \end{pmatrix}$$

is DD and singular, however,

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

is DD and singular.

Definition 3.46 (Reducible/irreducible matrices)

$A \in \mathcal{M}_n(\mathbb{C})$ is **reducible** if there exists a permutation matrix $P \in \mathcal{M}_n(\mathbb{R})$ and $r \in N = \{1, \ldots, n\}$ such that

$$PAP^T = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix}$$

where $A_{11} \in \mathcal{M}_r$, $A_{22} \in \mathcal{M}_{n-r}$. If there is no such P, then we say that A is **irreducible**

Remark 3.47

If $A \in \mathcal{M}_1$, then A irreducible if $a_{11} \neq 0$

In the reducible case, we can continue the process and find a matrix P (permutation) such that

$$PAP^T = \begin{pmatrix} R_{11} & R_{12} & \dots & R_{1m} \\ 0 & R_{22} & \dots & R_{2m} \\ & & \ddots & \\ 0 & & \dots & R_{nm} \end{pmatrix}$$

with the diagonal block R_{ii} irreducible. This is the **normal reduced form** of A

Remark 3.48

Establishing irreducibility this way is hard. If no obvious permutation of rows and columns gives rise to a matrix in reduced form, then deciding on irreducibility requires to exhaust all possible permutation matrices to assert none exists. There are n! permutation matrices of size $n \times n$...

Let $A \in \mathcal{M}_n(\mathbb{C})$. Let $\{v_1, \dots, v_n\}$ be n distinct points called **vertices**

For any (i,j), $i,j \in N$, for which $a_{ij} \neq 0$, connect v_j to v_j using a directed arc $\overrightarrow{v_i v_j}$

If $a_{ii} \neq 0$, there is a loop from v_i to v_j

The collection of all the directed arcs (and loops) obtained thusly is called the **directed graph** (or **digraph**) associated to A and is denoted $\mathcal{G}(A)$

A **directed path** in $\mathcal{G}(A)$ is a collection of directed arcs from v_i to v_j, i.e.,

$$\overrightarrow{v_{i_1} v_{i_2}}, \ldots, \overrightarrow{v_{i_{n-1}} v_{i_n}}$$

Along a directed path

$$\prod_{k=1}^{n-1} a_{i_k} a_{i_{k+1}} \neq 0$$

Remark 3.49

Given a graph $\mathcal{G}$, the matrix A such that $\mathcal{G}(A) = \mathcal{G}$ is the **adjacency matrix** *of $\mathcal{G}$*

Definition 3.50

Let $\mathcal{G}$ be a digraph with vertex set $\{v_1, \ldots, v_n\}$. $\mathcal{G}$ is **strongly connected** if for all ordered pairs (v_i, v_j) of vertices, there is a directed path from v_i to v_j in $\mathcal{G}$

Remark 3.51

If $\mathcal{G}(A)$ is strongly connected, then A cannot have a row with only zero off-diagonal entries. Indeed, suppose $\mathcal{G}(A)$ is strongly connected. Without loss of generality, assume row 1 in A has only zero off-diagonal entries. Then because of the way $\mathcal{G}(A)$ is constructed, this means there are no directed arcs terminating in v_1 and as a consequence, there is no directed path terminating in v_1, contradicting strong connectedness of $\mathcal{G}(A)$.

Remark 3.52

$\mathcal{G}(A)$ is strongly connected if and only if for any permutation matrix P, we have that $\mathcal{G}(P^T AP)$ is strongly connected. [Because permutation is a relabelling of vertices.]

Theorem 3.53

Let $A \in \mathcal{M}_n(\mathbb{C})$. Then A is irreducible if and only $\mathcal{G}(A)$ is strongly connected

Definition 3.54 (Irreducibly diagonally dominant matrix)

$A \in \mathcal{M}_n(\mathbb{C})$ is **irreducibly diagonally dominant** (IDD) if A is irreducible, diagonally dominant, i.e.,

$$\forall i \in N, \quad |a_{ii}| \geq r_i(A)$$

and there exists $i \in N$ for which diagonal dominance is strict, i.e., there exists i such that $|a_{ii}| = r_i(A)$.

Theorem 3.55 (Taussky 1949 [?])

For any $A \in \mathcal{M}_n(\mathbb{C})$, A IDD $\Rightarrow$ A non-singular

Another result of Taussky

Theorem 3.56

Let $A \in \mathcal{M}_n(\mathbb{C})$ *be irreducible. Suppose* $\lambda \in \sigma(A)$ *be such that* $\forall i \in N$, $\lambda \notin \operatorname{Int} \Gamma_i(A)$
Then

$$\forall i \in N, \quad |\lambda - a_{ii}| = r_i(A) \tag{2}$$

In particular, if $\lambda \in \partial\Gamma(A)$ *[the boundary of* $\Gamma(A)$*] for some* $\lambda \in \sigma(A)$, *then* (2) *holds for* λ

References I