MATH 4370/7370 – Linear Algebra and Matrix Analysis

Factorisations, canonical forms and decompositions

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Outline

Definition 4.1

Let $\mathbf{x}_1, \ldots, \mathbf{x}_k \in \mathbb{C}^n$. We say that $\mathbf{x}_1, \ldots, \mathbf{x}_k$ is an **orthogonal list** if $\mathbf{x}_i^* \mathbf{x}_j = 0$ for all $i \neq j$. If, in addition, we have that $\mathbf{x}_i^* \mathbf{x}_i = 1$, then we say that the list is **orthonormal**

Theorem 4.2

Every orthonormal list of vectors in $\mathbb{C}^n$ is linearly independent

Remark 4.3

In Theorem 4.2, if we have "only" orthogonal vectors, we need to replace "list of vectors" by "list of non-zero vectors" in the statement

Definition 4.4

Let $U \in \mathcal{M}_n$, we say that U is an **unitary matrix** if $U^*U = \mathbb{I}$. Furthermore, we say that $U \in \mathcal{M}_n(\mathbb{R})$ is a **(real) orthogonal matrix** if $U^TU = \mathbb{I}$

Theorem 4.5

Let $U \in \mathcal{M}_n$. TFAE:

1. *U is unitary*
2. *U is non-singular and $U^* = U^{-1}$*
3. $UU^* = \mathbb{I}$
4. *U^* is unitary*
5. *the columns of U are orthonormal*
6. *the rows of U are orthonormal*
7. *for all $\mathbf{x} \in \mathbb{C}^n$ we have $\|\mathbf{x}\|_2 = \|U\mathbf{x}\|_2$*

Definition 4.6

A **linear transformation** $T : \mathbb{C}^n \to \mathbb{C}^n$ is a **Euclidean isometry** if $\|\mathbf{x}\|_2 = \|T\mathbf{x}\|_2$ for all $\mathbf{x} \in \mathbb{C}^n$

Corollary 4.7

Let $U \in \mathcal{M}_n$. U is a Euclidean isometry if and only if U is unitary

Remark 4.8

Let U, $V \in \mathcal{M}_n$ are unitary matrices (respectively real orthogonal), then UV is unitary (respectively real orthogonal).
Indeed, U, V unitary $\Leftrightarrow U^{-1}, V^{-1}$ exist and $U^{-1} = U^*, V^{-1} = V^*$. Then

$$\begin{aligned} UV \ \ unitary \ &\Leftrightarrow (UV)^* UV = \mathbb{I} \\ &\Leftrightarrow V^* U^* UV = \mathbb{I} \\ &\Leftrightarrow \mathbb{I} = \mathbb{I} \end{aligned}$$

Notation: $\mathrm{GL}(n, \mathbb{F})$ is the general linear group, where the elements are non-singular matrices in $\mathcal{M}_n(\mathbb{F})$

Theorem 4.9

The set of unitary (respectively real orhogonal) matrices in $\mathcal{M}_n$ forms a group, the $n \times n$ unitary (respectively real orthogonal) subgroup of $\mathrm{GL}(n, \mathbb{C})$ (respectively $\mathrm{GL}(n, \mathbb{R})$)

Theorem 4.10 (Selection Principle)

Suppose that we have a sequence of unitary matrices $U_1, U_2 \ldots, \in \mathcal{M}_n$. Then there exists a subsequence $U_{k_1}, U_{k_2} \ldots$ such that the entries of U_{k_i} converge to entries of a unitary matrix as $i \to \infty$

Lemma 4.11

Let $U \in \mathcal{M}_n$ be a unitary matrix partitioned as

$$U = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix},$$

with $U_{ii} \in \mathcal{M}_k$. Then $\operatorname{rank} U_{12} = \operatorname{rank} U_{21}$ and $\operatorname{rank} U_{22} = \operatorname{rank} U_{11} + n - 2k$. If, furthermore, $U_{21} = 0$ and $U_{12} = 0$, then U_{11} and U_{22} are unitary

Theorem 4.12 (QR factorisation)

Let $A \in \mathcal{M}_{nm}$

1. *If* $n \geq m$*, there is a* $Q \in \mathcal{M}_{nm}$ *with orthogonormal columns and upper triangular* $R \in \mathcal{M}_m$ *with non-negative main diagonal entries such that* $A = QR$
2. *If* $\operatorname{rank} A = m$ *then the factors* Q *and* R *in (1) are uniquely determined and the main diagonal entries of* R *are all positive*
3. *If* $n = m$*, Then the factor* Q *in (1) is unitary*
4. *There is a unitary* $Q \in \mathcal{M}_n$ *and an upper triangular* $R \in \mathcal{M}_{nm}$ *with nonnegative diagonal entries such that* $A = QR$
5. *If* A *is real, then* Q *and* R *are in (1), (2), (3), and (4) may be taken to be real*

For a unitary matrix U, $U^* = U$, so the transformation $A \mapsto U^*AU$ is a **similarity transformation**, provided that U is unitary. This is a **unitary similarity**

Definition 4.13 (Unitarily similar matrices)

Let $A, B \in \mathcal{M}_n$. We say that A is **unitarily similar** to B if there exists $U \in \mathcal{M}_n$ unitary such that

$$A = U^*BU$$

If U can be taken real (i.e., if U is real orthogonal) than A is real orthogonal similar to B (if $A = U^TBU$)

Remark 4.14

1. *Unitary similarity is an equivalence relation*
2. *Unitary similarity implies similarity. However, the converse is not true*
3. *Similarity is a change of bases. Unitary similarity is a change of orthonormal bases*

Definition 4.15 (Householder matrix)

Let $0 \neq \omega \in \mathbb{C}^n$. The Householder matrix $U_\omega \in \mathcal{M}_n$ is

$$U_\omega = \mathbb{I} - 2(\omega^*\omega)^{-1}\omega\omega^*$$

Remark 4.16

1. *If* $\|\omega\| = 1$ *then* $U_\omega = \mathbb{I} - 2\omega\omega^*$
2. *Householder matrix are unitary and Hermitian, thus* $U_\omega^{-1} = U_\omega$.
3. *The eigenvalues of a Householder matrix are* $-1, 1, \ldots, 1$ *and* $|U_\omega| = 1$

Theorem 4.17

Let $\mathbf{x}, \mathbf{y} \in \mathbb{C}^n$ *and assume that* $\|\mathbf{x}\|_2 = \|\mathbf{y}\|_2 > 0$

- *If* $\mathbf{y} = e^{i\theta}\mathbf{x}$ *for some* $\theta \in \mathbb{R}$ *[*$\mathbf{x}, \mathbf{y}$ *are linearly dependent], define* $U(\mathbf{y}, \mathbf{x}) = e^{i\theta}\mathbb{I}$
- *Otherwise, let* $\phi \in [0, 2\pi)$ *be such that* $\mathbf{x}^*\mathbf{y} = e^{i\phi}|\mathbf{x}^*\mathbf{y}|$ *(taking* $\phi = 0$ *if* $\mathbf{x}^*\mathbf{y} = 0$*). Let* $\omega = e^{i\phi}\mathbf{x} - \mathbf{y}$ *and define*

$$U(\mathbf{y}, \mathbf{x}) = e^{i\phi} U_\omega$$

where $U_\omega = \mathbb{I} - 2(\omega^*\omega)^{-1}\omega\omega^*$ *is Householder*

1. $U(\mathbf{y}, \mathbf{x})$ *unitary and essentially Hermitian*
2. $U(\mathbf{y}, \mathbf{x})\mathbf{x} = \mathbf{y}$
3. $U(\mathbf{y}, \mathbf{x})\mathbf{z} \perp \mathbf{y}$*, when* $\mathbf{z} \perp \mathbf{y}$
4. *If* $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$*, then* $U(\mathbf{y}, \mathbf{x})$ *is real and* $U(\mathbf{y}, \mathbf{x}) = \mathbb{I}$ *if* $\mathbf{y} = \mathbf{x}$ *and* $U(\mathbf{y}, \mathbf{x}) = U_{\mathbf{x}-\mathbf{y}} \in \mathcal{M}_n(\mathbb{R})$ *otherwise*

Remark 4.18

For all $A \in \mathcal{M}_n$, $U(y,x)^* A U(y,x) = U_\omega^* A U_\omega$. This is called a Householder transformation.

Theorem 4.19 (Schur's Form)

Let $A \in \mathcal{M}_n$ with eigenvalues $\lambda_1, \dots, \lambda_n$ in any prescribed order (including multiplicities). Let $x \in \mathbb{C}^n$, $\|x\| = 1$, be such that $Ax = \lambda_1 x$

1. *There exists $U = [x\, u_2 \dots u_n] \in \mathcal{M}_n$ unitary such that $U^* A U = T$, where T is upper triangular such that $t_i i = \lambda_i$, $i = 1, \dots, n$.*
2. *If $A \in \mathcal{M}_n(\mathbb{R})$ and has real eigenvalues, then x can be chosen to be real and there exists*

$$Q = [x\, q_2 \dots q_n] \in \mathcal{M}_n(\mathbb{R})$$

real orthogonal and such that $Q^T A Q = T$, with T upper triangular with $t_{ii} = \lambda_1$ $i = 1, \dots, n$.

Theorem 4.20 (Schur version 2)

Let $A \in \mathcal{M}_n$ with eigenvalues $\lambda_1, \dots, \lambda_n$ (including mutiplicities). Then there esists $U \in \mathcal{M}_n$ such that

$$U^* A U = \begin{pmatrix} \lambda_1 & * & \dots & * \\ 0 & \lambda_2 & & \vdots \\ 0 & & \ddots & * \\ 0 & & & \lambda_n \end{pmatrix}$$

Remark 4.21

The decomposition is not unique

Theorem 4.22

Let $U \in \mathcal{M}_n, A,\ B \in \mathcal{M}_n$. Suppose A is unitarily similar to B, then

$$\sum_{i,j=1}^{n} |a_{ij}|^2 = \sum_{i,j} |b_{ij}|^2$$

Corollary 4.23

Let $A \in \mathcal{M}_n$ have eigenvalues $\lambda_1, \ldots, \lambda_n$, $T = UAU^*$ upper triangular. Then

$$\sum_{i=1}^{n} |\lambda_1|^2 = \sum_{i,j=1}^{n} |a_{ij}|^2 - \sum_{i<j} |t_{ij}|^2 \leq \sum_{i,j=1} |a_{ij}|^2 = \mathrm{tr} AA^*$$

with equality if T is diagonal.

Theorem 4.24 (Cayley-Hamilton)

Let $A \in \mathcal{M}_n$ and $p_A(t)$ is the characteristic polynomial of A, then $p_A(A) = 0$.

Theorem 4.25 (Sylvester's theorem – pole placement)

Assume $A \in \mathcal{M}_n$ has eigenvalues $\lambda_1, \ldots, \lambda_n$ with multiplicities $n_1, \ldots, n_d$ $(\sum_{i=1}^{d} n_i = n)$. Then A is unitary similar to a $d \times d$ block upper triangular matrix T, where $T_{i,j} \in M_{n_i m_j}$, $T_{ij} = 0$ if $i > i$, T_{ii} upper triangular with diagonal λ_i, $T_{ii} = \lambda\mathbb{I} + R_i$, R_i strictly upper triangular, and A is similar to a matrix to $\bigoplus_{i=1}^{d} T_{ii}$ [standard similarity, not unitary]

Theorem 4.26

(Every square matrix is almost diagonalisble) Let $A \in \mathcal{M}_n$ *for all* $\varepsilon > 0$, *there exists* $A(\varepsilon)[a_{ij}(\varepsilon)] \in \mathcal{M}$ *with distinct eigenvalues such that*

$$\sum_{i,j} |a_{ij} - a_{ij}(\varepsilon)|^2 < \varepsilon$$

Theorem 4.27

If $A \in \mathcal{M}_n$ *for all* $\varepsilon > 0$ *there exists* $S(\varepsilon) \in \mathcal{M}_n$ *non-singular such that*

$$S^{-1}(\varepsilon)AS(\varepsilon) = T(\varepsilon),$$

where $T(\varepsilon)$ *is upper triangular and* $|t_{ij}(\varepsilon)| < \varepsilon$ *for all* i, j, *with* $i < j$.

Lemma 4.28

Let $(A_k)_{k\in\mathbb{N}}$ a sequence of matrices such that $\lim\limits_{k\to\infty} A_k = A$ (entry-wise). Then there exists $k_1 < k_2 < \ldots$ and $U_{k_i} \in \mathcal{M}$ such that

1. $T_i = U_{k_i}^* A_{k_i} U_{k_i}$ *upper triangular*
2. $U + \lim\limits_{i\to\infty} U_{k_i}$ *exists and is unitary*
3. $T = U^* A U$ *upper triangular*
4. $\lim\limits_{i\to\infty} T_i = T$

Theorem 4.29

Let $(A_k)_{k\in\mathbb{N}}$ a sequence of matrices such that $\lim_{k\to\infty} A_k = A$ (entry-wise). Then let

$$\lambda(A) = \begin{bmatrix}\lambda_1(A) & \dots & \lambda_n(A)\end{bmatrix}^T$$

and

$$\lambda(A_k) = \begin{bmatrix}\lambda_1(A_k) & \dots & \lambda_n(A_k)\end{bmatrix}^T$$

be presentations of the eigenvalues of A and A_k. Define

$$S_n\{\pi \mid \pi \text{ is a permutation of } \{1, \dots, n\}\}.$$

Then for all $\varepsilon > 0$ there exists $N(\varepsilon) \in \mathbb{N} \setminus \{0\}$ such that

$$\min_{\pi\in S_n} \max_{i=1,\dots} \{|\lambda_{\pi(i)}(A_k) - \lambda_i(A)|\} \le \varepsilon \qquad \forall k \ge N(\varepsilon)$$

Recall that if $\mathbf{x}, \mathbf{y}$ are two (column) vectors in $\mathbb{F}^n$, then $\mathbf{x}\mathbf{y}^*$ is a rank 1 matrix in $\mathcal{M}_n(\mathbb{F})$. (Show it as an exercise.) The following is a famous result that quantifies the effect on the spectrum of a matrix of a perturbation built thusly

Theorem 4.30 (Brauer)

Suppose $A \in \mathcal{M}_n$ has eigenvalues $\lambda, \lambda_2, \ldots, \lambda_n$. Let $\mathbf{x}$ be an eigenvector associated to λ. Then for every vector $\mathbf{v} \in \mathbb{C}^n$, the eigenvalues of $A + \mathbf{x}^*\mathbf{v}$ are $\lambda + \mathbf{v}^*\mathbf{x}, \lambda_2, \ldots, \lambda_n$.

Definition 4.31 (Normal matrix)

A matrix $A \in \mathcal{M}_n$ is **normal** if $AA^* = A^*A$

All unitary, Hermitian or skew-Hermitian and diagonal matrices are normal

Theorem 4.32

Let $A \in \mathcal{M}_n$ with eigenvalues $\lambda_1, \ldots, \lambda_n$. TFAE:

1. *A is normal*
2. *A is unitary diagonalisable*
3. $\sum_{i,j} |a_{i,j}|^2 = \sum_{i} |\lambda_i|^2$
4. *A has n orthogonal eigenvectors*

Theorem 4.33

Let $A \in \mathcal{M}_n$ be a hermitian matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$. Let

$$\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$$

Then

1. $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$
2. *A is unitary diagonalisable*
3. *there exists $U \in \mathcal{M}_n$ such that $A = U\Lambda U^*$*

Definition 4.34

A **Jordan block** $J_k(\lambda)$ is a $k \times k$ upper triagular matrix of the form

$$\begin{pmatrix} \lambda & 1 & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & & \lambda & 1 \\ 0 & \dots & 0 & \lambda \end{pmatrix}$$

Theorem 4.35

Let $A \in \mathcal{M}_n$ then there exists $S \in \mathcal{M}_n$ non-singular such that

$$A = S^{-1} \begin{bmatrix} J_{n_1}(\lambda_1) & & 0 \\ & \ddots & \\ 0 & & J_{n_k}(\lambda_k) \end{bmatrix} S^{-1} = S \bigoplus_{i=1}^{k} J_{n_i}(\lambda_i) S^{-1}$$

Theorem 4.36

Let $A \in \mathcal{M}_n$ with real eigenvalues. Then there exists a basis of generalised eigenvectors for $\mathbb{R}^n$, and if $\{v_1, \dots, v_n\}$ is a basis of generalised eigenvectors of $\mathbb{R}^n$, then $P = \begin{bmatrix} v_1 & \dots & v_n \end{bmatrix}$ is non-singular and $A = D + N$ where $P^{-1}DP = \mathrm{diag}(\lambda_1, \dots, \lambda_n)$ and $N = A - D$ is nilpotent[1] of order $k \leq n$, and D and N commute.

References I