MATH 4370/7370 – Linear Algebra and Matrix Analysis

Essentially nonnegative matrices and M-matrices

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Outline

Essentially nonnegative matrices

Z-matrices

Class K_0

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The Perron-Frobenius can be applied not only to nonnegative matrices, but also to matrices that are *essentially nonnegative*, in the sense that they are nonnegative except perhaps along the main diagonal

Definition 7.1

A matrix $A \in \mathcal{M}_n$ is **essentially nonnegative** (or **quasi-positive**) if there exist $\alpha \in \mathbb{R}$ such that $A + \alpha\mathbb{I} \geq 0$

Remark 7.2

An essentially nonnegative matrix A has non-negative off-diagonal entries. The sign of the diagonal entries is not relevant

Remark 7.3

Irreducibility of a matrix is not affected by the nature of its diagonal entries. Indeed, consider an essentially nonnegative matrix A. The existence of a directed path in $G(A)$ does not depend on the existence of "self-loops". The same is not true of primitive matrices, where the presence of negative entries on the main diagonal has an influence on the values of A^k and thus ultimately, on the capacity to find k such that $A^k > 0$

So we can apply the "weak" versions of the Perron-Frobenius Theorem (the imprimitive cases in Theorem **??**) to $A + \alpha\mathbb{I}$, which is a nonnegative matrix (potentially irreducible). One important ingredient is a result that was proved as Theorem **??**. Namely, that perturbations of the entire diagonal by the same scalar lead to a shift of the spectrum; this is summarised as

$$\sigma(A + \alpha\mathbb{I}) = \{\lambda_1 + \alpha, \ldots, \lambda_n + \alpha, \quad \lambda_i \in \sigma(A)\}$$

Definition 7.4 (Spectral abscissa)

Let $A \in \mathcal{M}_n$. The **spectral abscissa** of A, $s(A)$, is

$$s(A) = \max\{\mathrm{Re}(\lambda), \lambda \in \sigma(A)\}$$

Theorem 7.5

Let $A \in \mathcal{M}_n(\mathbb{R})$ be essentially nonnegative. Then $s(A)$ is an eigenvalue of A and is associated to a nonnegative eigenvector. If, additionally, A is irreducible, then $s(A)$ is simple and is associated to a positive eigenvector

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Definition 7.6

A matrix is of class Z_n if it is in $\mathcal{M}_n(\mathbb{R})$ and such that $a_{i,j} \leq 0$, $i \neq j$, $i, j = 1, \ldots, n$

$$Z_n = \{A \in \mathcal{M}_n : a_{i,j} \leq 0, i \neq j\}$$

We also say that $A \in Z_n$ has the **Z-sign pattern**

Theorem 7.7 ([?])

Let $A \in Z_n$. TFAE

1. *There is a nonnegative vector x such that $Ax > 0$*
2. *There is a positive vector x such that $Ax > 0$*
3. *There is a diagonal matrix $\operatorname{diag}(D) > 0$ such that the entries in $AD = [w_{ik}]$ are such that*
$$w_{ii} > \sum_{k \neq i} |w_{ik}| \forall i$$
4. *For any $B \in Z_n$ such that $A \geq A$, then B is nonsingular*
5. *Every real eigenvalue of any principal submatrix of A is positive.*
6. *All principal minors of A are positive*

Theorem 7.7 (Continued)

7. *For all* $k = 1, \ldots, n$*, the sum of all principal minors is positive*
8. *Every real eigenvalue of* A *is positive*
9. *There exists a matrix* $C \geq 0$ *and a number* $k > \rho(A)$ *such that* $A = k\mathbb{I} - C$
10. *There exists a splitting* $A = P - Q$ *of the matrix* A *such that* $P^{-1} \geq 0$*,* $Q \geq 0$*, and* $\rho(P^{-1}Q < 1)$
11. A *is nonsingular and* $A^{-1} \geq 0$
12. ...

18 *The real part of any eigenvalue of* A *is positive*

Notation: $A \in Z_n$ such that any (and therefore all) of these properties holds is a matrix of class K (or a nonsingular M-matrix).

Theorem 7.8

Let $A \in Z = \bigcap_{i=1,\ldots} Z_n$ *be symmetric. Then* $A \in K$ *if and only if* A *is positive define.*

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Theorem 7.9

Let $A \in Z_n$. TFAE

1. $A + \varepsilon \mathbb{I} \in K$ *for all* $\varepsilon > 0$
2. *Every real eigenvalue of a principal submatrix of A is nonnegative*
3. *All principal minors of A are nonnegative*
4. *The sum of all principal minors of order $k = 1, \ldots, n$ is nonnegative*
5. *Every real eigenvaue of A is nonegative*
6. *There exists $C \geq 0$ and $k \geq \rho(C)$ such that $A = k\mathbb{I} - C$*
7. *Every eigenvalue of A has nonnegative real part*

$A \in Z_n$ such that any of these properties holds is a matrix of class K_0

Theorem 7.10

Let $A \in Z_n$. Assume $A \in K_0$. Then $A \in K \iff A$ nonsingular

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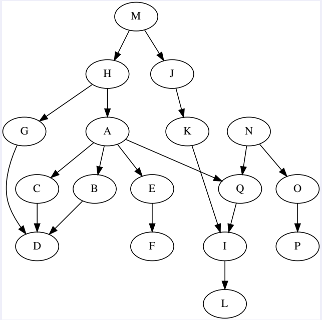
Definition 7.11 (Signature matrix)

A **signature matrix** is is a diagonal matrix S with diagonal entries ± 1

Theorem 7.12 ([?])

Let $A \in \mathcal{M}_n$. Then for each fixed letter $\mathcal{C}$ representing one of the following conditions, conditions $\mathcal{C}_i$ are equivalent for each i. Moreover, letting $\mathcal{C}$ then represent any of the equivalent conditions $\mathcal{C}_i$, the following implication tree holds:



If $A \in Z_n$, each of the following conditions is equivalent to the statement "A is a nonsingular M-matrix"

Theorem 7.12 (Continued)

(A_1) *All the principal minors of A are positive*

(A_2) *Every real eigenvalue of each principal submatrix of A is positive*

(A_3) *For each $\mathbf{x} \neq \mathbf{0}$ there exists a positive diagonal matrix D such that*

$$\mathbf{x}^T A D \mathbf{x} > 0$$

(A_4) *For each $\mathbf{x} \neq \mathbf{0}$ there exists a nonnegative diagonal matrix D such that*

$$\mathbf{x}^T A D \mathbf{x} > 0$$

(A_5) *A does not reverse the sign of any vector; that is, if $\mathbf{x} \neq \mathbf{0}$ and $\mathbf{y} = A\mathbf{x}$, then for some subscript i, $x_i y_i > 0$*

(A_6) *For each signature matrix S, there exists an $\mathbf{x} \gg \mathbf{0}$ such that*

$$SAS\mathbf{x} \gg \mathbf{0}$$

Theorem 7.12 (Continued)

(B_7) *The sum of all the $k \times k$ principal minors of A is positive for $k = 1, \ldots, n$*

(C_8) *A is nonsingular and all the principal minors of A are nonnegative*

(C_9) *A is nonsingular and every real eigenvalue of each principal submatrix of A is nonnegative*

(C_{10}) *A is nonsingular and $A + D$ is nonsingular for each positive diagonal matrix D*

(C_{11}) *$A + D$ is nonsingular for each nonnegative diagonal matrix D*

(C_{12}) *A is nonsingular and for each $\mathbf{x} \neq \mathbf{0}$ there exists a nonnegative diagonal matrix D such that*

$$\mathbf{x}^T D\mathbf{x} \neq 0 \quad and \quad \mathbf{x}^T AD\mathbf{x} > 0$$

(C_{13}) *A is nonsingular and if $\mathbf{x} \neq \mathbf{0}$ and $\mathbf{y} = A\mathbf{x}$, then for some subscript i, $x_i \neq 0$ and $x_i y_i \geq 0$.*

(C_{14}) *A is nonsingular and for each signature matrix S there exists a vector $\mathbf{x} > \mathbf{0}$ such that*

$$SAS\mathbf{x} \geq \mathbf{0}$$

Theorem 7.12 (Continued)

(D_{15}) *$A + \alpha\mathbb{I}$ is nonsingular for each $\alpha \geq 0$*

(D_{16}) *Every real eigenvalue of A is positive*

(E_{17}) *All the leading principal minors of A are positive*

(E_{18}) *There exists lower and upper triangular matrices L and U, respectively, with positive diagonals such that*

$$A = LU$$

(F_{19}) *There exists a permutation matrix P such that PAP^T satisfies (E_{17}) or (E_{18})*

Theorem 7.12 (Continued)

(G_{20}) *A is* **positive stable**; *that is, the real part of each eigenvalue of A is positive*

(G_{21}) *There exists a symmetric positive definite matrix W such that*

$$AW + WA^T$$

is positive definite.

(G_{22}) $A + \mathbb{I}$ *is nonsingular and*

$$G = (A + \mathbb{I})^{-1}(A - \mathbb{I})$$

is convergent

Theorem 7.12 (Continued)

(G_{23}) *$A + \mathbb{I}$ is nonsingular and for*

$$G = (A + \mathbb{I})^{-1}(A - \mathbb{I})$$

there exists a positive definite matrix W such that

$$W - G^T W G$$

is positive definite

Theorem 7.12 (Continued)

(H_{24}) *There exists a positive diagonal matrix D such that*

$$AD + DA^T$$

is positive definite

(H_{25}) *The exists a positive diagonal matrix E such that for $B = E^{-1}AE$, the matrix*

$$(B + B^T)/2$$

is positive definite

(H_{26}) *For each positive semidefinite matrix Q, the matrix QA has a positive diagonal element*

Theorem 7.12 (Continued)

(I_{27}) *A is* **semipositive**; *that is, there exists* $\mathbf{x} \gg \mathbf{0}$ *with* $A\mathbf{x} \gg \mathbf{0}$

(I_{28}) *There exists* $\mathbf{x} > \mathbf{0}$ *with* $A\mathbf{x} \gg \mathbf{0}$

(I_{29}) *There exists a positive diagonal matrix* D *such that* AD *has all positive row sums*

(J_{30}) *There exists* $\mathbf{x} \gg \mathbf{0}$ *with* $A\mathbf{x} > \mathbf{0}$ *and*

$$\sum_{j=1}^{n} a_{ij} x_j > 0, \quad i = 1, \ldots, n$$

(K_{31}) *There exists a permutation matrix* P *such that* PAP^T *satisfies* (J_{30})

Theorem 7.12 (Continued)

(L_{32}) *There exists* $\mathbf{x} \gg \mathbf{0}$ *with* $\mathbf{y} = A\mathbf{x} > \mathbf{0}$ *such that if* $y_{i_0} = 0$*, then there exists a sequence of indices* $i_1, \ldots, i_r$ *with* $a_{i_{j-1}i_j} \neq 0$, $j = 1, \ldots, r$ *and with* $y_{i_r} \neq 0$

(L_{33}) *There exists* $\mathbf{x} \gg \mathbf{0}$ *with* $\mathbf{y} = A\mathbf{x} > \mathbf{0}$ *such that the matrix* $\hat{A} = [\hat{a}_{ij}]$ *defined by*

$$\hat{a}_{ij} = \begin{cases} 1 & \text{if } a_{ij} \neq 0 \text{ or } y_i \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

is irreducible

Theorem 7.12 (Continued)

(M_{34}) *There exists* $\mathbf{x} \gg \mathbf{0}$ *such that for each signature matrix* S

$$SAS\mathbf{x} \gg \mathbf{0}$$

(M_{35}) *A has all positive diagonal elements and there exists a positive diagonal matrix D such that AD is* **strictly diagonally dominant**; *that is*

$$a_{ii}d_i > \sum_{j \neq i} |a_{ij}d_j|, \qquad i = 1, \ldots, n$$

(M_{36}) *A has all positive diagonal elements and there exists a positive diagonal matrix E such that $E^{-1}AE$ is strictly diagonally dominant*

Theorem 7.12 (Continued)

(M_{37}) *A has all positive diagonal elements and there exists a positive diagonal matrix D such that AD is* **lower semistrictly diagonally dominant**; *that is,*

$$a_{ii}d_i \geq \sum_{j \neq i} |a_{ij}d_j|, \qquad i = 1, \ldots, n$$

and

$$a_{ii}d_i > \sum_{j=1}^{i-1} |a_{ij}d_j|, \qquad i = 2, \ldots, n.$$

Theorem 7.12 (Continued)

(N_{38}) *A is* **inverse-positive***; that is,* A^{-1} *exists and*

$$A^{-1} \geq 0$$

(N_{39}) *A is* **monotone***; that is,*

$$Ax \geq 0 \Rightarrow x \geq 0 \quad \text{for all } x \in \mathbb{R}^n$$

(N_{40}) *There exists inverse-positive matrices* B_1 *and* B_2 *such that*

$$B_1 \leq A \leq B_2$$

(N_{41}) *There exists an inverse-positive matrix* $B \geq A$ *such that* $I - B^{-1}A$ *is convergent*

(N_{42}) *There exists an inverse-positive matrix* $B \geq A$ *and A satisfies* (I_{27}), (I_{28}) *and* (I_{29})

Theorem 7.12 (Continued)

(N_{43}) *There exists an inverse-positive matrix $B \geq A$ and a nonsingular M-matrix C such that*

$$A = BC$$

(N_{44}) *There exists an inverse-positive matrix B and a nonsingular M-matrix C such that*

$$A = BC$$

(N_{45}) *A has a* **convergent regular splitting***; that is, A has a representation*

$$A = M - N, \quad M^{-1} \geq 0, \quad N \geq 0$$

where $M^{-1}N$ is convergent

Theorem 7.12 (Continued)

(N_{46}) *A has a* **convergent weak regular splitting**; *that is, A has a representation*

$$A = M - N, \quad M^{-1} \geq 0, \quad M^{-1}N \geq 0$$

where $M^{-1}N$ *is convergent*

(O_{47}) *Each weak regular splitting of A is convergent*

(P_{48}) *Every regular splitting of A is convergent*

(Q_{49}) *For each* $\mathbf{y} \geq \mathbf{0}$ *the set*

$$S_{\mathbf{y}} = \{\mathbf{x} \geq \mathbf{0} : A^T\mathbf{x} \leq \mathbf{y}\}$$

is bounded and A is nonsingular

(Q_{50}) $S_{\mathbf{0}} = \{\mathbf{0}\}$; *that is, the inequalities* $A^b x \leq \mathbf{0}$ *and* $\mathbf{x} \geq \mathbf{0}$ *have only the trivial solution* $\mathbf{x} = \mathbf{0}$ *and A is nonsingular*

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