MATH 4370/7370 – Linear Algebra and Matrix Analysis

Nonnegative matrices

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Outline

Definition 6.1 (Nonnegative/positive matrix)

A matrix $A \in \mathcal{M}_{mn}(\mathbb{R})$ is a **nonnegative matrix** if $a_{ij} \geq 0$ for all $i = 1, \ldots, m$ and $j = 1, \ldots, n$. We write $A \geq \mathbf{0}$ or, often here, $\mathbf{0} \leq A \in \mathcal{M}_n(\mathbb{R})$

A is a **positive matrix** if $a_{ij} > 0$ for all $i = 1, \ldots, m$ and $j = 1, \ldots, n$. We write $A > \mathbf{0}$ or, often here, $\mathbf{0} < A \in \mathcal{M}_n(\mathbb{R})$

Unless otherwise specified, all matrices here have real entries, so read $\mathcal{M}_n$ as meaning $\mathcal{M}_n(\mathbb{R})$

Another (nicer in my view) notation

Remark 6.2

In other references, you will see

- $A \geq \mathbf{0} \iff a_{ij} \geq 0$
- $A > \mathbf{0} \iff A \geq \mathbf{0}$ *and* $A \neq \mathbf{0}$ *(i.e., there exists* (i,j)*,* $a_{ij} > 0$*)* *[positive]*
- $A \gg \mathbf{0} \iff a_{ij} > 0$ *for all* i,j *[strongly positive]*

I favour this notation over the one used in these notes, but since the former is more common in matrix theory, I use the notation of Definition 6.1 here

Notation

Let $A, B \in \mathcal{M}_{mn}(\mathbb{R})$. Nonnegativity and positivity are used to define partial orders on $\mathcal{M}_{mn}(\mathbb{R})$

- $A \geq B \iff A - B \geq \mathbf{0}$
- $A > B \iff A - B > \mathbf{0}$

The same is used for vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$: $\mathbf{x} \geq \mathbf{y}$ and $\mathbf{x} > \mathbf{y}$ if, respectively, $\mathbf{x} - \mathbf{y} \geq \mathbf{0}$ and $\mathbf{x} - \mathbf{y} > \mathbf{0}$

Note that the order is only partial: if $A \geq \mathbf{0}$ and $B \geq \mathbf{0}$, for instance, it is not necessarily possible to decide on the ordering of A and B with respect to one another

Theorem 6.3

Let A and B be nonnegative matrices of appropriate sizes. Then $A+B$ and AB are nonnegative. If $A > \mathbf{0}$ and $B \geq \mathbf{0}$, $B \neq \mathbf{0}$, then $AB \geq \mathbf{0}$ and $AB \neq \mathbf{0}$

Corollary 6.4

Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ be such that $\mathbf{x} \geq \mathbf{y}$ and $\mathbf{0} \leq A \in \mathcal{M}_{mn}$. Then $A\mathbf{x} \geq A\mathbf{y}$. Assume additionally that $\mathbf{x} \geq \mathbf{y}$, $\mathbf{x} \neq \mathbf{y}$ and $A > \mathbf{0}$. Then $A\mathbf{x} > A\mathbf{y}$

Definition 6.5

Let $P, Q \in \mathcal{M}_{mn}(\mathbb{F})$. P and Q have the same **zero-nonzero structure** if for all i, j, $p_{ij} \neq 0 \iff q_{ij} \neq 0$

Zero-nonzero structure defines an equivalence relation. Therefore, as with all equivalence relations, one only needs one representative from the equivalence class. One typical representative is defined using Boolean matrices

Definition 6.6

A **Boolean matrix** is a matrix whose entries are Boolean $\{0, 1\}$ and use Boolean arithmetics:

- $0 + 0 = 0$
- $1 + 0 = 0 + 1 = 1$
- $1 + 1 = 1$
- $0 \cdot 0 = 0 \cdot 1 = 1 \cdot 0 = 0$
- $1 \cdot 1 = 1$

Definition 6.7

Let $A \in \mathcal{M}_{mn}(\mathbb{F})$. Then A_B denotes the **Boolean representation** of A, defined as follows. If $A = [a_{ij}]$, then $A_B = [\alpha_{ij}]$ with

$$\alpha_{ij} = \begin{cases} 1 & \text{if } a_{ij} \neq 0 \\ 0 & \text{if } a_{ij} = 0 \end{cases}$$

Theorem 6.15 (Perron-Frobenius)

Let $\mathbf{0} \leq A \in \mathcal{M}_n$ *be* **irreducible**. *Then the spectral radius* $\rho(A) = \max\{|\lambda|, \lambda \in \sigma(A)\}$ *is an eigenvalue of* A. *It is simple (has algebraic multiplicity 1), positive and is associated with a positive eigenvector. Furthermore, there is no nonnegative eigenvector associated to any other eigenvalue of* A

Remark 6.16

We often say that $\rho(A)$ *is the* **Perron root** *of* A*; the corresponding eigenvector is the* **Perron vector** *of* A

Lemma 6.17 (Perron)

Let $\mathbf{0} < A \in \mathcal{M}_n$. Then $\rho(A)$ is a positive eigenvalue of A and there is only one linearly independent eigenvector associated to $\rho(A)$, which can be taken to be positive

Lemma 6.18

Let $\alpha_1, \dots, \alpha_n \in \mathbb{R}_+ \setminus \{0\}$ and $v_1, \dots, v_n \in \mathbb{C}$. Then

$$\left| \sum_{i=1}^{n} \alpha_i v_i \right| \leq \sum_{i=1}^{n} \alpha |v_i| \tag{1}$$

with equality if and only if there exists $\eta \in \mathbb{C}$, $|\eta| = 1$, such that $\eta v_i \geq 0$ for all $i = 1, \dots, n$

Theorem 6.19

Let $A \in \mathcal{M}_n$ and $f(x)$ a polynomial. Then

$$\sigma(f(A)) = \{f(\lambda_1), \ldots, f(\lambda_n), \lambda_i \in \sigma(A)\}$$

If we have $g(\lambda_i) \neq 0$ for $\lambda_i \in \sigma(A)$, for some polynomial g, then the matrix $g(A)$ is non-singular and

$$\sigma\left(f(A)g(A)^{-1}\right) = \left\{\frac{f(\lambda_1)}{g(\lambda_1)}, \ldots, \frac{f(\lambda_n)}{g(\lambda_n)}, \lambda_i \in \sigma(A)\right\}$$

If $x \neq 0$ eigenvector of A associated to $\lambda \in \sigma(A)$, then x is also an eigenvector of $f(A)$ and $f(A)g(A)^{-1}$ associated to eigenvalue $f(\lambda)$ and $f(\lambda)/g(\lambda)$, respectively

Lemma 6.20 (Schur's lemma)

Let $A \in \mathcal{M}_n$ and $\lambda \in \sigma(A)$. Then λ is simple if and only if both the following conditions are statisfied:

1. *There exists only one linear independent eigenvector of A associated to λ, say $\mathbf{u}$, and thus only one linear independent eigenvector of A^T associated to λ, say $\mathbf{v}$*
2. *Vectors $\mathbf{u}$ and $\mathbf{v}$ in (1) satisfy $\mathbf{v}^T\mathbf{u} \neq 0$*

Definition 6.21

Let $\mathbf{0} \leq A \in \mathcal{M}_n$. We say that A is **primitive** with **primitivity index** $k \in \mathbb{N}_+ \setminus \{0\}$ if there exists $k \in \mathbb{N}_+ \setminus \{0\}$ such that

$$A^k > \mathbf{0}$$

with k the smallest integer for which this is true. We say that a matrix is **imprimitive** if it is not primitive

Remark 6.22

Primitivity implies irreducibility. The converse is not true

Theorem 6.23

A sufficient condition for primitivity of $\mathbf{0} \leq A \in \mathcal{M}_n$ is irreducibility with at least one positive diagonal entry

Here d is the index of imprimitivity (i.e., the number of eigenvalues that have the same modulus as $\lambda_p = \rho(A)$). If $d = 1$, then A is primitive. We have that $d = \gcd$ of all the lengths of closed walks in $G(A)$

Theorem 6.24

Let $\mathbf{0} \leq A \in \mathcal{M}_n$. If A is primitive, then $A^k > \mathbf{0}$ for some $0 < k \leq (n-1)n^n$

Theorem 6.25

Let $\mathbf{0} \leq A \in \mathcal{M}_n$ be primitive. Suppose the shortest simple directed cycle in $G(A)$ has length s, then the primitivity index of A is $\leq n + s(n-1)$

Theorem 6.26 (Wielandt's Theorem)

Let $\mathbf{0} \leq A \in \mathcal{M}_n$. A is primitive if and only if $A^{n^2-2n+2} > \mathbf{0}$

Theorem 6.27

Let $\mathbf{0} \leq A \in \mathcal{M}_n$ be irreducible. Suppose that A has d positive entries on the diagonal. Then the primitivity index of A is $\leq 2n - d - 1$

Theorem 6.28

Let $\mathbf{0} \leq A \in \mathcal{M}_n$. *Then there exists* $\mathbf{0} \neq \mathbf{v} \geq \mathbf{0}$ *such that* $A\mathbf{v} = \rho(A)\mathbf{v}$

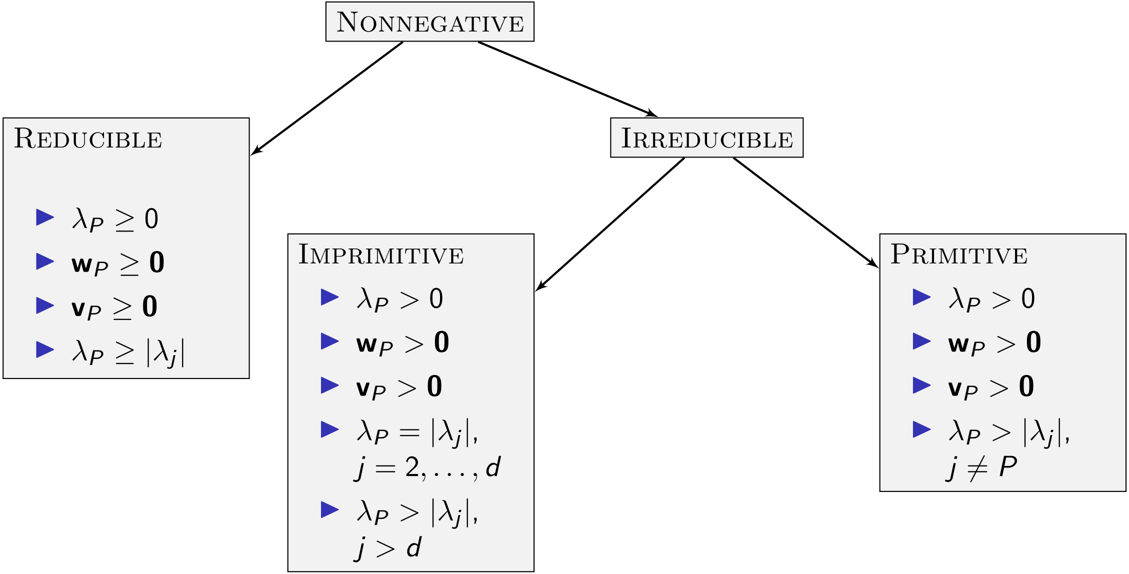
Let us restate the Perron-Frobenius theorem, taking into account the different cases. The classification in the result below follows the presentation in [**?**].

Theorem 6.29

Let $\mathbf{0} \leq A \in \mathcal{M}_n$. Denote λ_P the Perron root of A, i.e., $\lambda_P = \rho(A)$, $\mathbf{v}_P$ and $\mathbf{w}_P$ the corresponding right and left Perron vectors of A, respectively

Denote d the index of imprimitivity of A (with $d = 1$ when A is primitive) and $\lambda_j \in \sigma(A)$ the spectrum of A, with $j = 2, \ldots, n$ unless otherwise specified (assuming $\lambda_1 = \lambda_P$)

Conclusions of the Perron-Frobenius Theorem can be summarised as in the graph on the next slide


NONNEGATIVE
REDUCIBLE
$\lambda_P \geq 0$
$\mathbf{w}_P \geq \mathbf{0}$
$\mathbf{v}_P \geq \mathbf{0}$
$\lambda_P \geq |\lambda_j|$
IRREDUCIBLE
IMPRIMITIVE
$\lambda_P > 0$
$\mathbf{w}_P > \mathbf{0}$
$\mathbf{v}_P > \mathbf{0}$
$\lambda_P = |\lambda_j|$, $j = 2, \ldots, d$
$\lambda_P > |\lambda_j|$, $j > d$
PRIMITIVE
$\lambda_P > 0$
$\mathbf{w}_P > \mathbf{0}$
$\mathbf{v}_P > \mathbf{0}$
$\lambda_P > |\lambda_j|$, $j \neq P$

Theorem 6.30

Let $\mathbf{0} \leq A \in \mathcal{M}_n$ be irreducible and $h \in \mathbb{N}_+$. TFAE:

1. *There exists exactly h distinct eigenvalues such that $|\lambda| = \rho(A)$*
2. *There exists P a permutation matrix such that*

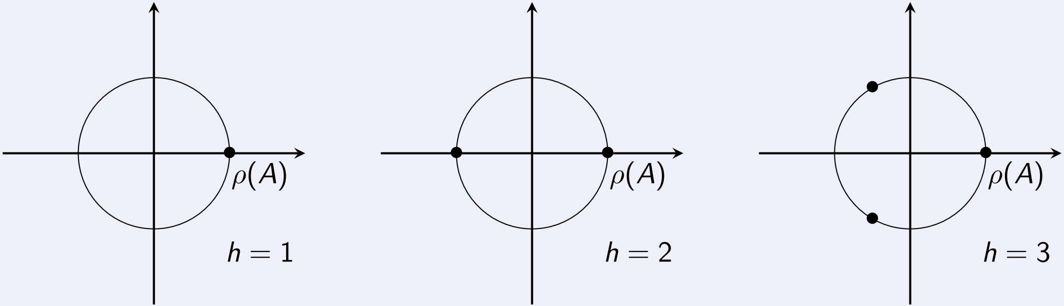
$$PAP^T = \begin{pmatrix} 0 & A_{12} & 0 & \dots & 0 \\ \vdots & & A_{23} & \ddots & \vdots \\ \vdots & & & \ddots & 0 \\ \vdots & & & & A_{h-1,h} \\ A_{h1} & 0 & \dots & \dots & 0 \end{pmatrix}$$

   where the diagonal blocks are square, and there does not exists other permutation matrix giving less than h horizontal blocks.

3. *The greatest common divisor of the lengths of all cycles in $G(A)$ is h*
4. *h is the maximal positive integer k such that $\sigma(e^{2\pi i/k}A) = \sigma(A)$*

Corollary 6.31

Let $\mathbf{0} \leq A \in \mathcal{M}_n$ be irreducible with exactly h distinct eigenvalues of modulus $\rho(A)$. Then these eigenvalues are the vertices of a regular polygon of h sides with centre at the origin and are of the vertices being $\rho(A)$



Remark 6.32

For Fiedler, a primitive matrix is defined as an irreducible nonnegative matrix such that $h = 1$

Theorem 6.33

Let $\mathbf{0} \leq A \in \mathcal{M}_n$, $n \geq 2$. TFAE

1. $A^n = \mathbf{0}$
2. *There exists* $\mathbb{N} \ni k > 0$ *such that* $A^k = \mathbf{0}$
3. $G(A)$ *is acyclic*
4. $\exists P$*, permutation matrix, s.t.* PAP^T *is upper-triangular with zeros on main diagonal*
5. $\rho(A) = 0$

Theorem 6.34

Let $\mathbf{0} \leq A \in \mathcal{M}_n$. Assume that A has a positive eigenvector. Then that eigenvector is the Perron vector and is associated to $\rho(A)$

Definition 6.35 (Stochastic matrix)

The matrix $A \in \mathcal{M}_n$ is **row stochastic** (sometimes **right stochastic**) if

- $A \geq \mathbf{0}$ [The matrix is nonnegative]
- $A\mathbb{1} = \mathbb{1}$, $\mathbb{1} = (1, \dots, 1)^T$ [All rows sum to 1]

The matrix $A \in \mathcal{M}_n$ is **column stochastic** (sometimes **left stochastic**) if

- $A \geq \mathbf{0}$ [The matrix is nonnegative]
- $\mathbb{1}^T A = \mathbb{1}^T$ [All columns sum to 1]

The matrix $A \in \mathcal{M}_n$ is **stochastic** if it is either row or column stochastic. The matrix is **doubly stochastic** if it is both row and column stochastic

Theorem 6.36

Let $A \in \mathcal{M}_n$ be stochastic. Then $\rho(A) = 1$

Theorem 6.37

Let $\mathbf{0} \leq P \in \mathcal{M}_n$. Assume that P has a positive eigenvector $\mathbf{u}$ and that $\rho(P) > 0$. Then there exists D, diagonal matrix with $\operatorname{diag}(D) > \mathbf{0}$, and $0 < k \in \mathbb{R}$ such that

$$A = kDPD^{-1}$$

is stochastic, with $k = \rho(P)^{-1}$

Theorem 6.38

Let $A, B \in \mathcal{M}_n$ be stochastic. Then AB is stochastic

Theorem 6.39

Let A be stochastic and primitive. Then $A^k \to \mathbb{1}\mathbf{v}^T$, $k \to \infty$, where $\mathbb{1}\mathbf{v}^T$ has rank 1 and $\mathbf{v}$ is the (left) eigenvector of A^T associated to $\rho(A) = 1$ and normalised so that $\mathbf{v}^T\mathbb{1} = 1$

Remark 6.40

This is a result that is used to compute the limit of a regular Markov chain

Definition 6.41

The matrix $\mathbf{0} \leq A \in \mathcal{M}_n$ is **doubly stochastic** if $A\mathbb{1} = \mathbb{1}$ and $\mathbb{1}^T A = \mathbb{1}^T$

Remark 6.42

Here $\rho(A) = 1$ is associated to $\mathbb{1}$ for A and for A^T

Consider E the Euclidean space. A set K of points in E is **convex** if A_1, A_2 points in K, $\lambda_1, \lambda_2 \in \mathbb{R}_+$ such that $\lambda_1 + \lambda_2 = 1$, then

$$\lambda_1 A_1 + \lambda_2 A_2 \in K.$$

A **convex polyhedron** K is the set of all points of the form

$$\sum_{i=1}^{N} \lambda_i A_i$$

where A_i are points in E and $\lambda_1 \in \mathbb{R}_+$

Let $A = [a_{ij}] \in \mathcal{M}_n(\mathbb{R})$. Consider this matrix as a point in E with coordinates $[a_{11}, a_{12}, \ldots, a_{nn}]$ (dim $E = n^2$)

Theorem 6.43

Let $A \in \mathcal{M}_n$, $A = [a_{ij}]$, if A is doubly stochastic, then this forms an $(n-1)^2$ dimensional subspace of $\tilde{E} = \mathbb{R}^{n^2}$

Theorem 6.44 (Birkhoff)

In the space $\tilde{E} = R^{n^2}$, the set of doubly stochastic matrices of order n is a convex polyhedron in E (the subspace of stochastic matrices). The vertices of the polyhedron are the points corresponding to all the permutation matrices

References I