

MATH 4370/7370 – Linear Algebra and Matrix Analysis

Essentially nonnegative matrices and M-matrices

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Outline

Essentially nonnegative matrices

Z-matrices

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The Perron-Frobenius can be applied not only to nonnegative matrices, but also to matrices that are *essentially nonnegative*, in the sense that they are nonnegative except perhaps along the main diagonal

Definition 7.1

A matrix $A \in \mathcal{M}_n$ is **essentially nonnegative** (or **quasi-positive**) if there exist $\alpha \in \mathbb{R}$ such that $A + \alpha \mathbb{I} \geq 0$

Remark 7.2

An essentially nonnegative matrix A has non-negative off-diagonal entries. The sign of the diagonal entries is not relevant

Remark 7.3

Irreducibility of a matrix is not affected by the nature of its diagonal entries. Indeed, consider an essentially nonnegative matrix A . The existence of a directed path in $G(A)$ does not depend on the existence of “self-loops”. The same is not true of primitive matrices, where the presence of negative entries on the main diagonal has an influence on the values of A^k and thus ultimately, on the capacity to find k such that $A^k > 0$

So we can apply the “weak” versions of the Perron-Frobenius Theorem (the imprimitive cases in Theorem ??) to $A + \alpha \mathbb{I}$, which is a nonnegative matrix (potentially irreducible). One important ingredient is a result that was proved as Theorem ?. Namely, that perturbations of the entire diagonal by the same scalar lead to a shift of the spectrum; this is summarised as

$$\sigma(A + \alpha \mathbb{I}) = \{\lambda_1 + \alpha, \dots, \lambda_n + \alpha, \quad \lambda_i \in \sigma(A)\}$$

Definition 7.4 (Spectral abscissa)

Let $A \in \mathcal{M}_n$. The **spectral abscissa** of A , $s(A)$, is

$$s(A) = \max\{\operatorname{Re}(\lambda), \lambda \in \sigma(A)\}$$

Theorem 7.5

Let $A \in \mathcal{M}_n(\mathbb{R})$ be essentially nonnegative. Then $s(A)$ is an eigenvalue of A and is associated to a nonnegative eigenvector. If, additionally, A is irreducible, then $s(A)$ is simple and is associated to a positive eigenvector

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Definition 7.6

A matrix is of class Z_n if it is in $\mathcal{M}_n(\mathbb{R})$ and such that $a_{i,j} \leq 0$, $i \neq j$, $i, j = 1, \dots, n$

$$Z_n = \{A \in \mathcal{M}_n : a_{i,j} \leq 0, i \neq j\}$$

We also say that $A \in Z_n$ has the **Z-sign pattern**

Theorem 7.7 ([?])

Let $A \in Z_n$. TFAE

1. There is a nonnegative vector x such that $Ax > 0$
2. There is a positive vector x such that $Ax > 0$
3. There is a diagonal matrix $\text{diag}(D) > 0$ such that the entries in $AD = [w_{ik}]$ are such that

$$w_{ii} > \sum_{k \neq i} |w_{ik}| \forall i$$

4. For any $B \in Z_n$ such that $A \geq B$, then B is nonsingular
5. Every real eigenvalue of any principal submatrix of A is positive.
6. All principal minors of A are positive

Theorem 7.7 (Continued)

7. *For all $k = 1, \dots, n$, the sum of all principal minors is positive*
8. *Every real eigenvalue of A is positive*
9. *There exists a matrix $C \geq 0$ and a number $k > \rho(A)$ such that $A = k\mathbb{I} - C$*
10. *There exists a splitting $A = P - Q$ of the matrix A such that $P^{-1} \geq 0$, $Q \geq 0$, and $\rho(P^{-1}Q) < 1$*
11. *A is nonsingular and $A^{-1} \geq 0$*
12. ...
18. *The real part of any eigenvalue of A is positive*

Notation: $A \in Z_n$ such that any (and therefore all) of these properties holds is a matrix of class K (or a nonsingular M -matrix).

Theorem 7.8

Let $A \in Z = \bigcap_{i=1, \dots} Z_n$ be symmetric. Then $A \in K$ if and only if A is positive definite.

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Theorem 7.9

Let $A \in Z_n$. TFAE

1. $A + \varepsilon \mathbb{I} \in K$ for all $\varepsilon > 0$
2. Every real eigenvalue of a principal submatrix of A is nonnegative
3. All principal minors of A are nonnegative
4. The sum of all principal minors of order $k = 1, \dots, n$ is nonnegative
5. Every real eigenvalue of A is nonnegative
6. There exists $C \geq 0$ and $k \geq \rho(C)$ such that $A = k\mathbb{I} - C$
7. Every eigenvalue of A has nonnegative real part

$A \in Z_n$ such that any of these properties holds is a matrix of class K_0

Theorem 7.10

Let $A \in Z_n$. Assume $A \in K_0$. Then $A \in K \iff A$ nonsingular

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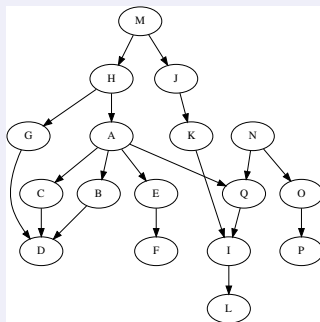
M-matrices

Definition 7.11 (Signature matrix)

A **signature matrix** is a diagonal matrix S with diagonal entries ± 1

Theorem 7.12 ([?])

Let $A \in \mathcal{M}_n$. Then for each fixed letter $\mathcal{C}$ representing one of the following conditions, conditions $\mathcal{C}_i$ are equivalent for each i . Moreover, letting $\mathcal{C}$ then represent any of the equivalent conditions $\mathcal{C}_i$, the following implication tree holds:



If $A \in Z_n$, each of the following conditions is equivalent to the statement “ A is a nonsingular M -matrix”

Theorem 7.12 (Continued)

(A₁) *All the principal minors of A are positive*

(A₂) *Every real eigenvalue of each principal submatrix of A is positive*

(A₃) *For each $\mathbf{x} \neq \mathbf{0}$ there exists a positive diagonal matrix D such that*

$$\mathbf{x}^T A D \mathbf{x} > 0$$

(A₄) *For each $\mathbf{x} \neq \mathbf{0}$ there exists a nonnegative diagonal matrix D such that*

$$\mathbf{x}^T A D \mathbf{x} > 0$$

(A₅) *A does not reverse the sign of any vector; that is, if $\mathbf{x} \neq \mathbf{0}$ and $\mathbf{y} = A\mathbf{x}$, then for some subscript i , $x_i y_i > 0$*

(A₆) *For each signature matrix S , there exists an $\mathbf{x} \gg \mathbf{0}$ such that*

$$S A S \mathbf{x} \gg \mathbf{0}$$

Theorem 7.12 (Continued)

- (B₇) *The sum of all the $k \times k$ principal minors of A is positive for $k = 1, \dots, n$*
- (C₈) *A is nonsingular and all the principal minors of A are nonnegative*
- (C₉) *A is nonsingular and every real eigenvalue of each principal submatrix of A is nonnegative*
- (C₁₀) *A is nonsingular and $A + D$ is nonsingular for each positive diagonal matrix D*
- (C₁₁) *$A + D$ is nonsingular for each nonnegative diagonal matrix D*
- (C₁₂) *A is nonsingular and for each $\mathbf{x} \neq \mathbf{0}$ there exists a nonnegative diagonal matrix D such that*

$$\mathbf{x}^T D \mathbf{x} \neq 0 \quad \text{and} \quad \mathbf{x}^T A D \mathbf{x} > 0$$

- (C₁₃) *A is nonsingular and if $\mathbf{x} \neq \mathbf{0}$ and $\mathbf{y} = A\mathbf{x}$, then for some subscript i , $x_i \neq 0$ and $x_i y_i \geq 0$.*
- (C₁₄) *A is nonsingular and for each signature matrix S there exists a vector $\mathbf{x} > \mathbf{0}$ such that*

$$S A S \mathbf{x} \geq \mathbf{0}$$

Theorem 7.12 (Continued)

(D_{15}) $A + \alpha \mathbb{I}$ is nonsingular for each $\alpha \geq 0$

(D_{16}) Every real eigenvalue of A is positive

(E_{17}) All the leading principal minors of A are positive

(E_{18}) There exists lower and upper triangular matrices L and U , respectively, with positive diagonals such that

$$A = LU$$

(F_{19}) There exists a permutation matrix P such that PAP^T satisfies (E_{17}) or (E_{18})

Theorem 7.12 (Continued)

(G₂₀) A is **positive stable**; that is, the real part of each eigenvalue of A is positive

(G₂₁) There exists a symmetric positive definite matrix W such that

$$AW + WA^T$$

is positive definite.

(G₂₂) $A + \mathbb{I}$ is nonsingular and

$$G = (A + \mathbb{I})^{-1}(A - \mathbb{I})$$

is convergent

Theorem 7.12 (Continued)

(G_{23}) $A + \mathbb{I}$ is nonsingular and for

$$G = (A + \mathbb{I})^{-1}(A - \mathbb{I})$$

there exists a positive definite matrix W such that

$$W - G^T W G$$

is positive definite

Theorem 7.12 (Continued)

(H₂₄) *There exists a positive diagonal matrix D such that*

$$AD + DA^T$$

is positive definite

(H₂₅) *There exists a positive diagonal matrix E such that for $B = E^{-1}AE$, the matrix*

$$(B + B^T)/2$$

is positive definite

(H₂₆) *For each positive semidefinite matrix Q , the matrix QA has a positive diagonal element*

Theorem 7.12 (Continued)

(I_{27}) A is **semipositive**; that is, there exists $\mathbf{x} \gg \mathbf{0}$ with $A\mathbf{x} \gg \mathbf{0}$

(I_{28}) There exists $\mathbf{x} > \mathbf{0}$ with $A\mathbf{x} \gg \mathbf{0}$

(I_{29}) There exists a positive diagonal matrix D such that AD has all positive row sums

(J_{30}) There exists $\mathbf{x} \gg \mathbf{0}$ with $A\mathbf{x} > \mathbf{0}$ and

$$\sum_{j=1}^n a_{ij}x_j > 0, \quad i = 1, \dots, n$$

(K_{31}) There exists a permutation matrix P such that PAP^T satisfies (J_{30})

Theorem 7.12 (Continued)

- (L₃₂) *There exists $\mathbf{x} \gg \mathbf{0}$ with $\mathbf{y} = A\mathbf{x} > \mathbf{0}$ such that if $y_{i_0} = 0$, then there exists a sequence of indices $i_1, \dots, i_r$ with $a_{i_{j-1}i_j} \neq 0$, $j = 1, \dots, r$ and with $y_{i_r} \neq 0$*
- (L₃₃) *There exists $\mathbf{x} \gg \mathbf{0}$ with $\mathbf{y} = A\mathbf{x} > \mathbf{0}$ such that the matrix $\hat{A} = [\hat{a}_{ij}]$ defined by*

$$\hat{a}_{ij} = \begin{cases} 1 & \text{if } a_{ij} \neq 0 \text{ or } y_i \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

is irreducible

Theorem 7.12 (Continued)

(M_{34}) *There exists $\mathbf{x} \gg \mathbf{0}$ such that for each signature matrix S*

$$SAS\mathbf{x} \gg \mathbf{0}$$

(M_{35}) *A has all positive diagonal elements and there exists a positive diagonal matrix D such that AD is **strictly diagonally dominant**; that is*

$$a_{ii}d_i > \sum_{j \neq i} |a_{ij}d_j|, \quad i = 1, \dots, n$$

(M_{36}) *A has all positive diagonal elements and there exists a positive diagonal matrix E such that $E^{-1}AE$ is strictly diagonally dominant*

Theorem 7.12 (Continued)

(M_{37}) *A has all positive diagonal elements and there exists a positive diagonal matrix D such that AD is **lower semistrictly diagonally dominant**; that is,*

$$a_{ii}d_i \geq \sum_{j \neq i} |a_{ij}d_j|, \quad i = 1, \dots, n$$

and

$$a_{ii}d_i > \sum_{j=1}^{i-1} |a_{ij}d_j|, \quad i = 2, \dots, n.$$

Theorem 7.12 (Continued)

(N₃₈) *A is **inverse-positive**; that is, A^{-1} exists and*

$$A^{-1} \geq 0$$

(N₃₉) *A is **monotone**; that is,*

$$Ax \geq 0 \Rightarrow x \geq 0 \quad \text{for all } x \in \mathbb{R}^n$$

(N₄₀) *There exists inverse-positive matrices B_1 and B_2 such that*

$$B_1 \leq A \leq B_2$$

(N₄₁) *There exists an inverse-positive matrix $B \geq A$ such that $I - B^{-1}A$ is convergent*

(N₄₂) *There exists an inverse-positive matrix $B \geq A$ and A satisfies (I_{27}) , (I_{28}) and (I_{29})*

Theorem 7.12 (Continued)

(N₄₃) *There exists an inverse-positive matrix $B \geq A$ and a nonsingular M-matrix C such that*

$$A = BC$$

(N₄₄) *There exists an inverse-positive matrix B and a nonsingular M-matrix C such that*

$$A = BC$$

(N₄₅) *A has a **convergent regular splitting**; that is, A has a representation*

$$A = M - N, \quad M^{-1} \geq 0, \quad N \geq 0$$

where $M^{-1}N$ is convergent

Theorem 7.12 (Continued)

(N₄₆) *A has a **convergent weak regular splitting**; that is, A has a representation*

$$A = M - N, \quad M^{-1} \geq 0, \quad M^{-1}N \geq 0$$

where $M^{-1}N$ is convergent

(O₄₇) *Each weak regular splitting of A is convergent*

(P₄₈) *Every regular splitting of A is convergent*

(Q₄₉) *For each $\mathbf{y} \geq \mathbf{0}$ the set*

$$S_{\mathbf{y}} = \{\mathbf{x} \geq \mathbf{0} : A^T \mathbf{x} \leq \mathbf{y}\}$$

is bounded and A is nonsingular

(Q₅₀) *$S_{\mathbf{0}} = \{\mathbf{0}\}$; that is, the inequalities $A^b \mathbf{x} \leq \mathbf{0}$ and $\mathbf{x} \geq \mathbf{0}$ have only the trivial solution $\mathbf{x} = \mathbf{0}$ and A is nonsingular*

References I