Graphs – Introduction (theory)

MATH 2740 – Mathematics of Data Science – Lecture 16

Julien Arino

julien.arino@umanitoba.ca

Department of Mathematics @ University of Manitoba

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The University of Manitoba campuses are located on original lands of Anishinaabeg, Ininew, Anisininew, Dakota and Dene peoples, and on the National Homeland of the Red River Métis. We respect the Treaties that were made on these territories, we acknowledge the harms and mistakes of the past, and we dedicate ourselves to move forward in partnership with Indigenous communities in a spirit of Reconciliation and collaboration.

Outline

Undirected graphs

Directed graphs

Matrices associated to a graph/digraph

Trees

Undirected graphs

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Matrices associated to a graph/digraph

Trees

Undirected graphs

- **Connectedness**
- Walks, trails, paths
- Complete, bipartite and other notable graphs
- Planar graphs
- Graph colouring

Connected vertices and graph, components

Definition 1 (Connected vertices)

Two vertices u and v in a graph G are **connected** if $u = v$, or if $u \neq v$ and there exists a path in G that links u and v

(For *path*, see Definition 14 later)

Definition 2 (Connected graph)

A graph is **connected** if every two vertices of G are connected; otherwise, G is **disconnected**

A necessary condition for connectedness

Theorem 3

A connected graph on p vertices has at least $p - 1$ edges

In other words, a connected graph G of order p has size$(G) \geq p - 1$

Connectedness is an equivalence relation

Denote $x \equiv y$ the relation "$x = y$, or $x \neq y$ and there exists a path in G connecting x and y". $\equiv$ is an equivalence relation since

1. $x \equiv y$ [reflexivity]
2. $x \equiv y \implies y \equiv x$ [symmetry]
3. $x \equiv y, y \equiv z \implies x \equiv z$ [transitivity]

Definition 4 (Connected component of a graph)

The classes of the equivalence relation $\equiv$ partition V into connected sub-graphs of G called **connected components** (or **components** for short) of G

A connected subgraph H of a graph G is a component of G if H is not contained in any connected subgraph of G having more vertices or edges than H

Vertex deletion & cut vertices

Definition 5 (Vertex deletion)

If $v \in V(G)$ is a vertex of G, the graph $G - v$ is the graph formed from G by removing v and all edges incident with v

Definition 6 (Cut-vertices)

Let G be a connected graph. Then v is a **cut-vertex** G if $G - v$ is disconnected

Edge deletion & bridges

Definition 7 (Edge deletion)

If e is an edge of G, the graph $G - e$ is the graph formed from G by removing e from G

Definition 8 (Bridge)

An edge e in a connected graph G is a **bridge** if $G - e$ is disconnected

Theorem 9

Let G be a connected graph. An edge e of G is a bridge of G $\iff$ e does not lie on any cycle of G

(For *cycle*, see Definition 17 later)

Undirected graphs

- Connectedness
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Walk

Definition 10 (Walk)

A **walk** in a graph $G = (V, E)$ is a non-empty alternating sequence $v_0 e_0 v_1 e_1 v_2 \ldots e_{k-1} v_k$ of vertices and edges in G such that $e_i = \{v_i, v_{i+1}\}$ for all $i < k$. This walk begins with v_0 and ends with v_k

Definition 11 (Length of a walk)

The **length** of a walk is equal to the number of edges in the walk

Definition 12 (Closed walk)

If $v_0 = v_k$, the walk is **closed**

Trail and path

Definition 13 (Trail)

If the edges in the walk are all distinct, it defines a **trail** in $G = (V, E)$

Definition 14 (Path)

If the vertices in the walk are all distinct, it defines a **path** in G

The sets of vertices and edges determined by a trail is a subgraph

Distance between two vertices

Definition 15 (Distance between two vertices)

The **distance** $d(u, v)$ in $G = (V, E)$ between two vertices u and v is the length of the shortest path linking u and v in G

If no such path exists, we assume $d(u, v) = \infty$

Circuit and cycle

Definition 16 (Circuit)

A trail linking u to v, containing at least 3 edges and in which $u = v$, is a **circuit**

Definition 17 (Cycle)

A circuit which does not repeat any vertices (except the first and the last) is a **cycle** (or **simple circuit**)

Definition 18 (Length of a cycle)

The **length of a cycle** is its number of edges

Definition 19 (Eulerian trail)

A walk in an undirected multigraph M that uses each edge **exactly once** is a **Eulerian trail** of M

Definition 20 (Traversable graph)

If a graph G has a Eulerian trail, then G is a **traversable graph**

Definition 21 (Eulerian circuit)

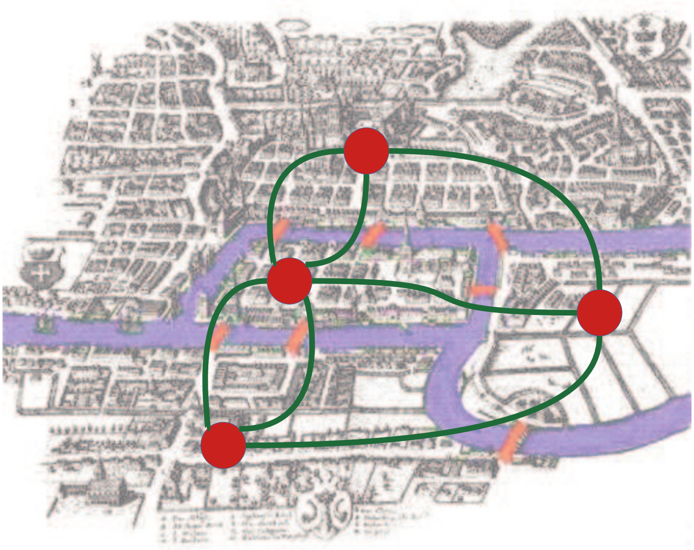
A circuit containing all the vertices and edges of a multigraph M is a **Eulerian circuit** of M

Definition 22 (Eulerian graph)

A graph (resp. multigraph) containing an Eulerian circuit is a **Eulerian graph** (resp. **Eulerian multigraph**)

Remember Euler's bridges of Königsberg?

Cross the 7 bridges in a single walk without recrossing any of them?



Theorem 23

A multigraph M is traversable $\iff$ M is connected and has exactly two odd vertices

Furthermore, any Eulerian trail of M begins at one of the odd vertices and ends at the other odd vertex

Theorem 24

A multigraph M is Eulerian $\iff$ M is connected and every vertex of M is even

Fleury's algorithm to find a Eulerian *trail*

For a connected graph with exactly 2 odd vertices

- Start at one of the odd vertices
- Marking your path as you move from vertex to vertex, travel along any edges you wish, but DO NOT travel along an edge that is a bridge for the graph formed by the EDGES THAT HAVE YET TO BE TRAVELED – unless you have to
- Continue until every edge has been traveled

RESULT: a Eulerian trail

Fleury's algorithm to find a Eulerian *circuit*

For a connected graph with no odd vertices

- Pick any vertex as a starting point
- Marking your path as you move from vertex to vertex, travel along any edges you wish, but DO NOT travel along an edge that is a bridge for the graph formed by the EDGES THAT HAVE YET TO BE TRAVELED – unless you have to
- Continue until you return to your starting point

RESULT: a Eulerian circuit

Definition 25 (Hamiltonian path)

A path containing all vertices of a graph G is a **Hamiltonian path** of G

Definition 26 (Traceable graph)

If a graph G has an Hamiltonian path, then G is a **traceable graph**

Definition 27 (Hamiltonian cycle)

A cycle containing all vertices of a graph G is a **Hamiltonian cycle** of G

Definition 28 (Hamiltonian graph)

A graph containing a Hamiltonian cycle is a **Hamiltonian graph**

Theorem 29 (Dirac's theorem)

If G is a graph of order $p \geq 3$ such that $deg(v) \geq p/2$ for every vertex v of G, then G is Hamiltonian

Theorem 30 (Ore's theorem)

If G is a graph of order $p \geq 3$ such that for all distinct nonadjacent vertices u and v of G,

$$deg(u) + deg(v) \geq p$$

then G is Hamiltonian

Undirected graphs

- Connectedness
- Walks, trails, paths
- **Complete, bipartite and other notable graphs**
- Planar graphs
- Graph colouring

Definition 31 (Complete graph)

A graph is complete if every two of its vertices are adjacent

Definition 32 (n-clique)

A simple, complete graph on n vertices is called an n-**clique** and is often denoted K_n

Note that a complete graph of order p is $(p-1)$-regular

Bipartite graph

Definition 33 (Bipartite graph)

A graph is **bipartite** if its vertices can be partitioned into two sets V_1 and V_2, such that no two vertices in the same set are adjacent. This graph may be written $G = (V_1, V_2, E)$

Definition 34 (Complete bipartite graph)

A bipartite graph in which every two vertices from the 2 different partitions are adjacent is called a **complete bipartite graph**

We often denote $K_{p,q}$ a simple, complete bipartite graph with $|V_1| = p$ and $|V_2| = q$

Some specific graphs

Definition 35 (Tree)

Any connected graph that has no cycles is a **tree**

Definition 36 (Cycle C_n)

For $n \geq 3$, the **cycle** C_n is a connected graph of order n that is a cycle on n vertices

Definition 37 (Path P_n)

The **path** P_n is a connected graph that consists of $n \geq 2$ vertices and $n - 1$ edges. Two vertices of P_n have degree 1 and the rest are of degree 2

Definition 38 (Star S_n)

The **star** of order n is the complete bipartite graph $K_{1,n-1}$ (1 vertex of degree $n - 1$ and $n - 1$ vertices of degree 1)

Undirected graphs

- Connectedness
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Planar graph

Definition 39 (Planar graph)

A graph is **planar** if it *can be* drawn in the plane with no crossing edges (except at the vertices). Otherwise, it is **nonplanar**

Definition 40 (Plane graph)

A **plane graph** is a graph *that is drawn* in the plane with no crossing edges. (This is only possible if the graph is planar)

(To see the difference, have you ever played this game?)

Let G be a plane graph

- the connected parts of the plane are called **regions**
- vertices and edges that are incident with a region R make up a **boundary** of R

Theorem 41 (Euler's formula)

Let G be a connected plane graph with p vertices, q edges, and r regions, then

$$p - q + r = 2$$

Corollary 42

Let G be a plane graph with p vertices, q edges, r regions, and k connected components, then

$$p - q + r = k + 1$$

Theorem 43

Let G be a connected planar graph with p vertices and q edges, where $p \geq 3$, then

$$q \leq 3p - 6.$$

(a maximal connected planar graph with p vertices has $q = 3p - 6$ edges)

Corollary 44

If G is a planar graph, then $\delta(G) \leq 5$, where $\delta(G)$ is the minimal degree of G. (every planar graph contains a vertex of degree less than 6)

Two well-known non-planar graphs

$K_{3,3}$ and K_5 are nonplanar

Theorem 45 (Kuratowski Theorem)

A graph G is planar $\iff$ it contains no subgraph isomorphic to K_5 or $K_{3,3}$ or any subdivision of K_5 or $K_{3,3}$

Note: If a graph G is nonplanar and G is a subgraph of G', then G' is also nonplanar

Undirected graphs

- Connectedness
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- Complete, bipartite and other notable graphs
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- **Graph colouring**

Definition 46 (Colouring of a graph G)

A **colouring** of a graph G is an assignment of colours to the vertices of G such that adjacent vertices have different colours

Definition 47 (n-colouring of G)

A **n-colouring** is a colouring of G using n colours

Definition 48 (n-colourable)

G is **n-colourable** if there exists a colouring of G that uses n colours

Definition 49 (Chromatic number)

The **chromatic number** $\chi(G)$ of a graph G is the minimal value n for which an n-colouring of G exists

Property 50

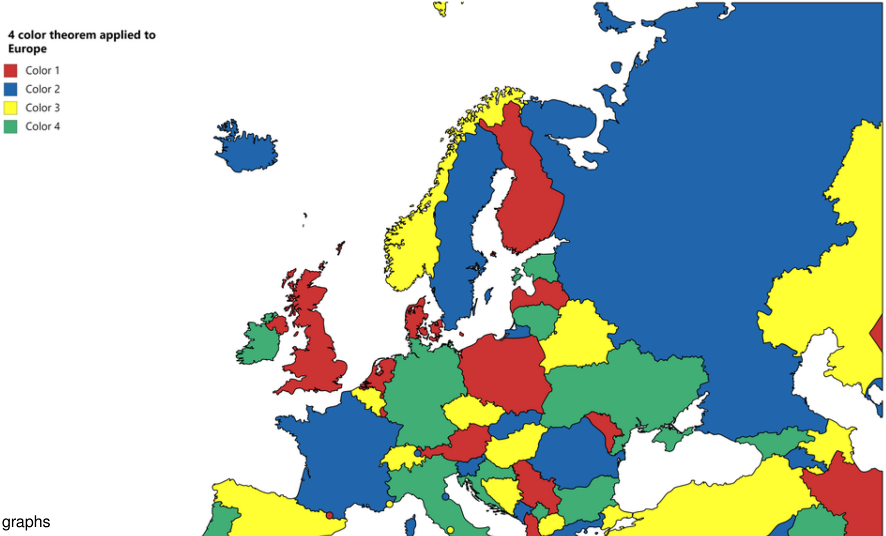
- $\chi(G) = 1 \iff G$ *have no edges*
- *If* $G = K_{n,m}$*, then* $\chi(G) = 2$
- *If* $G = K_n$*, then* $\chi(G) = n$
- *For any graph* G*,*

  $$\chi(G) \leq 1 + \Delta(G)$$

  where $\Delta(G)$ *is the maximum degree of* G
- *If* G *is a planar graph, then* $\chi(G) \leq 4$

“Real life” problem

What is the minimal number of colours to colour all states in the map so that two adjacent states have different colours?



“Real life” problem

What is the minimal number of colours to colour all states in the map so that two adjacent states have different colours?

Mathematical representation:

- vertices correspond to the states
- vertices are adjacent $\iff$ the two states are adjacent (sharing an isolated point such as the “Four Corners” does not count)

Mathematical problem

What is the chromatic number of the graph associated to the map?

Welch-Powell algorithm for colouring a graph G

1. Order the vertices of G by decreasing degree. (Such an ordering may not be unique since some vertices may have the same degree)

2. Use one colour to paint the first vertex and to paint, in sequential order, each vertex on the list that is not adjacent to a vertex previously painted with this colour

3. Start again at the top of the list and repeat the process, painting previously unpainted vertices using a second colour

4. Repeat with additional colours until all vertices have been painted

Undirected graphs

Directed graphs

Matrices associated to a graph/digraph

Trees

Definitions

Definition 51 (Digraph)

A directed graph (or **digraph**) is a pair $G = (V, A)$ of sets such that

- V is a set of points: $V = \{v_1, v_2, v_3, .., v_p\}$
- A is a set of ordered pairs of V: $A = \{(v_i, v_j), (v_i, v_k), \ldots, (v_n, v_p)\}$ or $A = \{v_i v_j, v_i v_k, \ldots, v_n v_p\}$

Definition 52 (Vertex)

The elements of V are the vertices of the digraph G. V or $V(G)$ is the vertex set of the digraph G

Definition 53 (Arc)

The elements of A are the **arcs** (directed edges) of the digraph G. A or $A(G)$ is the arc set of the digraph G

Digraph and binary relation

A (simple) digraph D can be defined in term of a vertex set V and an irreflexive relation R over V

The defining relation R of the digraph G need not be symmetric

Directed network

Definition 54 (Directed network)

A directed network is a digraph together with a function f,

$$f : A \to \mathbb{R},$$

which maps the arc set A into the set of real number. The value of the arc $uv \in A$ is $f(uv)$

Loops & Multiple arcs

Definition 55 (Loop)

A **loop** is an arc with both the same ends; *e.g.* (u, u) is a loop

Definition 56 (Multiple arcs)

Multiple arcs (or multi-arcs) are two or more arcs connecting the same two vertices

Multidigraph/Digraph

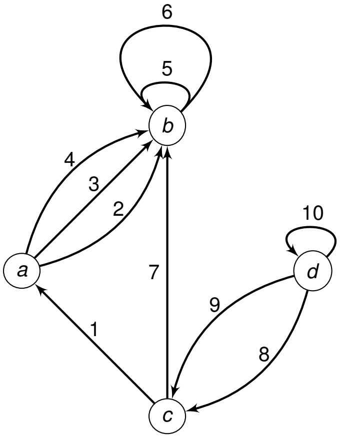
Definition 57 (Multidigraph)

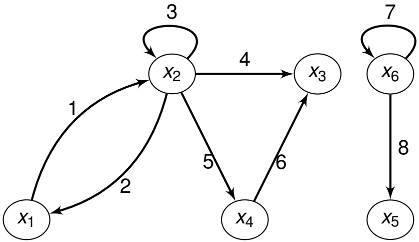
A **multidigraph** is a digraph which allows repetition of arcs or loops

Definition 58 (Digraph)

In a digraph, no more than one arc can join any pair of vertices

Examples





Let $G = (V, A)$ be a digraph

Definition 59 (Arc endpoints)

For an arc $u = (x, y)$, vertex x is the **initial endpoint**, and vertex y is the **terminal endpoint**

Definition 60 (Predecessor - Successor)

If $(u, v) \in A(G)$ is an arc of G, then

- u is a **predecessor** of v
- v is a **successor** of u

Definition 61 (Neighbours of a vertex)

Let $x \in V$ be a vertex. The **neighbours** of x is the set $\Gamma(x) = \Gamma_G^+(x) \cup \Gamma_G^-(x)$, where $\Gamma_G^+(x)$ and $\Gamma_G^-(x)$ are, respectively, the set of successors and predecessors of v

Sources and sinks

Definition 62 (Directed away - Directed towards)

If $a = (u, v) \in A(G)$ is an arc of G, then

- the arc a is said to be **directed away** from u
- the arc a is said to be **directed towards** v

Definition 63 (Source - Sink)

- Any vertex which has no arcs directed towards it is a **source**
- Any vertex which has no arcs directed away from it is a **sink**

Adjacent arcs

Definition 64 (Adjacent arcs)

Two arcs are **adjacent** if they have at least one endpoint in common

Arcs incident to a subset of arcs

Definition 65 (Arc incident out of $X \subset A(G)$)

If the initial endpoint of an arc u belongs to $X \subset A(G)$ and if the terminal endpoint of arc u does not belong to X, then u is said to be **incident out of** X; we write $u \in \omega^+(X)$

Similarly, we define an **arc incident into** X and the set $\omega^-(X)$

Finally, the set of arcs **incident to** X is denoted

$$\omega(X) = \omega^+(X) \cup \omega^-(X)$$

Definition 66 (Subgraph of G generated by $A \subset V$)

The **subgraph** of G generated by A is the graph with A as its vertex set and with all the arcs in G that have both their endpoints in A. If $G = (V, \Gamma)$ is a 1-graph, then the subgraph generated by A is the 1-graph $G_A = (A, \Gamma_A)$ where

$$\Gamma_A(x) = \Gamma(x) \cap A \qquad (x \in A)$$

Definition 67 (Partial graph of G generated by $V \subset U$)

The graph (X, V) whose vertex set is X and whose arc set is V. In other words, it is graph G without the arcs $U - V$

Definition 68 (Partial subgraph of G)

A partial subgraph of G is the subgraph of a partial graph of G

Directed graphs

Degrees in digraphs

Walks, paths, etc.

Connectivity in digraphs

Orientable graphs

Degree

Let v be a vertex of a digraph $G = (V, A)$

Definition 69 (Outdegree of a vertex)

The number of arcs directed away from a vertex v, in a digraph is called the **outdegree** of v and is written $d_G^+(v)$

Definition 70 (Indegree of a vertex)

The number of arcs directed towards a vertex v, in a digraph is called the **indegree** of v and is written $d_G^-(v)$

Definition 71 (Degree)

For any vertex v in a digraph, the **degree** of v is defined as

$$d_G(v) = d_G^+(v) + d_G^-(v)$$

Theorem 72

For any (di)graph, the sum of the degrees of the vertices equals twice the number of edges (arcs)

Corollary 73

In any (di)graph, the sum of the degrees of the vertices is a nonnegative even integer

Theorem 74

If G is a digraph with vertex set $V(G) = \{v_1, \ldots, v_p\}$ and q arcs, then

$$\sum_{i=1}^{p} d_G^+(v_i) = \sum_{i=1}^{p} d_G^-(v_i) = q$$

Definition 75 (Regular digraph)

A digraph G is r-regular if $d_G^+(v) = d_G^-(v) = r$ for all $v \in V(G)$

Symmetric/antisymmetric digraphs

Definition 76 (Symmetric digraph)

Let $G = (V, A)$ be a digraph with associated binary relation R. If R is *symmetric*, the digraph is symmetric

Definition 77 (Anti-symmetric digraph)

Let $G = (V, A)$ be a digraph with associated binary relation R. The digraph G is **anti-symmetric** if

$$xRy \implies y \not R x$$

Definition 78 (Symmetric multidigraph)

Let $G = (V, A)$ be a multidigraph. G is symmetric if $\forall x, y \in V(G)$, the number of arcs from x to y equals the number of arcs from y to x

Directed graphs

- Degrees in digraphs
- **Walks, paths, etc.**
- Connectivity in digraphs
- Orientable graphs

Walks

Let $G = (V, A)$ be a digraph.

Definition 79 (Directed walk)

A **directed walk** in a digraph G is a non-empty alternating sequence $v_0 a_0 v_1 a_1 v_2 \dots a_{k-1} v_k$ of vertices and arcs in G such that $a_i = (v_i, v_{i+1})$ for all $i < k$. This walk begins with v_0 and ends with v_k

Definition 80 (Length of a directed walk)

The length of a directed walk is equal to the number of arcs in the directed walk

Definition 81 (Closed walk)

If $v_0 = v_k$, the walk is closed

Trails

Let $G = (V, A)$ be a digraph.

Definition 82 (Directed trail)

A directed walk in G in which all arcs are distinct is a **directed trail** in G

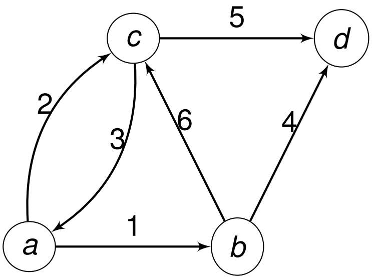
Definition 83 (Directed path)

A directed walk in G in which all vertices are distinct is a **directed path** in G

Definition 84 (Directed cycle)

A closed walk is a **directed cycle** if it contains at least three vertices and all its vertices are distinct except for $v_0 = v_k$

Examples of directed cycles



Cycles:

- $\mu^1 = (1, 6, 2) = [abca]$
- $\mu^2 = (1, 6, 3) = [abca]$
- $\mu^3 = (2, 3) = [aca]$
- $\mu^4 = (1, 4, 5, 2) = [abdca]$
- $\mu^5 = (6, 5, 4) = [acdb]$
- $\mu^6 = (1, 4, 5, 3) = [abdca]$

Directed graphs

- Degrees in digraphs
- Walks, paths, etc.
- **Connectivity in digraphs**
- Orientable graphs

Definitions

Definition 85 (Underlying graph)

Given a digraph, the undirected graph with each arc replaced by an edge is called the **underlying graph**

Definition 86 (Weakly connected digraph)

If the underlying graph is a connected graph, then the digraph is **weakly connected**

Definition 87 (Strongly connected digraph)

A digraph G is **strongly connected** if for every two distinct vertices u and v of G, there exists a directed path from u to v

Definition 88 (Disconnected digraph)

A digraph is said to be **disconnected** if it is not weakly connected

Strong connectedness is an equivalence relation

Denote $x \equiv y$ the relation "$x = y$, or $x \neq y$ and there exists a directed path in G from x to y". $\equiv$ is an equivalence relation since

1. $x \equiv y$ [reflexivity]
2. $x \equiv y \implies y \equiv x$ [symmetry]
3. $x \equiv y, y \equiv z \implies x \equiv z$ [transitivity]

Definition 89 (Connected component of a graph)

Sets of the form

$$A(x_0) = \{x : x \in V, x \equiv x_0\}$$

are equivalence classes. They partition V into strongly connected sub-digraphs of G called **strongly connected components** (or **strong components**) of G

A strong component in G is a maximal strongly connected subdigraph of G

Theorem 90 (Properties)

Let $G = (V, A)$ be a digraph

- *If G is strongly connected, it has only one strongly connected component*
- *The strongly connected components partition the vertices $V(G)$, with every vertex in exactly one strongly connected component*

Algorithm for determining strongly connected components in $G = (V, A)$

- Determine the strongly connected component $C(v)$ containing the vertex v; if $V - C(v)$ is non-empty, re-do the same operation on the sub-digraph $G' = (V - C(v), A')$
- To determine $C(v)$, the strongly connected component containing v: let v be a vertex of a digraph , which is not already in any strongly connected component
  1. Mark the vertex v with $\pm$
  2. Mark with $+$ all successors (not already marked with $+$) of a vertex marked with $+$
  3. Mark with $-$ all predecessors (not already marked with $-$) of a vertex marked with $-$
  4. Repeat until no more possible marking with $+$ or $-$

  All vertices marked with $\pm$ belong to the same strongly connected component $C(v)$ containing the vertex v

Condensation of a digraph

Definition 91 (Condensation of a digraph)

The condensation G^* of a digraph G is a digraph having as vertices the strongly connected components (SCC) of G and such that there exists an arc in G^* from a SCC C_i to another SCC C_j if there is an arc in G from some vertex of S_i to a vertex of S_j

Definition 92 (Articulation set)

For a connected graph, a set X of vertices is called an **articulation set** (or a **cutset**) if the subgraph of G generated by $V - X$ is not connected

Definition 93 (Stable set)

A set S of vertices is called a **stable set** if no arc joins two distinct vertices in S

Directed graphs

- Degrees in digraphs
- Walks, paths, etc.
- Connectivity in digraphs
- **Orientable graphs**

Orientation

Definition 94 (Orienting a graph)

Given a connected *graph*, we describe the act of assigning a direction to each edge (edge $\rightarrow$ arc) as **orienting the graph**

Definition 95 (Strong orientation)

If the digraph resulting from orienting a graph is strongly connected, the orientation is a **strong orientation**

Orientable graph

Definition 96 (Orientable graph)

A connected graph G is **orientable** if it admits a strong orientation

Theorem 97

A connected graph $G = (V, E)$ is orientable $\iff$ G contains no bridges

(in other words, iff every edge is contained in a cycle)

Matrices associated to a graph/digraph

There are multiple matrices associated to a graph/digraph

The branch of graph theory that studies the properties of matrices derived from graphs and uses of these matrices in determining graph properties is *spectral graph theory*

Graphs greatly simplify some problems in linear algebra and vice versa

Matrices associated to a graph/digraph

- **Adjacency matrices**
- **Other matrices associated to a graph/digraph**
- **Linking graphs and linear algebra**

Adjacency matrix (undirected case)

Let $G = (V, E)$ be a graph of order p and size q, with vertices $v_1, \ldots, v_p$ and edges $e_1, \ldots, e_q$

Definition 98 (Adjacency matrix)

The **adjacency matrix** is

$$M_A = M_A(G) = [m_{ij}]$$

is a $p \times p$ matrix in which

$$m_{ij} = \begin{cases} 1 & \text{if } v_i \text{ and } v_j \text{ are adjacent} \\ 0 & \text{otherwise} \end{cases}$$

Theorem 99 (Adjacency matrix and degree)

The sum of the entries in row i of the adjacency matrix is the degree of v_i in the graph

We often write $A(G)$ and, reciprocally, if A is an adjacency matrix, $G(A)$ the corresponding graph

G undirected $\implies A(G)$ symmetric

$A(G)$ has nonzero diagonal entries if G is not simple

Adjacency matrix (directed case)

Let $G = (V, A)$ be a digraph of order p with vertices $v_1, \dots, v_p$

Definition 100 (Adjacency matrix)

The **adjacency matrix** $M = M(G) = [m_{ij}]$ is a $p \times p$ matrix in which

$$m_{ij} = \begin{cases} 1 & \text{if arc } v_i v_j \in A \\ 0 & \text{otherwise} \end{cases}$$

Theorem 101 (Properties)

- *M is not necessarily symmetric*
- *The sum of any column of M is equal to the number of arcs directed towards v_j*
- *The sum of the entries in row i is equal to the number of arcs directed away from vertex v_i*
- *The $(i, j)-$entry of M^n is equal to the number of walks of length n from vertex v_i to v_j*

Definition 102 (Multiplicity of a pair)

The **multiplicity** of a pair x, y is the number $m_G^+(x, y)$ of arcs with initial endpoint x and terminal endpoint y. Let

$$
\begin{aligned}
m_G^-(x, y) &= m_G^+(y, x) \\
m_G(x, y) &= m_G^+(x, y) + m_G^-(x, y)
\end{aligned}
$$

If $x \neq y$, then $m_G(x, y)$ is number of arcs with both x and y as endpoints. If $x = y$, then $m_G(x, y)$ equals twice the number of loops attached to vertex x. If $A, B \subset V$, $A \neq B$, let

$$
\begin{aligned}
m_G^+(A, B) &= \{u : u \in U, u = (x, y), x \in A, y \in B\} \\
m_G(A, B) &= m_G^+(A, B) + m_G^+(A, B)
\end{aligned}
$$

Adjacency matrix of a multigraph

Definition 103 (Matrix associated with G)

If G has vertices $x_1, x_2, \ldots, x_n$, then the **matrix associated** with G is

$$a_{ij} = m_G^+(x_i, x_j)$$

Definition 104 (Adjacency matrix)

The matrix $a_{ij} + a_{ji}$ is the **adjacency matrix** associated with G

Adjacency matrix (multigraph case)

Definition 105 (Adjacency matrix of a multigraph)

G an ℓ-graph, then the adjacency matrix $M_A = [m_{ij}]$ is defined as follows

$$m_{ij} = \begin{cases} k & \text{if arc there are } k \text{ arcs } (i,j) \in U \\ 0 & \text{otherwise} \end{cases}$$

with $k \leq \ell$

G undirected $\implies M_A(G)$ symmetric

$M_A(G)$ has nonzero diagonal entries if G is not simple.

Weighted adjacency matrices

Sometimes, adjacency matrices (typically for 1-graphs) have real entries, usually positive

This means that the arcs/edges have been given a weight

Theorem 106 (Number of walks of length n)

Let A be the adjacency matrix of a graph $G = (V(G), E(G))$, where $V(G) = \{v_1, v_2, \ldots, v_p\}$. Then the $(i, j)-$entry of A^n, $n \geq 1$, is the number of different walks linking v_i to v_j of length n in G.

(two walks of the same length are equal if their edges occur in exactly the same order)
Example: let A be the adjacency matrix of a graph $G = (V(G), E(G))$.

- the $(i, i)-$entry of A^2 is equal to the degree of v_i.
- the $(i, i)-$entry of A^3 is equal to twice the number of C_3 containing v_i.

Matrices associated to a graph/digraph

- Adjacency matrices
- **Other matrices associated to a graph/digraph**
- Linking graphs and linear algebra

Incidence matrix (undirected case)

Let $G = (V, E)$ be a graph of order p, and size q, with vertices $v_1, \ldots, v_p$, and edges $e_1, \ldots, e_q$

Definition 107 (Incidence matrix)

The incidence matrix is

$$B = B(G) = [b_{ij}]$$

is that $p \times q$ matrix in which

$$b_{ij} = \begin{cases} 1 & \text{if } v_i \text{ is incident with } e_j \\ 0 & \text{otherwise} \end{cases}$$

Theorem 108 (Incidence matrix and degrees)

The sum of the entries in row i of the incidence matrix is the degree of v_i in the graph

Incidence matrix (directed case)

Let $G = (V, A)$ be a digraph of order p and size q, with vertices $v_1, \ldots, v_p$ and arcs $a_1, \ldots, a_q$

Definition 109 (Incidence matrix)

The **incidence matrix** $B = B(G) = [b_{ij}]$ is a $p \times q$ matrix in which

$$b_{ij} = \begin{cases} 1 & \text{if arc } a_j \text{ is directed away from a vertex } v_i \\ -1 & \text{if arc } a_j \text{ is directed towards a vertex } v_i \\ 0 & \text{otherwise} \end{cases}$$

Spectrum of a graph

We will come back to this later, but for now..

Definition 110 (Spectrum of a graph)

The **spectrum** of a graph G is the spectrum (set of eigenvalues) of its associated adjacency matrix $M(G)$

This is regardless of the type of adjacency matrix or graph

Degree matrix

Definition 111 (Degree matrix)

The **degree** matrix $D = [d_{ij}]$ for G is a $n \times n$ diagonal matrix defined as

$$d_{ij} = \begin{cases} d_G(v_i) & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

In an undirected graph, this means that each loop increases the degree of a vertex by two

In a directed graph, the term “degree” may refer either to indegree (the number of incoming edges at each vertex) or outdegree (the number of outgoing edges at each vertex)

Laplacian matrix

Definition 112 (Laplacian matrix)

$G = (V, A)$ a simple graph with n vertices. The **Laplacian** matrix is

$$L = D(G) - M(G)$$

where $D(G)$ is the degree matrix and $M(G)$ is the adjacency matrix

Laplacian matrix (continued)

G simple graph $\implies$ $M(G)$ only contains 1 or 0 and its diagonal elements are all 0

For directed graphs, either the indegree or outdegree is used, depending on the application

Elements of L are given by

$$\ell_{ij} = \begin{cases} d_G(v_i) & \text{if } i = j \\ -1 & \text{if } i \neq j \text{ and } v_i \text{ is adjacent to } v_j \\ 0 & \text{otherwise} \end{cases}$$

Distance matrix

Let G be a graph of order p with vertices $v_1, \ldots, v_p$

Definition 113 (Distance matrix)

The distance matrix $\Delta(G) = [d_{ij}]$ is a $p \times p$ matrix in which

$$\delta_{ij} = d_G(v_i, v_j)$$

Note $\delta_{ii} = 0$ for $i = 1, \ldots, p$

Property 114

- *M is not necessarily symmetric*
- *The sum of any column of M is equal to the number of arcs directed towards v_j*
- *The sum of the entries in row i is equal to the number of arcs directed away from vertex v_i*
- *The $(i,j)-$entry of M^n is equal to the number of walks of length n from vertex v_i to v_j*

Matrices associated to a graph/digraph

- Adjacency matrices
- Other matrices associated to a graph/digraph
- **Linking graphs and linear algebra**

Counting paths

Theorem 115

G a digraph and $M_A(G)$ its adjacency matrix. Denote $P = [p_{ij}]$ the matrix $P = M_A^k$. Then p_{ij} is the number of distinct paths of length k from i to j in G

Definition 116 (Irreducible matrix)

A matrix $A \in \mathcal{M}_n$ is **reducible** if $\exists P \in \mathcal{M}_n$, permutation matrix, s.t. $P^T AP$ can be written in block triangular form. If no such P exists, A is **irreducible**

Theorem 117

A irreducible $\iff$ $G(A)$ strongly connected

Theorem 118

Let A be the adjacency matrix of a graph G on p vertices. A graph G on p vertices is connected $\iff$

$$I + A + A^2 + \cdots + A^{p-1} = C$$

has no zero entries

Theorem 119

Let M be the adjacency matrix of a digraph D on p vertices. A digraph D on p vertices is strongly connected $\iff$

$$I + M + M^2 + \cdots + M^{p-1} = C$$

has no zero entries

Nonnegative matrix

$A = [a_{ij}] \in \mathcal{M}_n(\mathbb{R})$ **nonnegative** if $a_{ij} \geq 0 \; \forall i, j = 1, \ldots, n$; $\boldsymbol{v} \in \mathbb{R}^n$ nonnegative if $v_i \geq 0 \; \forall i = 1, \ldots, n$. **Spectral radius** of A

$$\rho(A) = \max_{\lambda \in \mathrm{Sp}(A)} \{|\lambda|\}$$

$\mathrm{Sp}(A)$ the **spectrum** of A

Perron-Frobenius (PF) theorem

Theorem 120 (PF – Nonnegative case)

$0 \leq A \in \mathcal{M}_n(\mathbb{R})$. *Then* $\exists \mathbf{v} \geq \mathbf{0}$ *s.t.*

$$A\mathbf{v} = \rho(A)\mathbf{v}$$

Theorem 121 (PF – Irreducible case)

Let $0 \leq A \in \mathcal{M}_n(\mathbb{R})$ *irreducible. Then* $\exists \mathbf{v} > \mathbf{0}$ *s.t.*

$$A\mathbf{v} = \rho(A)\mathbf{v}$$

$\rho(A) > 0$ *and with algebraic multiplicity 1. No nonnegative eigenvector is associated to any other eigenvalue of* A

Primitive matrices

Definition 122

$0 \leq A \in \mathcal{M}_n(\mathbb{R})$ **primitive** (with **primitivity index** $k \in \mathbb{N}_+^*$) if $\exists k \in \mathbb{N}_+^*$ s.t.

$$A^k > 0,$$

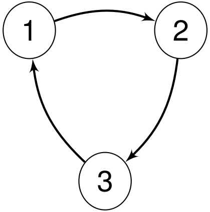
with k the smallest integer for which this is true. A **imprimitive** if it is not primitive

A primitive $\implies$ A irreducible; the converse is false

Theorem 123

$A \in \mathcal{M}_n(\mathbb{R})$ irreducible and $\exists i = 1, \ldots, n$ s.t. $a_{ii} > 0 \implies A$ primitive

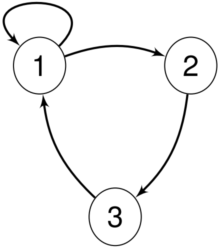
Here d is the index of imprimitivity (i.e., the number of eigenvalues that have the same modulus as $\lambda_p = \rho(A)$). If $d = 1$, then A is primitive. We have that $d = \gcd$ of all the lengths of closed walks in $G(A)$



Adjacency matrix

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

Closed walks in $G(A)$ (lengths): $1 \to 1$ (3), $2 \to 2$ (3), $2 \to 2$ (3) $\implies \gcd = 3 \implies d = 3$ (here, all eigenvalues have modulus 1)



$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

Closed walk $1 \to 1$ has length 1 $\implies$ gcd of lengths of closed walks is 1 $\implies$ A primitive

Let $\mathbf{0} \leq A \in \mathcal{M}_n$

Theorem 124

A primitive $\implies \exists\, 0 < k \leq (n-1)n^n$ *such that* $A^k > 0$

Theorem 125

If A is primtive and the shortest simple directed cycle in $G(A)$ *has length s, then the primitivity index is* $\leq n + s(n-1)$

Theorem 126

A primitive $\iff A^{n^2-2n+2} > \mathbf{0}$

Theorem 127

If A is irreducible and has d positive entries on the diagonal, then the primitivity index $\leq 2n - d - 1$

Theorem 128

0 $\leq A \in \mathcal{M}_n$, $\lambda_P = \rho(A)$ *the Perron root of A,* $\boldsymbol{v}_P$ *and* $\boldsymbol{w}_P$ *the corresponding right and left Perron vectors of A, respectively, d the index of imprimitivity of A (with* $d = 1$ *when A is primitive) and* $\lambda_j \in \sigma(A)$ *the spectrum of A, with* $j = 2, \ldots, n$ *unless otherwise specified (assuming* $\lambda_1 = \lambda_P$*)*

NONNEGATIVE → REDUCIBLE, IRREDUCIBLE

IRREDUCIBLE → IMPRIMITIVE, PRIMITIVE

REDUCIBLE

- $\lambda_P \geq 0$
- $\boldsymbol{w}_P \geq 0$
- $\boldsymbol{v}_P \geq 0$
- $\lambda_P \geq |\lambda_j|$

IMPRIMITIVE

- $\lambda_P > 0$
- $\boldsymbol{w}_P > 0$
- $\boldsymbol{v}_P > 0$
- $\lambda_P = |\lambda_j|$, $j = 2, \ldots, d$
- $\lambda_P > |\lambda_j|$, $j > d$

PRIMITIVE

- $\lambda_P > 0$
- $\boldsymbol{w}_P > 0$
- $\boldsymbol{v}_P > 0$
- $\lambda_P > |\lambda_j|$, $j \neq P$

Undirected graphs

Directed graphs

Matrices associated to a graph/digraph

Trees

Trees

Definition 129 (Forest, trees and branches)

- A connected graph with no cycle is a **tree**
- A tree is a connected acyclic graph, its edges are called **branches**
- A graph (connected or not) without any cycle is a **forest**. Each component is a tree

(A forest is a graph whose connected components are trees)

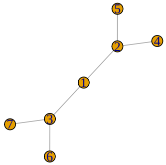
Is the "Karate graph" a tree?

```
is_acyclic(G)

## Error:  object 'G' not found

is_tree(G)

## Error:  object 'G' not found
```

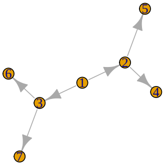
So we need friend to play with!

```
G_tu <- make_tree(7, 2, mode = "undirected")
G_td <- make_tree(7, 2)
```

An undirected tree



A (out) directed tree



Property 130

- *Every edge of a tree is a bridge*
- *Given two vertices u and v of a tree, there is an unique path linking u to v*
- *A tree with p vertices and q edges satisfies $q = p - 1$. Thus, a tree is minimally connected*

(First property: the deletion of any edge of a tree diconnects it)

Every edge of a tree is a bridge

```
E(G_tu)

## + 6/6 edges from bbcf4f1:
## [1] 1--2 1--3 2--4 2--5 3--6 3--7

bridges(G_tu)

## + 6/6 edges from bbcf4f1:
## [1] 2--4 2--5 1--2 3--6 3--7 1--3

all(sort(E(G_tu)) == sort(bridges(G_tu)))

## [1] TRUE
```

Spanning tree

Definition 131 (Spanning tree)

A **spanning tree** of a connected graph G is a subgraph of G that contains all the vertices of G and is a tree.

A graph may have many spanning trees

Minimal spanning tree

Definition 132 (Value of a spanning tree)

The **value of a spanning tree** T of order p is

$$\sum_{i=1}^{p-1} f(e_i)$$

where f is the function that maps the edge set into $\mathbb{R}$

Definition 133 (Minimal spanning tree)

Let G be an undirected network, and let T be a **minimal spanning tree** of G. Then T is a spanning tree whose the value is minimum

Algorithm to find a minimal spanning tree

Let $G = (V(G), E(G))$ be an undirected network and T be a minimal spanning tree

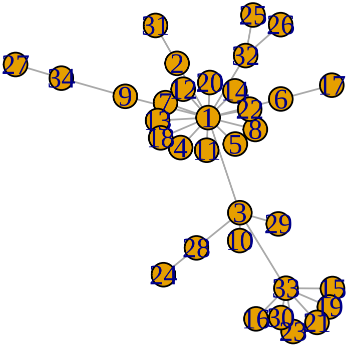
1. Sort the edges of G in increasing order by value
2. $T = (V(G), \emptyset)$
3. For each edge e in sorted order if the endpoints of e are disconnected in T add e to T

Finding a minimal spanning tree of the Karate graph

The function `mst` finds minimal spanning trees, using distances if no edge weights are provided

```
G_mst = mst(G)

## Error:  object 'G' not found
## Error in h(simpleError(msg, call)):  error in evaluating the
argument 'x' in selecting a method for function 'plot':  object
'G_mst' not found
```

A minimal spanning tree of the Karate graph



Minimal connector problem

- Model: a graph G such that edges represent all possible connections, and each edge has a positive value which represents its cost; an undirected network G
- Solution: a minimal spanning tree T of G
  - a spanning tree of G is a subgraph of G that contains all the vertices of G and is a tree.
  - the cost of the spanning tree is the sum of values of the edges of T
  - a spanning tree such that no other spanning tree has a smaller cost is a minimmal spanning tree.

Theorem 134 (Characterisation of trees)

$H = (V, U)$ a graph of order $|V| = n > 2$. The following are equivalent and all characterise a tree :

1. *H connected and has no cycles*
2. *H has $n - 1$ arcs and no cycles*
3. *H connected and has exactly $n - 1$ arcs*
4. *H has no cycles, and if an arc is added to H, exactly one cycle is created*
5. *H connected, and if any arc is removed, the remaining graph is not connected*
6. *Every pair of vertices of H is connected by one and only one chain*

Definition 135 (Pendant vertex)

A vertex is **pendant** if it is adjacent to exactly one other vertex

Theorem 136

A tree of order $n \geq 2$ has at least two pendant vertices

Theorem 137

A graph $G = (V, U)$ has a partial graph that is a tree $\iff$ G connected

Recall that a partial graph is a graph generated by a subset of the arcs (Definition 67 slide 39)

Spanning tree

The procedure in the proof of Theorem 137 gives a **spanning tree**

Can also build a spanning tree as follows:

- Consider any arc u_0
- Find arc u_1 that does not form a cycle with u_0
- Find arc u_2 that does not form a cycle with $\{u_0, u_1\}$
- Continue
- When you cannot continue anymore, you have a spanning tree

Theorem 138

G connected graph with ≥ 1 arc. TFAE

1. *G strongly connected*
2. *Every arc lies on a circuit*
3. *G contains no cocircuits*

Theorem 139

G graph with ≥ 1 arc. TFAE

1. *G is a graph without circuits*
2. *Each arc is contained in a cocircuit*

Theorem 140

If G is a strongly connected graph of order n, then G has a cycle basis of $\nu(G)$ circuits

Definition 141 (Node, anti-node, branch)

$G = (V, U)$ strongly connected without loops and > 1 vertex. For each $x \in V$, there is a path from it and a path to it so x has at least 2 incident arcs. Specifically,

- $x \in V$ with > 2 incident arcs is a **node**
- $x \in V$ with 2 incident arcs is an **anti-node**

A path whose only nodes are its endpoints is a **branch**

Definition 142 (Minimally connected graph)

G is **minimally connected** if it is strongly connected and removal of any arc destroys strong-connectedness

A minimally connected graph is 1-graph without loops

Definition 143 (Contraction)

$G = (V, U)$. The **contraction** of the set $A \subset V$ of vertices consists in replacing A by a single vertex a and replacing each arc into (resp. out of) A by an arc with same index into (resp. out of) a

Theorem 144

G minimally connected, $A \subset V$ generating a strongly connected subgraph of G. Then the contraction of A gives a minimally connected graph

Theorem 145

G a minimally connected graph, G' be the minimally connected graph obtained by the contraction of an elementary circuit of G. Then

$$\nu(G) = \nu(G') + 1$$

Theorem 146

G minimally connected of order $n \geq 2 \implies G$ has ≥ 2 anti-nodes

Theorem 147

$G = (V, U)$. Then the graph C' obtained by contracting each strongly connected component of G contains no circuits

Arborescences

Definition 148 (Root)

Vertex $a \in V$ in $G = (V, U)$ is a **root** if all vertices of G can be reached by paths *starting* from a

Not all graphs have roots

Definition 149 (Quasi-strong connectedness)

G is **quasi-strongly connected** if $\forall x, y \in V$, exists $z \in V$ (denoted $z(x, y)$ to emphasize dependence on x, y) from which there is a path to x and a path to y

Strongly connected $\implies$ quasi-strongly connected (take $z(x, y) = x$); converse not true

Quasi-strongly connected $\implies$ connected

Arborescence

Definition 150 (Arborescence)

An **arborescence** is a tree that has a root

Lemma 151

$G = (V, U)$ has a root $\iff$ G quasi-strongly connected

Theorem 152

H graph of order $n > 1$. TFAE (and all characterise an arborescence)

1. *H quasi-strongly connected without cycles*
2. *H quasi-strongly connected with $n-1$ arcs*
3. *H tree having a root a*
4. *$\exists a \in V$ s.t. all other vertices are connected with a by 1 and only 1 path from a*
5. *H quasi-strongly connected and loses quasi-strong connectedness if any arc is removed*
6. *H quasi-strongly connected and $\exists a \in V$ s.t.*

$$\begin{aligned} d_H^-(a) &= 0 \\ d_H^-(x) &= 1 \qquad \forall x \neq a \end{aligned}$$

7. *H has no cycles and $\exists a \in V$ s.t.*

$$\begin{aligned} d_H^-(a) &= 0 \\ d_H^-(x) &= 1 \qquad \forall x \neq a \end{aligned}$$

Theorem 153

G has a partial graph that is an arborescence $\iff$ G quasi-strongly connected

Theorem 154

$G = (V, E)$ simple connected graph and $x_1 \in V$. It is possible to direct all edges of E so that the resulting graph $G_0 = (V, U)$ has a spanning tree H s.t.

1. *H is an arborescence with root x_1*
2. *The cycles associated with H are circuits*
3. *The only elementary circuits of G_0 are the cycles associated with H*

Counting trees

Proposition 155

X a set with n distinct objects, $n_1, \ldots, n_p$ nonnegative integers s.t. $n_1 + \cdots + n_p = n$. The number of ways to place the n objects into p boxes $X_1, \ldots, X_p$ containing $n_1, \ldots, n_p$ objects respectively is

$$\binom{n}{n_1, \ldots, n_p} = \frac{n!}{n_1! \cdots n_p!}$$

Proposition 156 (Multinomial formula)

Let $a_1, \ldots, a_p \in \mathbb{R}$ be p real numbers, then

$$(a_1 + \cdots + a_p)^n = \sum_{n_1, \ldots, n_p \geq 0} \binom{n}{n_1, \ldots, n_p} (a_1)^{n_1} \cdots (a_p)^{n_p}$$

Theorem 157

Denote $T(n; d_1, \ldots, d_n)$ the number of distinct trees H with vertices $x_1, \ldots, x_n$ and with degrees $d_H(x_1) = d_1, \ldots, d_H(x_n) = d_n$. Then

$$T(n; d_1, \ldots, d_n) = \binom{n-2}{d_1 - 1, \ldots, d_n - 1}$$

Theorem 158

The number of different trees with vertices $x_1, \ldots, x_n$ is n^{n-2}

There is a whole industry of similar results (as well as for arborescences), but we will stop here. The main point is that we are talking about a large number of possibilities..