Review of first-year linear algebra

In MATH 2740, we rely on notions you acquired in MATH 1210/1220/1300. We also use some material from first-year calculus

So let us (briefly) go over material in these courses

I also add (for some of you) a few things that will be handy and establish some terminology that we use throughout the course

OUTLINE

Sets and logic

Complex numbers

Vectors and vector spaces

Linear systems and matrices

Matrix arithmetic

Diagonalisation

Linear independence/Bases/Dimension

- Linear algebra in a nutshell

Sets and elements

Definition 1 (Set)

A **set** X is a collection of **elements**

We write $x \in X$ or $x \notin X$ to indicate that the element x belongs to the set X or does not belong to the set X, respectively

Definition 2 (Subset)

Let X be a set. The set S is a **subset** of X, which is denoted $S \subset X$, if all its elements belong to X

Not used here but worth noting: we say S is a **proper subset** of X and write $S \subsetneq X$, if it is a subset of X and not equal to X

Quantifiers

A shorthand notation for "for all elements x belonging to X" is $\forall x \in X$

For example, if $X = \mathbb{R}$, the *field* of real numbers, then $\forall x \in \mathbb{R}$ means "for all real numbers x"

A shorthand notation for "there exists an element x in the set X" is $\exists x \in X$

$\forall$ and $\exists$ are **quantifiers**

Intersection and union of sets

Let X and Y be two sets

Definition 3 (Intersection)

The intersection of X and Y, $X \cap Y$, is the set of elements that belong to X **and** to Y,

$$X \cap Y = \{x : x \in X \textbf{ and } x \in Y\}$$

Definition 4 (Union)

The union of X and Y, $X \cup Y$, is the set of elements that belong to X **or** to Y,

$$X \cup Y = \{x : x \in X \textbf{ or } x \in Y\}$$

In mathematics, or=and/or in common parlance. We also have an **exclusive or** (xor)

A teeny bit of logic

In a logical sense, a **proposition** is an assertion (or statement) whose truth value (true or false) can be asserted. For example, a theorem is a proposition that has been shown to be true. "The sky is blue" is also a proposition

Let A be a proposition. We generally write

$$A$$

to mean that A is true, and

$$\text{not } A$$

to mean that A is false. not A is the **contraposition** of A (or not A is the contraposite of A)

A teeny bit of logic (cont.)

Let A, B be propositions. Then

- $A \Rightarrow B$ (read A implies B) means that whenever A is true, then so is B
- $A \Leftrightarrow B$, also denoted A if and only if B (A iff B for short), means that $A \Rightarrow B$ **and** $B \Rightarrow A$
  We also say that A and B are **equivalent**

Let A and B be propositions. Then

$$(A \Rightarrow B) \Leftrightarrow (\text{not } B \Rightarrow \text{not } A)$$

Necessary or sufficient conditions

Suppose we want to establish whether a given statement P is true, depending on the truth value of a statement H. Then we say that

- H is a **necessary condition** if $P \Rightarrow H$
  (It is necessary that H be true for P to be true; so whenever P is true, so is H)

- H is a **sufficient condition** if $H \Rightarrow P$
  (It suffices for H to be true for P to also be true)

- H is a **necessary and sufficient condition** if $H \Leftrightarrow P$, i.e., H and P are equivalent

Playing with quantifiers

For the quantifiers $\forall$ (for all) and $\exists$ (there exists),

$$\exists \text{ is the contraposite of } \forall$$

Therefore, for example, the contraposite of

$$\forall x \in X, \exists y \in Y$$

is

$$\exists x \in X, \forall y \in Y$$

Complex numbers

Definition 5 (Complex numbers)

A **complex number** is an ordered pair (a, b), where $a, b \in \mathbb{R}$. Usually written $a + ib$ or $a + bi$, where $i^2 = -1$ (i.e., $i = \sqrt{-1}$)
The set of all complex numbers is denoted $\mathbb{C}$,

$$\mathbb{C} = \{a + ib : a, b \in \mathbb{R}\}$$

Definition 6 (Addition and multiplication on $\mathbb{C}$)

Letting $a + ib$ and $c + id \in \mathbb{C}$, addition on $\mathbb{C}$ is defined by

$$(a + ib) + (c + id) = (a + c) + i(b + d)$$

and multiplication on $\mathbb{C}$ is defined by

$$(a + ib)(c + id) = (ac - bd) + i(ad + bc)$$

Latter is easy to obtain using regular multiplication and $i^2 = -1$

Properties

$\forall \alpha, \beta, \gamma \in \mathbb{C}$,

$\alpha + \beta = \beta + \alpha$ and $\alpha\beta = \beta\alpha$ [commutativity]

$(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$ and $(\alpha\beta)\gamma = \alpha(\beta\gamma)$ [associativity]

$\gamma + 0 = \gamma$ and $\gamma 1 = \gamma$ [identities]

$\forall \alpha \in \mathbb{C}$, $\exists \beta \in \mathbb{C}$ unique s.t. $\alpha + \beta = 0$ [additive inverse]

$\forall \alpha \neq 0 \in \mathbb{C}$, $\exists \beta \in \mathbb{C}$ unique s.t. $\alpha\beta = 1$ [multiplicative inverse]

$\gamma(\alpha + \beta) = \gamma\alpha + \gamma\beta$ [distributivity]

Additive & multiplicative inverse, subtraction, division

Definition 7

Let $\alpha, \beta \in \mathbb{C}$

- $-\alpha$ is the **additive inverse** of α, i.e., the unique number in $\mathbb{C}$ s.t. $\alpha + (-\alpha) = 0$
- **Subtraction** on $\mathbb{C}$:
$$\beta - \alpha = \beta + (-\alpha)$$
- For $\alpha \neq 0$, $1/\alpha$ is the **multiplicative inverse** of α, i.e., the unique number in $\mathbb{C}$ s.t.
$$\alpha(1/\alpha) = 1$$
- **Division** on $\mathbb{C}$:
$$\beta/\alpha = \beta(1/\alpha)$$

Definition 8 (Real and imaginary parts)

Let $z = a + ib$. Then $\mathrm{Re}\, z = a$ is **real part** and $\mathrm{Im}\, z = b$ is **imaginary part** of z

If ambiguous, write $\mathrm{Re}\,(z)$ and $\mathrm{Im}\,(z)$

Definition 9 (Conjugate and Modulus)

Let $z = a + ib \in \mathbb{C}$. Then

- **Complex conjugate** of z is

$$\bar{z} = a - ib$$

- **Modulus** (or **absolute value**) of z is

$$|z| = \sqrt{a^2 + b^2} \geq 0$$

Properties of complex numbers

Let $w, z \in \mathbb{C}$, then

- $z + \bar{z} = 2\mathrm{Re}\ z$
- $z - \bar{z} = 2i\mathrm{Im}\ z$
- $z\bar{z} = |z|^2$
- $\overline{w + z} = \bar{w} + \bar{z}$ and $\overline{wz} = \bar{w}\bar{z}$
- $\bar{\bar{z}} = z$
- $|\mathrm{Re}\ z| \leq |z|$ and $|\mathrm{Im}\ z| \leq |z|$
- $|\bar{z}| = |z|$
- $|wz| = |w|\ |z|$
- $|w + z| \leq |w| + |z|$ **[triangle inequality]**

Solving quadratic equations

Consider the polynomial

$$P(x) = a_0 + a_1 x + a_2 x^2$$

where $x, a_0, a_1, a_2 \in \mathbb{R}$. Letting

$$\Delta = a_1^2 - 4 a_0 a_2$$

you know that if $\Delta > 0$, then

$$P(x) = 0$$

has two distinct *real* solutions,

$$x_1 = \frac{-a_1 - \sqrt{\Delta}}{2a_2} \quad \text{and} \quad x_2 = \frac{-a_1 + \sqrt{\Delta}}{2a_2}$$

if $\Delta = 0$, then there is a (multiplicity 2) unique *real* solution

$$x_1 = \frac{-a_1}{2a_2}$$

while if $\Delta < 0$, there is no solution

Solving quadratic equations with complex numbers

Consider the polynomial

$$P(x) = a_0 + a_1 x + a_2 x^2$$

where $x, a_0, a_1, a_2 \in \mathbb{R}$. If instead of seeking $x \in \mathbb{R}$, we seek $x \in \mathbb{C}$, then the situation is the same, except when $\Delta < 0$

In the latter case, note that

$$\sqrt{\Delta} = \sqrt{(-1)(-\Delta)} = \sqrt{-1}\sqrt{-\Delta} = i\sqrt{-\Delta}$$

Since $\Delta < 0$, $-\Delta > 0$ and the square root is the usual one

Solving quadratic equations with complex numbers

To summarize, consider the polynomial

$$P(x) = a_0 + a_1 x + a_2 x^2$$

where $x, a_0, a_1, a_2 \in \mathbb{R}$. Letting

$$\Delta = a_1^2 - 4a_0 a_2$$

Then

$$P(x) = 0$$

has two solutions,

$$x_{1,2} = \frac{-a_1 \pm \sqrt{\Delta}}{2a_2}$$

where, if $\Delta < 0$, $x_1, x_2 \in \mathbb{C}$ and take the form

$$x_{1,2} = \frac{-a_1 \pm i\sqrt{-\Delta}}{2a_2}$$

Why this matters

Recall (we will come back to this later) that to find the *eigenvalues* of the matrix

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

we seek λ solutions to $\det(A - \lambda\mathbb{I}) = 0$, i.e., λ solutions to

$$|A - \lambda\mathbb{I}| = \begin{vmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{vmatrix} = (a_{11} - \lambda)(a_{22} - \lambda) - a_{12}a_{21} = 0$$

i.e., λ solutions to

$$\lambda^2 - (a_{11} + a_{22})\lambda + a_{11}a_{22} - a_{12}a_{21} = 0$$

Why this matters (cont.)

Let

$$P(\lambda) = \lambda^2 - (a_{11} + a_{22})\lambda + a_{11}a_{22} - a_{12}a_{21}$$

From previous discussion, letting

$$\begin{aligned} \Delta &= (a_{11} + a_{22})^2 - 4(a_{11}a_{22} - a_{12}a_{21}) \\ &= a_{11}^2 + a_{22}^2 + 2a_{11}a_{22} - 4a_{11}a_{22} + 4a_{12}a_{21} \\ &= a_{11}^2 + a_{22}^2 - 2a_{11}a_{22} + 4a_{12}a_{21} \\ &= (a_{11} - a_{22})^2 + 4a_{12}a_{21} \end{aligned}$$

we have two (potentially equal) solutions to $P(\lambda) = 0$

$$x_{1,2} = \frac{a_{11} + a_{22} \pm \sqrt{\Delta}}{2}$$

that are complex if $\Delta < 0$

Example: $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

Vectors

A **vector** $\boldsymbol{v}$ is an ordered n-tuple of real or complex numbers

Denote $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ (real or complex numbers). For $v_1, \ldots, v_n \in \mathbb{F}$,

$$\boldsymbol{v} = (v_1, \ldots, v_n) \in \mathbb{F}^n$$

is a vector. $v_1, \ldots, v_n$ are the **components** of $\boldsymbol{v}$

If unambiguous, we write v. Otherwise, $\boldsymbol{v}$ or $\vec{v}$

Vector space

Definition 10 (Vector space)

A **vector space** over $\mathbb{F}$ is a set V together with two binary operations, **vector addition**, denoted $+$, and **scalar multiplication**, that satisfy the relations:

1. $\forall \boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in V$, $\boldsymbol{u} + (\boldsymbol{v} + \boldsymbol{w}) = (\boldsymbol{u} + \boldsymbol{v}) + \boldsymbol{w}$
2. $\forall \boldsymbol{v}, \boldsymbol{w} \in V$, $\boldsymbol{v} + \boldsymbol{w} = \boldsymbol{w} + \boldsymbol{v}$
3. $\exists 0 \in V$, the zero vector, such that $\boldsymbol{v} + 0 = \boldsymbol{v}$ for all $\boldsymbol{v} \in V$
4. $\forall \boldsymbol{v} \in V$, there exists an element $\boldsymbol{w} \in V$, the additive inverse of $\boldsymbol{v}$, such that $\boldsymbol{v} + \boldsymbol{w} = 0$
5. $\forall \alpha \in \mathbb{R}$ and $\forall \boldsymbol{v}, \boldsymbol{w} \in V$, $\alpha(\boldsymbol{v} + \boldsymbol{w}) = \alpha\boldsymbol{v} + \alpha\boldsymbol{w}$
6. $\forall \alpha, \beta \in \mathbb{R}$ and $\forall \boldsymbol{v} \in V$, $(\alpha + \beta)\boldsymbol{v} = \alpha\boldsymbol{v} + \beta\boldsymbol{v}$
7. $\forall \alpha, \beta \in \mathbb{R}$ and $\forall \boldsymbol{v} \in V$, $\alpha(\beta\boldsymbol{v}) = (\alpha\beta)\boldsymbol{v}$
8. $\forall \boldsymbol{v} \in V$, $1\boldsymbol{v} = \boldsymbol{v}$

Norms

Definition 11 (Norm)

Let V be a vector space over $\mathbb{F}$, and $\boldsymbol{v} \in V$ be a vector. The **norm** of $\boldsymbol{v}$, denoted $\|\boldsymbol{v}\|$, is a function from V to $\mathbb{R}_+$ that has the following properties:

1. For all $\boldsymbol{v} \in V$, $\|\boldsymbol{v}\| \geq 0$ with $\|\boldsymbol{v}\| = 0$ iff $\boldsymbol{v} = 0$
2. For all $\alpha \in \mathbb{F}$ and all $\boldsymbol{v} \in V$, $\|\alpha \boldsymbol{v}\| = |\alpha| \, \|\boldsymbol{v}\|$
3. For all $\boldsymbol{u}, \boldsymbol{v} \in V$, $\|\boldsymbol{u} + \boldsymbol{v}\| \leq \|\boldsymbol{u}\| + \|\boldsymbol{v}\|$

Let V be a vector space (for example, $\mathbb{R}^2$ or $\mathbb{R}^3$)

The **zero element** (or **zero vector**) is the vector $0 = (0, \ldots, 0)$

The **additive inverse** of $\boldsymbol{v} = (v_1, \ldots, v_n)$ is $-\boldsymbol{v} = (-v_1, \ldots, -v_n)$

For $\boldsymbol{v} = (v_1, \ldots, v_n) \in V$, the length (or Euclidean norm) of $\boldsymbol{v}$ is the **scalar**

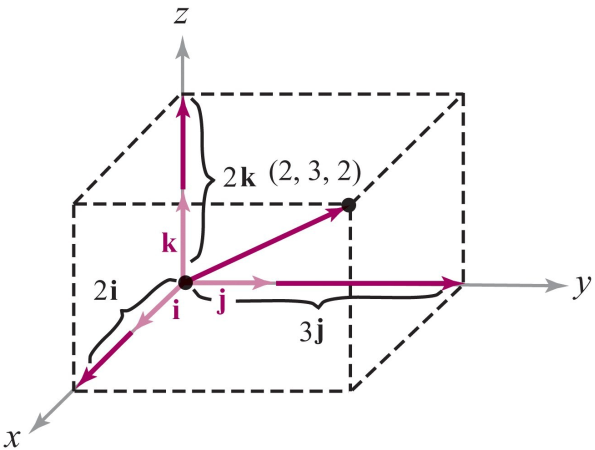
$$\|\boldsymbol{v}\| = \sqrt{v_1^2 + \cdots + v_n^2}$$

To **normalize** the vector $\boldsymbol{v}$ consists in considering $\tilde{\boldsymbol{v}} = \boldsymbol{v}/\|\boldsymbol{v}\|$, i.e., the vector in the same direction as $\boldsymbol{v}$ that has unit length

Standard basis vectors

Vectors $\boldsymbol{i} = (1, 0, 0)$, $\boldsymbol{j} = (0, 1, 0)$ and $\boldsymbol{k} = (0, 0, 1)$ are the **standard basis vectors** of $\mathbb{R}^3$. A vector $\boldsymbol{v} = (v_1, v_2, v_3)$ can then be written

$$\boldsymbol{v} = v_1\boldsymbol{i} + v_2\boldsymbol{j} + v_3\boldsymbol{k}$$



For V ($\mathbb{R}^n$), the standard basis vectors are usually denoted $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n$, with

$$\boldsymbol{e}_k = (\underbrace{0, \ldots, 0}_{k-1}, 1, \underbrace{0, \ldots, 0}_{n-k+1})$$

Dot product

Definition 12 (Dot product)

Let $\boldsymbol{a} = (a_1, \ldots, a_n) \in \mathbb{R}^n$, $\boldsymbol{b} = (b_1, \ldots, b_n) \in \mathbb{R}^n$. The **dot product** of $\boldsymbol{a}$ and $\boldsymbol{b}$ is the **scalar**

$$\boldsymbol{a} \bullet \boldsymbol{b} = \sum_{i=1}^{n} a_i b_i = a_1 b_1 + \cdots + a_n b_n$$

The dot product is a special case of **inner product**

Properties of the dot product

Theorem 13

For $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \in \mathbb{R}^n$ *and* $\alpha \in \mathbb{R}$,

- $\boldsymbol{a} \bullet \boldsymbol{a} = \|\boldsymbol{a}\|^2$ *(so* $\boldsymbol{a} \bullet \boldsymbol{a} \geq 0$, *with* $\boldsymbol{a} \bullet \boldsymbol{a} = 0$ *iff* $\boldsymbol{a} = 0$*)*
- $\boldsymbol{a} \bullet \boldsymbol{b} = \boldsymbol{b} \bullet \boldsymbol{a}$ *($\bullet$ is commutative)*
- $\boldsymbol{a} \bullet (\boldsymbol{b} + \boldsymbol{c}) = \boldsymbol{a} \bullet \boldsymbol{b} + \boldsymbol{a} \bullet \boldsymbol{c}$ *($\bullet$ distributive over $+$)*
- $(\alpha \boldsymbol{a}) \bullet \boldsymbol{b} = \alpha(\boldsymbol{a} \bullet \boldsymbol{b}) = \boldsymbol{a} \bullet (\alpha \boldsymbol{b})$
- $0 \bullet \boldsymbol{a} = 0$

Some results stemming from the dot product

Theorem 14

If θ is the angle between the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, then

$$\boldsymbol{a} \bullet \boldsymbol{b} = \|\boldsymbol{a}\| \, \|\boldsymbol{b}\| \, \cos\theta$$

Corollary 15 (Cauchy-Schwarz inequality)

For any two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, we have

$$|\boldsymbol{a} \bullet \boldsymbol{b}| \leq \|\boldsymbol{a}\| \, \|\boldsymbol{b}\|$$

with equality if and only if $\boldsymbol{a}$ is a scalar multiple of $\boldsymbol{b}$, or one of them is 0.

Theorem 16

$\boldsymbol{a}$ and $\boldsymbol{b}$ are orthogonal if and only if $\boldsymbol{a} \bullet \boldsymbol{b} = 0$.

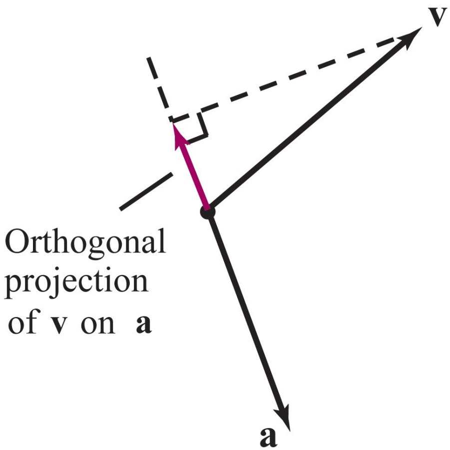
Scalar and vector projections

Scalar projection of $\boldsymbol{v}$ onto $\boldsymbol{a}$ (or component of $\boldsymbol{v}$ along $\boldsymbol{a}$):

$$\text{comp}_{\boldsymbol{a}}\boldsymbol{v} = \frac{\boldsymbol{a}\bullet\boldsymbol{v}}{\|\boldsymbol{a}\|}$$

Vector (or orthogonal) projection of $\boldsymbol{v}$ onto $\boldsymbol{a}$:

$$\text{proj}_{\boldsymbol{a}}\boldsymbol{v} = \left(\frac{\boldsymbol{a}\bullet\boldsymbol{v}}{\|\boldsymbol{a}\|}\right)\frac{\boldsymbol{a}}{\|\boldsymbol{a}\|} = \frac{\boldsymbol{a}\bullet\boldsymbol{v}}{\|\boldsymbol{a}\|^2}\,\boldsymbol{a}$$



Linear systems

Definition 17 (Linear system)

A **linear system** of m equations in n unknowns takes the form

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ \vdots & & \vdots & & \vdots & & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_n \end{array} \tag{1}$$

The a_{ij}, x_j and b_j could be in $\mathbb{R}$ or $\mathbb{C}$, although here we typically assume they are in $\mathbb{R}$

The aim is to find $x_1, x_2, \ldots, x_n$ that satisfy all equations simultaneously

Theorem 18 (Nature of solutions to a linear system)

A linear system can have

- *no solution*
- *a unique solution*
- *infinitely many solutions*

Operations on linear systems

You learned to manipulate linear systems using

- Gaussian elimination
- Gauss-Jordan elimination

with the aim to put the system in **row echelon form** (REF) or **reduced row echelon form** (RREF)

Matrices and linear systems

Writing

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}, \quad \boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \quad \text{and} \quad \boldsymbol{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

where A is an $m \times n$ **matrix**, $\boldsymbol{x}$ and $\boldsymbol{b}$ are n (column) **vectors** (or $n \times 1$ matrices), then the linear system in the previous slide takes the form

$$A\boldsymbol{x} = \boldsymbol{b}$$

Notation for vectors

We usually assume vectors are column vectors and thus write, e.g.,

$$\boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = (x_1, x_2, \ldots, x_n)^T$$

Here, ^{T} is the **transpose operator** (more on this soon)

Consider the system

$$A\boldsymbol{x} = \boldsymbol{b}$$

If $\boldsymbol{b} = 0$, the system is **homogeneous** and always has the solution $\boldsymbol{x} = 0$ and so the "no solution" option in Theorem 18 goes away

Definition 19 (Matrix)

An m-by-n or $m \times n$ matrix is a rectangular array of elements of $\mathbb{R}$ or $\mathbb{C}$ with m rows and n columns,

$$A = [a_{ij}] = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}$$

We always list indices as "row,column"

We denote $\mathcal{M}_{mn}(\mathbb{F})$ or $\mathbb{F}^{mn}$ the set of $m \times n$ matrices with entries in $\mathbb{F} = \{\mathbb{R}, \mathbb{C}\}$. Often, we omit $\mathbb{F}$ in $\mathcal{M}_{mn}$ if the nature of $\mathbb{F}$ is not important

When $m = n$, we usually write $\mathcal{M}_n$

Basic matrix arithmetic

Let $A \in \mathcal{M}_{mn}, B \in \mathcal{M}_{mn}$ be matrices (of the same size) and $c \in \mathbb{F} = \{\mathbb{R}, \mathbb{C}\}$ be a scalar

- **Scalar multiplication**

$$cA = [ca_{ij}]$$

- **Addition**

$$A + B = [a_{ij} + b_{ij}]$$

- **Subtraction** (addition of $-B = (-1)B$ to A)

$$A - B = A + (-1)B = [a_{ij} + (-1)b_{ij}] = [a_{ij} - b_{ij}]$$

- **Transposition** of A gives a matrix $A^T = \mathcal{M}_{nm}$ with

$$A^T = [a_{ji}], \quad j = 1, \ldots, n, \quad i = 1, \ldots, m$$

Matrix multiplication

The (matrix) **product** of A and B, AB, requires the "inner dimensions" to match, i.e., the number of columns in A must equal the number of rows in B

Suppose that is the case, i.e., let $A \in \mathcal{M}_{mn}$, $B \in \mathcal{M}_{np}$. Then the i, j entry in $C := AB$ takes the form

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

Recall that the matrix product is not commutative, i.e., in general, $AB \neq BA$ (when both those products are defined, i.e., when $A, B \in \mathcal{M}_n$)

Special matrices

Definition 20 (Zero and identity matrices)

The **zero** matrix is the matrix 0_{mn} whose entries are all zero. The **identity** matrix is a square $n \times n$ matrix $\mathbb{I}_n$ with all entries on the main diagonal equal to one and all off diagonal entries equal to zero

Definition 21 (Symmetric matrix)

A square matrix $A \in \mathcal{M}_n$ is **symmetric** if $\forall i, j = 1, \ldots, n$, $a_{ij} = a_{ji}$. In other words, $A \in \mathcal{M}_n$ is symmetric if $A = A^T$

Properties of symmetric matrices

Theorem 22

1. *If* $A \in \mathcal{M}_n$*, then* $A + A^T$ *is symmetric*
2. *If* $A \in \mathcal{M}_{mn}$*, then* $AA^T \in \mathcal{M}_m$ *and* $A^TA \in \mathcal{M}_n$ *are symmetric*

X symmetric $\iff$ $X = X^T$, so use $X =$ the matrix whose symmetric property you want to check

1. True if $A + A^T = (A + A^T)^T$. We have

$$(A + A^T)^T = A^T + (A^T)^T = A^T + A = A + A^T$$

2. AA^T symmetric if $AA^T = (AA^T)^T$. We have

$$(AA^T)^T = (A^T)^T A^T = AA^T$$

A^TA works similarly

Determinants

Definition 23 (Determinant)

Let $A \in \mathcal{M}_n$ with $n \geq 2$. The **determinant** of A is the *scalar*

$$\det(A) = |A| = \sum_{j=1}^{n} a_{ij} C_{ij}$$

where $C_{ij} = (-1)^{i+j} \det(A_{ij})$ is the (i,j)-**cofactor** of A and A_{ij} is the submatrix of A from which the ith row and jth column have been removed

This is a cofactor expansion along the ith row
This is a recursive formula: it gives result in terms of n $\mathcal{M}_{n-1}$ matrices, to which it must in turn be applied, all the way down to

$$\det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

Two special matrices and their determinants

Definition 24

$A \in \mathcal{M}_n$ is **upper triangular** if $a_{ij} = 0$ when $i > j$, **lower triangular** if $a_{ij} = 0$ when $j > i$, **triangular** if it is *either* upper or lower triangular and **diagonal** if it is *both* upper and lower triangular

When A diagonal, we often write $A = \text{diag}(a_{11}, a_{22}, \ldots, a_{nn})$

Theorem 25

Let $A \in \mathcal{M}_n$ be triangular or diagonal. Then

$$\det(A) = \prod_{i=1}^{n} a_{ii} = a_{11} a_{22} \cdots a_{nn}$$

Inversion/Singularity

Definition 26 (Matrix inverse)

$A \in \mathcal{M}_n$ is **invertible** (or **nonsingular**) if $\exists A^{-1} \in \mathcal{M}_n$ s.t.

$$AA^{-1} = A^{-1}A = \mathbb{I}$$

A^{-1} is the **inverse** of A. If A^{-1} does not exist, A is **singular**

Theorem 27

Let $A \in \mathcal{M}_n$, $\boldsymbol{x}, \boldsymbol{b} \in \mathbb{F}^n$. Then

- *A invertible $\iff \det(A) \neq 0$*
- *If A invertible, A^{-1} is unique*
- *If A invertible, then $A\boldsymbol{x} = \boldsymbol{b}$ has the unique solution $\boldsymbol{x} = A^{-1}\boldsymbol{b}$*

Revisiting matrix arithmetic

With addition, subtraction, scalar multiplication, multiplication, transposition and inversion, you can perform arithmetic on matrices essentially as on scalar, if you bear in mind a few rules

- The sizes have to be compatible
- The order is important since matrix multiplication is not commutative
- Transposition and inversion change the order of products:

$$(AB)^T = B^T A^T \text{ and } (AB)^{-1} = B^{-1} A^{-1}$$

Eigenvalues / Eigenvectors / Eigenpairs

Definition 28

Let $A \in \mathcal{M}_n$. A vector $\boldsymbol{x} \in \mathbb{F}^n$ such that $\boldsymbol{x} \neq 0$ is an **eigenvector** of A if $\exists \lambda \in \mathbb{F}$ called an **eigenvalue**, s.t.

$$A\boldsymbol{x} = \lambda \boldsymbol{x}$$

A couple $(\lambda, \boldsymbol{x})$ with $\boldsymbol{x} \neq 0$ s.t. $A\boldsymbol{x} = \lambda \boldsymbol{x}$ is an **eigenpair**

If $(\lambda, \boldsymbol{x})$ eigenpair, then for $c \neq 0$, $(\lambda, c\boldsymbol{x})$ also eigenpair since $A(c\boldsymbol{x}) = cA\boldsymbol{x} = c\lambda\boldsymbol{x}$ and dividing both sides by c..

Similarity

Definition 29 (Similarity)

$A, B \in \mathcal{M}_n$ are **similar** $(A \sim B)$ if $\exists P \in \mathcal{M}_n$ invertible s.t.

$$P^{-1}AP = B$$

Theorem 30 ($\sim$ is an equivalence relation)

$A, B, C \in \mathcal{M}_n$, then

- $A \sim A$ *($\sim$ **reflexive**)*
- $A \sim B \implies B \sim A$ *($\sim$ **symmetric**)*
- $A \sim B$ *and* $B \sim C \implies A \sim C$ *($\sim$ **transitive**)*

Similarity (cont.)

Theorem 31

$A, B \in \mathcal{M}_n$ with $A \sim B$. Then

- $\det A = \det B$
- *A invertible $\iff$ B invertible*
- *A and B have the same eigenvalues*

Diagonalisation

Definition 32 (Diagonalisability)

$A \in \mathcal{M}_n$ is **diagonalisable** if $\exists D \in \mathcal{M}_n$ diagonal s.t. $A \sim D$

In other words, $A \in \mathcal{M}_n$ is diagonalisable if there exists a diagonal matrix $D \in \mathcal{M}_n$ and a nonsingular matrix $P \in \mathcal{M}_n$ s.t. $P^{-1}AP = D$

Could of course write $PAP^{-1} = D$ since P invertible, but $P^{-1}AP$ makes more sense for computations

Theorem 33

$A \in \mathcal{M}_n$ diagonalisable $\iff$ A has n linearly independent eigenvectors

Corollary 34 (Sufficient condition for diagonalisability)

$A \in \mathcal{M}_n$ has all its eigenvalues distinct $\implies$ A diagonalisable

For $P^{-1}AP = D$: in P, put the linearly independent eigenvectors as columns and in D, the corresponding eigenvalues

Linear combination and span

Definition 35 (Linear combination)

Let V be a vector space. A **linear combination** of a set $\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_k\}$ of vectors in V is a *vector*

$$c_1 \boldsymbol{v}_1 + \cdots + c_k \boldsymbol{v}_k$$

where $c_1, \dots, c_k \in \mathbb{F}$

Definition 36 (Span)

The set of all linear combinations of a set of vectors $\boldsymbol{v}_1, \dots, \boldsymbol{v}_k$ is the **span** of $\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_k\}$,

$$\operatorname{span}(\boldsymbol{v}_1, \dots, \boldsymbol{v}_k) = \{c_1 \boldsymbol{v}_1 + \cdots + c_k \boldsymbol{v}_k : c_1, \dots, c_k \in \mathbb{F}\}$$

Finite/infinite-dimensional vector spaces

Theorem 37

The span of a set of vectors in V is the smallest subspace of V containing all the vectors in the set

Definition 38 (Set of vectors spanning a space)

If $\text{span}(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k) = V$, we say $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k$ **spans** V

Definition 39 (Dimension of a vector space)

A vector space V is **finite-dimensional** if some set of vectors in it spans V. A vector space V is **infinite-dimensional** if it is not finite-dimensional

Linear (in)dependence

Definition 40 (Linear independence/Linear dependence)

A set $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k\}$ of vectors in a vector space V is **linearly independent** if

$$(c_1 \boldsymbol{v}_1 + \cdots + c_k \boldsymbol{v}_k = 0) \Leftrightarrow (c_1 = \cdots = c_k = 0)\,,$$

where $c_1, \ldots, c_k \in \mathbb{F}$. A set of vectors is **linearly dependent** if it is not linearly independent.

If linearly dependent, assume w.l.o.g. that $c_1 \neq 0$, then

$$\boldsymbol{v}_1 = -\frac{c_2}{c_1}\boldsymbol{v}_2 - \cdots - \frac{c_k}{c_1}\boldsymbol{v}_k$$

i.e., $\boldsymbol{v}_1$ is a linear combination of the other vectors in the set

Theorem 41

Let V be a finite-dimensional vector space. Then the ***cardinal*** *(number of elements) of every linearly independent set of vectors is less than or equal to the number of elements in every spanning set of vectors*

E.g., in $\mathbb{R}^3$, a set with 4 or more vectors is automatically linearly dependent

Basis

Definition 42 (Basis)

Let V be a vector space. A **basis** of V is a set of vectors in V that is both linearly independent and spanning

Theorem 43 (Criterion for a basis)

A set $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ of vectors in a vector space V is a basis of V $\iff$ $\forall \mathbf{v} \in V$, $\mathbf{v}$ can be written uniquely in the form

$$\mathbf{v} = c_1 \mathbf{v}_1 + \cdots + c_k \mathbf{v}_k,$$

where $c_1, \ldots, c_k \in \mathbb{F}$

Plus/Minus Theorem

Theorem 44 (Plus/Minus Theorem)

S a nonempty set of vectors in vector space V

- *If S is linearly independent and $V \ni \mathbf{v} \notin \operatorname{span}(S)$, then $S \cup \{\mathbf{v}\}$ is linearly independent*
- *If $\mathbf{v} \in S$ is linear combination of other vectors in S, then $\operatorname{span}(S) = \operatorname{span}(S - \{\mathbf{v}\})$*

More on bases

Theorem 45 (Basis of finite-dimensional vector space)

Every finite-dimensional vector space has a basis

Theorem 46

Any two bases of a finite-dimensional vector space have the same number of vectors

Definition 47 (Dimension)

The **dimension** $\dim V$ of a finite-dimensional vector space V is the number of vectors in any basis of the vector space

Theorem 48 (Dimension of a subspace)

Let V be a finite-dimensional vector space and $U \subset V$ be a subspace of V. Then $\dim U \leq \dim V$

Constructing bases

Theorem 49

Let V be a finite-dimensional vector space. Then every linearly independent set of vectors in V with $\dim V$ elements is a basis of V

Theorem 50

Let V be a finite-dimensional vector space. Then every spanning set of vectors in V with $\dim V$ elements is a basis of V

To finish: the "famous" "growing result"

Theorem 51

Let $A \in \mathcal{M}_n$. The following statements are equivalent (TFAE)

1. *The matrix A is invertible*
2. $\forall \boldsymbol{b} \in \mathbb{F}^n$, $A\boldsymbol{x} = \boldsymbol{b}$ *has a unique solution* ($\boldsymbol{x} = A^{-1}\boldsymbol{b}$)
3. *The only solution to* $A\boldsymbol{x} = 0$ *is the trivial solution* $\boldsymbol{x} = 0$
4. $RREF(A) = \mathbb{I}_n$
5. *The matrix A is equal to a product of elementary matrices*
6. $\forall \boldsymbol{b} \in \mathbb{F}^n$, $A\boldsymbol{x} = \boldsymbol{b}$ *has a solution*
7. *There is a matrix* $B \in \mathcal{M}_n$ *such that* $AB = \mathbb{I}_n$
8. *There is an invertible matrix* $B \in \mathcal{M}_n$ *such that* $AB = \mathbb{I}_n$
9. $\det(A) \neq 0$
10. 0 *is not an eigenvalue of* A